ON BEHAVIORAL IMPLEMENTATION

by

ECE YEGANE

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APPROVED BY:

Doç. Dr. Mehmet Barlo ....
(Thesis Supervisor)

Prof. Dr. Özgür Kıbrıs ....

Prof. Dr. Alpay Filiztekin ....

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ABSTRACT

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Ece YEGANE

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Thesis Supervisor: Assoc. Prof. Mehmet BARLO

Keywords: Behavioral implementation, Nash implementation, bounded rationality.

This thesis studies implementation under complete information with boundedly rational agents. We introduce a notion called maximal choice set and define an associated maximal choice monotonicity condition on social choice rules. We show that this condition is necessary for Nash implementation when choice correspondences satisfy Sen's property α. Moreover, we prove that maximal choice monotonicity and no veto-power suffice for Nash implementation when agents have unique maximal choice sets. Our results, following Maskin (1999), provide some convenience when evaluating the Nash implementability of social choice correspondences without checking for the existence of the associated Moore and Repullo sets.

ÖZET

DAVRANIŞSAL UYGULAMA ÜZERİNE

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Anahtar Kelimeler: Davranışsal uygulama, Nash uygulaması, kısıtlı rasyonellik.

Bu tezde tam bilgi ve kısıtlı akılcılık altında uygulama analizi incelenmektedir. Maksimal seçim kümeleri kurgusunu sunup sosyal seçim kuralları üzerine alakalı bir tekdüzelik kavramı tanımlıyoruz. Bu tekdüzelik nosyonunun Sen'in α özelliği ile beraber, sosyal seçim kurallarının Nash dengesi altında uygulanması için gerekli olduğunu kanıtlıyoruz. Devamında, alakalı tekdüzelik nosyonunun ve veto yetkisinin yokluğu ile beraber maksimal seçim kümelerinin tekliği varsayımının sosyal seçim kurallarının Nash dengesi altında yeterli olduğunu gösteriyoruz. Elde ettiğimiz sonuçlar, alakalı Moore ve Repullo kümelerinin varlığını kontrol etmeden Maskin (1999) tarafından kurulan Nash dengesi altında sosyal seçim kurallarının uygulanabilirliğinin değerlendirilmesi açısından elle tutulur kolaylık sağlamaktadır.

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Contents

1 Introduction

A social planner wishes to implement a pre-determined social choice rule that selects the socially optimal outcomes based on the preferences of individuals, by designing the institution through which the members of the society will interact. Implementation theory works on the question of designing a game form (or a mechanism) such that when individuals play it, the equilibrium outcomes of the game will coincide with the outcomes of the social choice rule in question. One way to accomplish this goal is to obtain preferences of the individuals and choose the socially optimal outcomes that is dependent on these preferences accordingly. However, it may be costly for the planner to elicit these preferences. More importantly, if individuals know the social choice rule the planner abides by, they may have an incentive to falsely report their preferences. Hence, in implementation theory, it is assumed that the individual preferences are unknown to the social planner and the goal of the planner is to design a game so that the equilibrium actions of individuals are reliable to select the true socially optimal outcome.

Most of the previous results on implementation work under the assumption of rational individual preferences. However, it is now well-known that individual behavior may not be consistent with a rational preference relation (e.g. Tversky (1969), Thaler (1980)). As a matter of fact, people are prone to cognitive biases such as status quo bias, endowment effect, attraction effect, which typically result in violations of the standard rationality assumptions in economics (e.g. Huber, Payne, and Puto (1982), Kahneman, Knetsch, and Thaler (1991)). The insights from behavioral economics and psychology on the systematic deviations from rationality have received wide attention in the popular culture. Moreover, there is a recent tendency among economists as well as policymakers on effectively using these behavioral insights to help people with making better decisions for themselves. In particular, Thaler and Sunstein (2008)'s book *Nudge* has received wide media attention and became influential in guiding government policies. For instance, the Behavioral Insights Team, also known as the *Nudge Unit*, has been established in 2010 in the United Kingdom in order to improve the government policies using ideas and recommendations from the behavioral sciences [1]. In the United States, President Obama has released an executive order in 2015, emphasizing the importance of behavioral insights to deliver better policy outcomes at lower costs and at the same time encouraging executive departments and agencies to incorporate these insights into policy-making [2]. In 2017, a behavioral public policy unit with a similar purpose has also been established in Turkey [3]. Considering the attention behavioral insights has received from decision makers, the analysis of implementation can be consequently improved with the new information on how people actually make decisions and behave. Therefore, relaxing the rationality assumptions in the usual implementation framework is a relevant and important question to analyze.

In this thesis, we introduce a notion which we call *maximal choice set*. A maximal choice set of an alternative x of an agent is a set of alternatives from which the alternative x is chosen from and there is no other set strictly containing the maximal choice set with the property that x is still being chosen by that agent. Our notion of a maximal choice set is related to Sugden (2004)'s opportunity criterion. Sugden (2004) formulates the opportunity criterion in order to make normative judgments on resource allocations in environments without using the standard assumptions of rationality. In that regard, the opportunity criterion attaches value directly to the abundance of opportunities provided to individuals while keeping them responsible from how they use these opporunities. In the maximal choice set definition, we

[1]Please refer to https://www.theguardian.com/society/2010/sep/09/cameron-nudge-unit-economic-behaviour

[2]Please refer to https://obamawhitehouse.archives.gov/the-press-office/2015/09/15/executive-order-using-behavioral-science-insights-better-serve-american

[3]Please refer to https://www.aa.com.tr/tr/ekonomi/davranissal-kamu-politikalari-ekonomi-bakanliginda-hayata-geciyor/971671

have a similar purpose: to offer the richest set of alternatives while making sure that the alternative with which the maximal choice set is defined will be chosen from this maximal set.

We define our environment using choice correspondences instead of preference relations while relaxing the rationality assumptions imposed on choice. By using the maximal choice set structure, we define a maximal choice monotonicity condition akin to Maskin monotonicity. Existence of a rational preference relation (meaning that behavior can be represented by a binary relation that is reflexive, complete and transitive) requires that the choice correspondence must satisfy two axioms property α and property β as formulated by Sen (1971). Property α restricts choice correspondences as the choice set shrinks while property β imposes a restriction as the choice set expands. In words, property α requires that if an alternative is chosen from a set S, then it must be chosen whenever that alternative is available in any subset of S. On the other hand, a choice correspondence satisfies property β if whenever two alternatives are chosen from a set S, then in any superset of S, if one of the alternatives chosen in S is chosen, then the other alternative chosen in S must be chosen as well. We use Nash equilibrium as the equilibrium concept to determine the strategic outcomes of the mechanism we wish to implement. However, as we allow for choice behavior possibly inconsistent to represent via a rational preference relation, the Nash equilibrium we use is locally defined using the outcome function of the implementing mechanism.

For the necessity side of implementation, we show that maximal choice monotonicity is a necessary condition when choice correspondences satisfy property α. For the sufficiency part of implementation, we show that our notion of maximal choice monotonicity and the no-veto power property suffice for implementation under the assumption that each agent's maximal choice set for every alternative is unique.

The uniqueness of maximal choice sets can be obtained by imposing Sen's property γ on choice correspondences. Property γ demands that if an alternative is chosen from two sets, then it must be chosen from the union of these two sets as well. If a choice correspondence satisfy property α and property β, then it satisfies property γ; but the other way around is not necessarily true. Therefore, uniqueness of maximal choice sets is a weaker requirement that property γ. However, when property γ is satisfied, maximal choice set has a special structure: it is equal to the union of every choice set where the alternative itself is chosen from.

As game theory is the study of strategic interactions, it has a major role in the study of implementation. In fact, implementation theory is often viewed as the reverse engineering approach to game theory for the following reason: game theory models a certain environment as a game and analyzes the possible outcomes of that game using an equilibrium concept. On the other hand, implementation theory starts with the set of outcomes aimed to be attained by the society and which usually have socially plausible properties. The goal of implementation is to design the game form through which individuals will interact and the set of equilibrium outcomes of that game will be equivalent to the outcomes desired to be achieved in the first place. From the relation between implementation theory and game theory, it follows that the information structure and the equilibrium concept which is used have profound effects on the question of implementation. In this thesis, we assume complete information, meaning that the members of the society know the true state of the world which determines the individual preferences. We use Nash equilibrium as the solution concept, in accordance with the complete information assumption. That is, when we question the implementability of a social choice rule, we inquire whether a it is implementable as a Nash equilibrium outcome of some game.

We now summarize previous literature on Nash implementation which is related to our analysis. In Section 2 of Chapter 4, we analyze in detail how our results compare to previous studies on behavioral implementation. Maskin (1999) showed that two conditions on social choice rules, monotonicity and no veto power, are sufficient for Nash implementation and monotonicity set is a necessary condition for Nash implementation.[4] Maskin's seminal paper assumed that agents have an ordering over the set of alternatives, which implies that agents have rational preferences *i.e.* preference relations that satisfy reflexiveness, completeness

[4]The first version of Maskin's paper started circulating as a working paper in (1977) but it was published in 1999. Hence the reverse chronological order.

and transitivity assumptions. Existence of a preference relation satisfying the standard rationality assumptions requires the underlying choice correspondence to satisfy property α and property β.

Hurwicz (1986) noticed that Maskin (1999)'s proof did not actually use either of the completeness and transitivity assumptions. Consequently, he showed that the results in Maskin (1999) can be extended to environments where preferences are not necessarily complete or transitive. Furthermore, Hurwicz (1986) was the first study to introduce the choice framework into implementation theory. One limitation in this study is that the domain of the choice correspondences is restricted to one or two element subsets of the set of alternatives. There are several different definitions that can be used when jumping back and forth between choice correspondences and preference relations. One can define the preference over two alternatives by looking at the choices over pairs or by investigating whether one of them is chosen in some set where the other alternative is also present. Whenever choice correspondence satisfy property α and property γ, an associated preference relation exists and the two definitions relating choice correspondences to preference relations will be equivalent. Hence, the extension of Hurwicz (1986)'s restriction to a setting where choice correspondences are defined from all nonempty subsets to all nonempty subsets would require the axioms of property α and property γ.

Moore and Repullo (1990) supply a condition called condition μ which is both necessary and sufficient for Nash implementation. The setting of Moore and Repullo (1990) is more general than Maskin (1999), because Maskin monotonicity and no veto power together imply condition μ but not vice versa. However, condition μ relies on the existence of two families of sets and proving whether such sets exist can be complex. In order to overcome this issue, Sjostrom (1991) provides an algorithm which can be applied to check for the existence of the sets as required in condition μ.

Later on, Korpela (2012) generalizes Moore and Repullo (1990)'s condition μ to condition λ which is sufficient for Nash implementation without any assumptions on choice correspondences. Moreover, Korpela (2012) shows that condition λ is a necessary condition for Nash implementation when choice correspondences satisfy Sen's property α. De Clippel (2014) provides a more general framework by separating the three requirements in condition μ to three conditions; namely consistency, strong consistency and unanimity. Furthermore, De Clippel (2014) shows that the consistency is a necessary condition for Nash implementation without any restrictions on choice correspondences. De Clippel (2014) also shows that consistency, strong consistency and unanimity together are sufficient for Nash implementation.

As both Korpela (2012) and De Clippel (2014) provide conditions following Moore and Repullo (1990), their necessary and sufficient conditions are subject to the same problem of checking for the existence of these arbitrary sets. In order to overcome this problem, Korpela (2012) applies the algorithm of Sjostrom (1991) to his setting. However, the application of the algorithm in Sjostrom (1991) requires choice correspondences to satisfy Sen's property α and γ. When choice correspondences satisfy these two properties, we can obtain the associated base relation from each of the choice correspondences and Maskin (1999)'s or Hurwicz (1986)'s implementation proof will work in exactly the same way. Then, we can directly check for Maskin monotonicity and no veto power to see whether a social choice rule is implementable which is easier than using an algorithm. In that regard, we emphasize that the maximal choice set structure we use allows us to define necessary and sufficient conditions following Maskin (1999) without the added complexity of checking for the existence of appropriate choice sets that is present in Korpela (2012) and De Clippel (2014)'s setting.

This thesis is organized as follows: Chapter 2 provides two motivating examples incorporating behavior which violates the standard rationality assumptions. Chapter 3 introduces the preliminaries for choice correspondences and implementation. Chapter 4 presents our necessity and sufficiency results, as well as previous findings on implementation. Chapter 5 concludes.

2 Examples with attraction effect

Decoy alternatives can cause preference reversals when they are introduced in the choice set, resulting in a violation of standard rationality assumptions on choice correspondences. When the decoy option is inferior to one of the alternatives, say x, in terms of all relevant attributes and at the same time, compared to another alternative say y, it is superior in some attributes and inferior in others, the introduction of the decoy can affect the choice of the individual in favor of x, the non-dominated alternative. This is referred to as the attraction effect or the asymmetric domination effect.

The following example due to Herne (1997), demonstrates how the presence of a decoy alternative causes attraction effect in a policy-making context and how to implement socially optimal goals in this situation. In September 1993, Finland took the decision of building a new nuclear power plant to parliamentary vote with some people opposing to the construction of a nuclear power plant. The main argument of the opponents of the new nuclear power plant was to use an alternative production system where energy is generated by small capacity power plants such as solar or wind power. However, even though those who are opposed the new nuclear power plant did not consider coal as one of the alternatives to nuclear power, supporters of the nuclear power plant compared coal with nuclear power. In this case, coal is deliberately used by the supporters of nuclear power as a decoy alternative, in the sense that it is not intended to be implemented but it is still present in the consideration set in order to increase the attractiveness of nuclear power for the following reason: Nuclear power dominates coal as it is more environment friendly and reliable in terms of the stability of energy production and price. On the other hand, solar power is better for the environment when compared to both nuclear power and coal. But in terms of reliability, high costs of solar panels and intermittency of the energy source make it less appealing in comparison to both nuclear power and coal. That is, coal is dominated by nuclear power in both dimensions while it is dominated by solar power in one dimension and it dominates solar power in the other. Hence, coal is asymmetrically dominated by nuclear power in the presence of solar power.

Motivated by these findings, we present two examples in both of which the social choice rule we aim to implement consists of the Bernheim-Rangel optimal alternatives: An alternative x is *Bernheim-Rangel optimal* if there is no other alternative y where x is never chosen in any subset of the set of available alternatives containing both x and y by any individual in the society. While both examples feature two states, in the first and simple example we limit the set of all available alternatives to nuclear power, solar power and coal and identify a mechanism which implements the social choice rule given by Bernheim-Rangel optimal alternatives in Nash equilibrium. Even though this example is a clear display of Nash implementation of a well-established social choice rule with agents who do not satisfy the standard assumptions of rationality (neither property α nor γ holds in this example), it involves choice correspondences that do not satisfy a critical assumption, the uniqueness of maximal choice sets, which we employ in our sufficiency result. To that regard, the second example involves four alternatives and choice correspondences that satisfy the uniqueness of maximal choice sets while not obeying properties α and γ.

2.1 A simple example with three alternatives and two states

Suppose that there are three relevant alternatives; solar power, nuclear power and coal. Meanwhile, there are two possible states of the world: the first one (θ) is the "usual" state of the world, while the second (θ') concerns a situation where experts are warning of the dangers and consequences of a potential earthquake prompting public discussions and news about the risk of a catastrophic nuclear power plant accidents.

We argue that the following summarizes this situation: Let $X = \{\mathbf{c}oal, \mathbf{s}olar, \mathbf{n}uclear\}$ be the set of alternatives, while the set of states is given by $\Theta = \{\theta, \theta'\}$. The choice correspondences of agents are given as follows:

S	$C_1(S,\theta)$	$C_1(S,\theta')$	$C_2(S,\theta)$	$C_2(S,\theta')$
c,n,s	n	s	n	c
n,s	s	s	s	s
c,n	n	c	n	c
c,s	s	c,s	s	c

It maybe useful to point out that the above table completely describes agents' choice in every state. For example, agent 1 chooses alternative n in state θ from the choice set c, n, while agent 2 chooses alternative c in state θ' from the choice set c, n, s.

In state θ, both agents have the same choice correspondence and their behavior is subject to attraction effect as described above; and in θ', both agents find nuclear power undesirable. Notice that, nuclear power is chosen from the entire set of alternatives in θ, but it is not chosen when only nuclear power and solar power are offered. Hence, the choice correspondences of both agents at θ violates property α. Moreover, solar power is chosen when coal and solar power are available as well as when nuclear power and solar power are available. However, solar power is not chosen from the union of these two sets. Thus, the choice correspondences of both agents at θ violate property γ.

We consider the behavioral welfare analysis criterion proposed by Bernheim and Rangel (2009) as the social choice rule to be implemented. An alternative $x \in X$ is *BR-efficient* in θ if there is no other alternative $y \in X$ with $x \notin C_i(S, \theta)$ for any non-empty subset S of X with $x, y \in S$ for all agents $i = 1, 2$. The social choice rule f^{BR} selects all BR-efficient alternatives at a given state of the world.

The set of $f^{BR} - optimal$ alternatives in state θ are nuclear power and solar energy, while in θ' solar energy and coal are the $f^{BR} - optimal$ alternatives.

We denote agent i's maximal choice set of alternative x in state θ by $O_i(x, \theta)$. Then, each agent's maximal choice sets of all alternatives in every state of the world can be derived from their choice correspondences and are given as follows:

	$O_1(\cdot, \theta)$	$O_2(\cdot, \theta)$	$O_1(\cdot, \theta')$	$O_2(\cdot, \theta')$
s	$\{n, s\}$ and $\{c, s\}$	$\{n, s\}$ and $\{c, s\}$	$\{c, n, s\}$	$\{n, s\}$
n	$\{c, n, s\}$	$\{c, n, s\}$	$\{n\}$	$\{n\}$
c	$\{c\}$	$\{c\}$	$\{c, n\}$ and $\{c, s\}$	$\{c, n, s\}$

Now, consider the following simple mechanism where the strategy set of agent 1 is $S_1 = \{A, B, C, D\}$ and that of 2 is $S_2 = \{L, M, R\}$ while the letters inside the matrix show the alternative assigned by the mechanism to agents' corresponding strategies, the so-called outcome function.

1/2	L	M	R
A	s	n	c
B	n	s	c
C	s	c	s
D	s	s	s

The strategy profiles $(C, L), (D, L), (D, R), (A, M)$, and (C, R) form a Nash equilibrium in state θ. The outcome of these strategy profiles are either s or n which are f^{BR}-optimal in θ. Similarly, the following strategy profiles form a Nash equilibrium at θ': $(A, R), (B, R), (D, L), (D, M)$, and (D, R) yielding an outcome of either s or c both of which are the only f^{BR}-optimal alternatives in state θ'.

As the set of f^{BR}-optimal outcomes coincide exactly with the set of Nash equilibrium outcomes of the mechanism, we say that this mechanism implements f^{BR} in Nash equilibrium.

The maximal choice sets of alternatives are not unique in this example. As it is shown, this does not prevent us from implementing socially optimal alternatives. However, our sufficiency direction of our Nash implementation result calls for the uniqueness of maximal choice sets.

Therefore, we present the following example where we implement BR-efficient alternatives when choice correspondences in the society are such that property α and uniqueness of maximal choice sets hold, while property γ is violated.

2.2 An example with four alternatives and two states

Let the set of alternatives be $X = \{\mathbf{c}oal, \mathbf{n}uclear, \mathbf{s}olar, \mathbf{w}ind\}$ while the set of states is given by $\Theta = \{\theta, \theta'\}$. The choice correspondences of agents are given in the following table:

S	$C_1(S,\theta)$	$C_2(S,\theta)$	$C_1(S,\theta')$	$C_2(S,\theta')$
c,n,s,w	n	n	s,w	c
c,n,s	n	n	s	c
c,n,w	n	n	w	c
c,s,w	s	s,w	s,w	c
n,s,w	w	n	s,w	s,w
c,n	n	n	c	c
c,s	s	s	s	c
c,w	c	c	w	c
n,s	n	n	s	s
n,w	n	n	w	w
s,w	w	s,w	s,w	s,w

Agent 1 in state θ is subject to attraction effect as it was described in the first example. That is, the presence of an inferior alternative, coal, increases the attractiveness of nuclear power. Hence, when nuclear power and coal are both available, agent 1 chooses nuclear power. Moreover, agent 1 in state θ chooses nuclear power from the sets nuclear power and solar power as well as nuclear power and wind power. However, nuclear power is not chosen from the union of these two sets. Hence, this choice correspondence violates property γ. Moreover, even though agent 1 chooses m from the choice set $\{c, n, s, w\}$ in state θ, he does not choose n from $\{n, s, w\}$. As the first choice set contains the second, property α is violated.

However, in spite of the violation of properties α and γ, each agent's maximal choice set of each alternative is unique. We present the resulting maximal choice sets as follows:

	$O_1(\cdot, \theta)$	$O_2(\cdot, \theta)$	$O_1(\cdot, \theta')$	$O_2(\cdot, \theta')$
c	$\{c, w\}$	$\{c, w\}$	$\{c, n\}$	$\{c, n, s, w\}$
s	$\{c, s, w\}$	$\{c, s, w\}$	$\{c, n, s, w\}$	$\{n, s, w\}$
n	$\{c, n, s, w\}$	$\{c, n, s, w\}$	$\{n\}$	$\{n\}$
w	$\{n, s, w\}$	$\{c, s, w\}$	$\{c, n, s, w\}$	$\{n, s, w\}$

Next, consider the following mechanism where the strategy spaces are given by $S_1 = \{E, F\}$ and $S_2 = \{A, B, C, D\}$ and the letters inside of the matrix shows the alternative assigned by the mechanism to agents' corresponding strategies.

1/2	A	B	C	D
E	c	s	w	c
F	w	n	s	n

Notice that, the strategy profiles (E, C), (F, B), (F, D) are Nash equilibria in θ, resulting in an outcome of either n or w equaling to the *BR-efficient* alternatives in θ. Similarly, (F, A), (F, C), (E, D) are Nash equilibria in θ', resulting in outcomes w, n and c, which are *BR-efficient* alternatives in θ'. Therefore, we say that this mechanism implements f^{BR} in Nash equilibrium.

3 Preliminaries

Let $N = \{1, ..., n\}$ be a finite set of agents with $n \geq 3$; X, a finite set of alternatives; $\Theta = \Theta_1 \times \Theta_2 \times ... \times \Theta_n$, the set of all states; and $C(\theta) = (C_1(\theta), ..., C_n(\theta))$, the individual choice correspondences of members of the society when the realized state is $\theta = (\theta_1, \theta_2, ..., \theta_n)$. Letting $\mathcal{X}$ be the set of all nonempty subsets of X, a choice correspondence of an agent i in state θ is a mapping $C_i(\theta) : \mathcal{X} \to \mathcal{X}$ such that $\forall S \in \mathcal{X}, C_i(S, \theta) \subseteq S$. In summary, the environment is denoted by $G = (N, X, \Theta, C(\theta)_{\{\theta \in \Theta\}})$.

3.1 Choice Correspondences

We next define three properties of choice correspondences as formulated by Sen (1971) that are critical in the analysis of classical rationality.

Property α requires that for any $S, T \in \mathcal{X}$ and for any $x \in S$, if $S \subseteq T$ and $x \in C(T)$, then it must be that $x \in C(S)$. In fact, it imposes a restriction on choice correspondences in terms of what is chosen when contracting the larger set to a smaller set. That is, if an alternative is chosen from a larger set, then it must be chosen from any other set that is obtained from removing some elements from the larger set.

On the other hand, property β demands that for any $S, T \in \mathcal{X}$ and for all $x, y \in C(S)$, if $S \subseteq T$, then whenever $x \in C(T)$, it must be that $y \in C(T)$. So, it restricts choice correspondences when expanding from smaller sets to larger sets. If two alternatives are chosen from a set S, then in any other superset of S, if one of the alternatives in $C(S)$ is chosen, then the other alternative in $C(S)$ must be chosen as well.

Property α and β are equivalent to Samuelson's Weak Axiom of Revealed Preference: If $x, y \in S \cap T, x \in C(S)$ and $y \in C(T)$, then $x \in C(T)$. Hence, when we say that an agent's choice behavior is *rational*, we mean that it satisfies property α and β.

Finally, property γ calls for the following: Let M be the any class of sets from $\mathcal{X}$ and let K be the union of every set in M. Then any x that belongs to $C(S)$ for every $S \in M$ must belong to $C(K)$. Similar to property β, γ is an expansion consistency property. If an alternative x is chosen from multiple sets, then it must be chosen from the union of these sets. It is trivial to show that property α and β together imply property γ.

The *preference relation obtained from the choice correspondence* $C_i(\theta)$, denoted by $R_i^C(\theta) \subset X \times X$, is a binary relation defined on X using the following condition:

$$xR_i^C(\theta)y \text{ if and only if there exist } S \in \mathcal{X} \text{ such that } y \in S \text{ and } x \in C_i(S, \theta). \tag{3.1}$$

Given a choice correspondence $C_i(\theta)$ and its associated preference relation $R_i^C(\theta)$, the *lower contour set of agent i of an alternative x* is denoted by $L_i(x, \theta)$ and equals to $\{y \in X : xR_i(\theta)y\}$.

The *choice correspondence induced by a binary relation* $R \subset X \times X$, denoted by $C_i^R(\theta) : \mathcal{X} \to \mathcal{X}$, is defined as follows:

$$C_i^R(S, \theta) := \{x \in S \mid xR_i(\theta)y \text{ for all } y \in S\}. \tag{3.2}$$

The following result is a restatement of Theorem 9 of Sen (1971, p.314), which shows that when choice correspondences satisfy property α and γ, we can consistently go back and forth between choice correspondences and the associated revealed preference relations.

Theorem 1 *$C^R = C$ if and only if $C : \mathcal{X} \to \mathcal{X}$ satisfies properties α and γ.*

Proof. Suppose that $C^R = C$. Take any $x \in C(S)$. By the definition of C^R, it must be that xRy for all $y \in S$. Take any set $T \subset S$ with $x \in T$, then xRy for all $y \in T$. Thus, by the definition of C^R, $x \in C(T)$. This implies that C satisfies property α. Now take any sets, S, T, such that $x \in C(S)$ and $x \in C(T)$. Then, xRy for all $y \in S$ and all $y \in T$. Moreover, xRy for any $y \in S \bigcup T$. Hence $x \in C^R(S \bigcup T)$. Thus, C satisfies property γ. For the converse, take any C which satisfies property α and γ. Let $x \in C^R(S)$, by definition, xRy for all $y \in S$. As C^R satisfies property α, $x \in C(\{x, y\})$ for any $y \in S$. By property γ, $x \in C(S)$. ■

When a choice correspondence satisfies property α and property γ, we can define the associated preference relation from the choice correspondence using definition 3.1 and then reobtain the same choice correspondence using definition 3.2. Sen (1971) refers to such choice correspondences as *normal*.

3.2 Choice Sets

Next, we introduce the notion of maximal choice sets which we will employ in our construction and proofs:

Definition 1 *A maximal choice set of an alternative $x \in X$ in state $\theta \in \Theta$ of agent $i \in N$, denoted by $O_i(x, \theta)$, is $S \in \mathcal{X}$ such that $x \in C_i(S, \theta)$ and there is no set $\tilde{S} \in \mathcal{X}$ strictly containing S with $x \in C_i(\tilde{S}, \theta)$.*

Our notion of a maximal choice set is related to Sugden (2004)'s opportunity criterion. Sugden (2004) formulates the opportunity criterion in order to make normative judgments on resource allocations in environments without using the classical assumptions of rationality. In that regard, the opportunity criterion attaches value directly to the abundance of opportunities provided to individuals. In the maximal choice set definition, we have a similar purpose: to offer the richest set of alternatives while making sure that the alternative with which the maximal choice set is defined will be chosen from this maximal set.

It should be pointed out that for every alternative $x \in X$ and state $\theta \in \Theta$ and agent $i \in N$, a maximal choice set of an alternative $x \in X$ in state $\theta \in \Theta$ of agent $i \in N$ must exist. This is due to the following: $x \in C_i(\{x\}, \theta)$ for all x, θ, i. If $\{x\}$ is not a maximal choice set it means there is $S \neq \{x\}$ and $x \in C_i(S, \theta)$. If, in turn, S is not a maximal choice set then there is $\tilde{S}$ strictly containing S with $x \in C_i(\tilde{S}, \theta)$. Noticing that the strict containment relation is transitive (hence, acyclic) and X being finite establishes the desired observation. However, given x, θ, i, we may have two distinct maximal choice sets; i.e. $O_i(x, \theta)$ is not necessarily uniquely determined. The following simple example illustrates.

Example 1 *Suppose there are three alternatives, $X = \{a, b, c\}$ and consider the following choice correspondence, $C(\{a, b, c\}) = a$, $C(\{a, b\}) = b$, $C(\{b, c\}) = b$, $C(\{a, c\}) = c$ and all alternatives are trivially chosen when they are the only available alternative. Alternative b is chosen from two sets, $\{a, b\}$ and $\{b, c\}$ with none of the two sets strictly containing the other. Therefore, the maximal choice set of b is not unique and is equal to either $\{a, b\}$ or $\{b, c\}$.*

We emphasize that the uniqueness of maximal choice sets is crucial in the sufficiency direction of our construction.

Assumption 1 *For every given $i \in N$ and $x \in X$ and $\theta \in \Theta$, $\#|O_i(x, \theta)| = 1$.*

One way to guarantee the uniqueness of maximal choice sets is to impose property γ on the profile of choice correspondence. Notice that, when the choice correspondence of agent i in state θ satisfies property γ, then given an alternative x, $O_i(x, \theta)$ is uniquely determined and equals to $\bigcup\limits_{S \in \mathcal{X}} \{S : x \in C_i(S, \theta)\}$. However, the reverse of the statement is not necessarily true, that is, in a given state, existence of unique maximal choice sets for every alternative for every agent i does not imply property γ. To illustrate, we provide the following example.

Example 2 *In the choice correspondence below, associated maximal choice set of each alternative is unique, $O(x) = \{x, y, z, t\}$, $O(y) = \{y, z, t\}$, $O(z) = \{x, y, z\}$, $O(t) = \{t, z\}$. However, $C\{x, y\} = x$ and $\{x, z\} = x$ but $C\{x, y, z\} = z$. Hence, this choice correspondence violates property γ.*

S	$C(S)$
x, y, z, t	x
x, y, z	z
x, y, t	x
x, z, t	x
y, z, t	y
x, y	x
x, z	x
x, t	x
y, z	y
y, t	y
t, z	t

Therefore, uniqueness of maximal choice sets is a weaker requirement on choice correspondences than property γ. When choice correspondences satisfy properties α and γ, then by Theorem 1, there exists an associated reflexive and complete preference relation (not necessarily transitive) from which we could build the same choice correspondences using normality given in condition 3.2. Hence, we have the following result:

Lemma 1 *Assume that choice correspondences satisfy property α and γ. Then, for any $\theta \in \Theta$, $x \in X$, $i \in N$, $O_i(x, \theta) = L_i(x, \theta)$.*

Proof. Let $y \in O_i(x, \theta)$. Then, there exists an $S \in \mathcal{X}$ where $x, y \in S$ and $x \in C(S)$. Thus, by (3.1), $xR_i(\theta)y$; so $y \in L_i(x, \theta)$. Now, let $y \in L_i(x, \theta)$. Then, for some $S \in \mathcal{X}$, it must be that $x, y \in S$ and $x \in C(S)$; so $S \subseteq O_i(x, \theta)$; hence, $y \in O_i(x, \theta)$. ■

The fact that maximal choice sets and lower contour sets are not equivalent in general can easily be seen from Example 1, where $L(b) = \{a, b, c\}$ which is not a maximal choice set of b.

We have to confess that when property α and uniqueness of maximal choice sets hold at the same time, property γ is also satisfied. This observation is presented as follows:

Lemma 2 *Suppose that $C : \mathcal{X} \to \mathcal{X}$ satisfies property α and maximal choice set of each alternative is unique, then C satisfies property γ.*

Proof. Suppose not and C does not satisfy property γ. Then there exist an alternative x and sets S and T in $\mathcal{X}$ with $x \in C(S)$ and $x \in C(T)$ but $x \notin C(S \cup T)$. Notice that $x \notin C(S \cup T)$ implies $S \nsubseteq T$ and $T \nsubseteq S$. Because the maximal choice set of every alternative is unique, there must exist a set K with $S \subset K$, $T \subset K$ and $x \in C(K)$, implying that $(S \cup T) \subset K$. As $x \in C(K)$, by property α, it must be $x \in C(S \cup T)$ which is a contradiction. ■

3.3 Implementation

A *social choice rule* $f : \Theta \to \mathcal{X}$ assigns a nonempty subset of alternatives to each state $\theta \in \Theta$. Given a state θ, the alternatives $f(\theta) \in \mathcal{X}$ will be referred to as *f-optimal alternatives* in state θ.

A *mechanism* $\mu = (S, g)$ specifies a strategy (message) space $S_i \neq \emptyset$ for each agent $i \in N$ and an outcome function $g : S \to X$ where $S = \times_{i \in N} S_i$. Opportunity sets in a Nash implementation framework is introduced by De Clippel (2014) and are given as follows: Given a mechanism and a strategy profile for players $j \neq i$, agent i's opportunity set consists of all the alternatives that agent i can obtain via the outcome function of the mechanism by changing his strategies while the other agents are to play according to the given strategy profile. Hence, the opportunity set of an agent depends on the strategies of the other players and the mechanism in question. Formally, given a mechanism μ and a strategy profile for other players $j \neq i$, i.e. $s_{-i} \in S_{-i} = \prod_{j \neq i} S_j$, we define the associated *opportunity set* by $A_i^{\mu}(s_{-i}) = \{g(s_i, s_{-i}) \in X : s_i \in S_i\}$, which is trivially non-empty as g is a function mapping S into X. Next, using the associated opportunity set of a mechanism μ, we supply the definition of a Nash equilibrium of the game induced by that very mechanism as follows: Given a mechanism μ, a strategy profile $s^* \in S$ is a *Nash equilibrium* at θ of the mechanism μ if for all $i \in N$ we have that $g(s_i^*, s_{-i}^*) \in C_i(A_i^{\mu}(s_{-i}^*), \theta)$.

We now provide the definition of Nash implementation: f is said to be *Nash implementable in pure strategies* if there exists a mechanism $\mu = (S, g)$ such that:

1. For every $\theta \in \Theta$ and $x \in f(\theta)$, there exists a Nash equilibrium strategy profile $s^* \in S$ such that $g(s^*) = x$; and

2. For every $\theta \in \Theta$, if $s^* \in S$ is Nash equilibrium of $\mu = (S, g)$, then $g(s^*) \in f(\theta)$.

Namely, the set of f-optimal alternatives at θ and the set of Nash equilibria of the mechanism μ coincide for all $\theta \in \Theta$.

4 Nash Implementation

4.1 Maximal Implementation

In this section, we show that Maskin's well-known implementation results extend to our setting. To that regard, first we define our *maximal choice monotonicity* condition using maximal choice sets and show that it is a necessary condition for Nash implementation when choice correspondences satisfy property α. Then, we formulate our version of the no veto power property and prove that maximal choice monotonicity combined with no veto power under Assumption 1 suffice for Nash implementation without the need to restrict attention to properties α and/or γ.

Definition 2 *The social choice rule $f : \theta \twoheadrightarrow X$ is maximal choice monotone if for all $x \in f(\theta)$ and for all $i \in N$,*

$$O_i(x, \theta) \subseteq O_i(x, \theta') \textit{ for some } \theta' \in \Theta \textit{ implies } x \in f(\theta').$$

Maximal choice monotonicity condition requires that, if an alternative is f-optimal in some state θ, and any maximal choice set of this f-optimal alternative in state θ for every i is a subset of a maximal choice set of the same f-optimal alternative for every agent i in some other state θ', then that alternative must be f-optimal in θ' as well.

The necessity direction of implementation is provided by the following theorem:

Theorem 2 *Suppose that the choice correspondence of every player satisfies property α in every state. Then, f is monotone whenever it is implementable in Nash equilibrium.*

Proof. Suppose that f is implementable in Nash equilibrium by some mechanism $\mu = (S, g)$. Let $\theta, \theta' \in \Theta$ and $x \in f(\theta)$ and assume that $O_i(x, \theta) \subseteq O_i(x, \theta')$ for all $i \in N$. Then, there exists a Nash equilibrium of μ at θ, denoted by s^* such that $g(s^*) = x \in f(\theta)$ and $x \in C_i(A_i^\mu(s_{-i}^*), \theta)$ for all $i \in N$. In fact, $A_i^\mu(s_{-i}^*) \subseteq O_i(x, \theta)$ because $x \in A_i^\mu(s_{-i}^*)$ and $x \in C_i(A_i^\mu(s_{-i}^*), \theta)$ and $O_i(x, \theta)$ is a maximal non-empty subset of X such that $x \in C_i(O_i(x, \theta), \theta)$. Since $O_i(x, \theta) \subseteq O_i(x, \theta')$ for all $i \in N$, we have $A_i^\mu(s_{-i}^*) \subseteq S$ where S is one of the maximal choice sets at θ'. Moreover, by construction of maximal choice sets, $x \in C_i(O_i(x, \theta'), \theta')$ for all $i \in N$. As choice correspondence of every agent satisfies property α, we have, $x \in C_i(A_i^\mu(s_{-i}^*), \theta')$ for all $i \in N$. Hence, s^* also forms a Nash equilibrium of μ in state θ'. As μ implements f in Nash equilibrium, we have $g(s^*) = x \in f(\theta')$. ■

Next, we define our version of the no veto power condition:

Definition 3 *The social choice rule $f : \theta \twoheadrightarrow X$ satisfies no veto power property if for all $\theta \in \Theta$ and for all $x \in X$ and for all $i \in N$,*

$$O_j(x, \theta) = X \text{ for all } j \in N \setminus \{i\} \text{ implies } x \in f(\theta).$$

No veto power property demands that if the maximal choice set of an alternative is equal to the entire set of all alternatives for $n - 1$ individuals in an n-member society for a given state, then that alternative must be f-optimal in that state.

Now, we are ready to state and prove our sufficiency result:

Theorem 3 *If $n \geq 3$ and $f : \Theta \twoheadrightarrow X$ is an n-person social choice rule satisfying maximal choice monotonicity and no veto power property and Assumption 1 holds, then it is implementable in Nash equilibrium.*

Proof. We use the standard Maskin-Repullo type mechanism, $\mu = (g, S)$, which is defined as follows: Define the strategy space for each player $i \in N$ as $S_i = \Theta \times X \times \mathbb{N}$, where Θ is the set of states, X is the set of outcomes and $\mathbb{N}$ is the set of natural numbers. Let the associated outcome function g be defined by

1. if $s_i = (\theta, x, n)$ for all $i \in N$ and $x \in f(\theta)$, then $g(s) = x$;

2. if for all $i \in N \setminus \{j\}$ for some $j \in N$ we have $s_i = (\theta, x, n) \neq (\theta', x', m) = s_j$ with $x \in f(\theta)$, then $g(s) = x'$ whenever $x' \in O_j(x, \theta)$ and $g(s) = x$ otherwise;

3. In the remaining cases, $g(s) = x$ where x is the second component of the strategy profile announced by the player who has the highest index among those who reported the highest integer.

First, we show that for any $\theta \in \Theta$ and any $x \in f(\theta)$, there is a corresponding strategy profile which constitutes a Nash equilibrium in θ: Let $\theta \in \Theta$ and $x \in f(\theta)$ and consider the strategy profile $s_1 = s_2 = ... = s_n = (\theta, x, n)$ which falls into (1) in the definition of g, so the outcome is $g(s) = x$. Notice that, any player $i \in N$ can unilaterally deviate by announcing a different strategy $(\theta', x', n') \neq (\theta, x, n)$ and due to (2) of the definition of g obtain any $x' \in O_i(x, \theta)$. In other words, opportunity sets in this state of the mechanism takes the form of maximal choice sets. By construction, $x \in O_i(x, \theta)$ where $g(s) = x$ and $O_i(x, \theta) = A_i^{\mu}(s_{-i})$. Consequently for all $i \in N$, we have $g(s) \in C_i(A_i^{\mu}(s_{-i}), \theta)$. Hence, $(s_1, s_2, ..., s_n)$ forms a Nash equilibrium at θ.

In what follows, we show that any Nash equilibrium of the game induced by mechanism μ is f-optimal.

Observe that there can be three distinct type of Nash equilibria arising from the three rules of the mechanism. First, suppose that s^* is a Nash equilibrium that falls into (1) with $s_1^* = s_2^* = ... = s_n^* = (\theta', x, n)$ and $x \in f(\theta')$. In this state of the mechansim, opportunity sets take the form of maximal choice sets and the mechanism assigns the outcome x to this strategy profile. If θ' is the true state of the world, we immediately have $x \in f(\theta')$. Now suppose that the true state of the world is $\theta \neq \theta'$. By hypothesis, s^* is a Nash equilibrium in state θ' while the true state is θ, so it must be that $x \in C_i(O_i(x, \theta'), \theta)$ for all $i \in N$. By letting $S = O_i(x, \theta')$ we observe that $x \in C_i(S, \theta)$ implies by the defining property of the maximal choice sets that $S \subseteq O_i(x, \theta)$; hence, $O_i(x, \theta') \subseteq O_i(x, \theta)$. Thus, $x \in f(\theta')$ and $O_i(x, \theta') \subseteq O_i(x, \theta)$ and maximal choice monotonicity imply that $x \in f(\theta)$.

Next, suppose that s^* is a Nash equilibrium at θ that falls into (2). That is, $s_i = (\theta, x, n)$ for all $i \in N \setminus \{j\}$ for some $j \in N$ and $s_j = (\theta', x', n') \neq (\theta, x, n)$. Let the outcome of this strategy profile be $\bar{x}$. Then, opportunity set of agent j is equal to $O_j(x, \theta)$ while opportunity set of any other agent $i \neq j$ is equal X, as any $i \neq j$ can unilaterally deviate to case (3) and obtain any alternative. Hence, $O_i(\bar{x}, \theta) = X$ for all $i \in N \setminus \{j\}$. Then, no veto power property implies $\bar{x} \in f(\theta)$.

A similar argument can be made for the set of Nash equilibria that falls into case (3): Then, s^* is a Nash equilibrium with $g(s^*) = x$. In this situation, the opportunity set of any agent is equal to the set of all alternatives. Hence, it must be that $x \in O_i(x, \theta) = X$ for all $i \in N$. Then, by the no veto power property, we conclude that $x \in f(\theta)$. ■

At this point, it is necessary to explain why do we need the Assumption 1 in our sufficiency result. To understand exactly why we need this assumption, we should take a closer look at the part of the sufficiency proof where we show that any Nash equilibrium at θ' with an outcome x that falls into rule (1) must be f-optimal. Under such a strategy profile, any agent i can unilaterally deviate to rule (2) of the mechanism μ, which assigns one of the associated maximal choice sets of x to every agent i. If maximal choice sets were to be multiple, then in order to use maximal choice monotonicity, we should have shown that *any* maximal choice set of x at θ' is a subset of some maximal choice set of x at θ for all $i \in N$. However, the rule employed in our implementing mechanism does not allow for this; we can only show that *one* maximal choice set of x at θ, that is used within the mechanism μ, is a subset of one maximal choice set of x at θ' for any $i \in N$. Hence, in order to be able to use our maximal choice monotonicity notion with the Maskin-Repullo type mechanism defined as in our sufficiency proof, we emphasize that we need the assumption of unique maximal choice sets.

4.2 Relation to Other Results

In this section, we provide definitions of comparable notions from Nash implementation literature.

For the necessity part, we wish to emphasize that Assumption 1 is not needed for the necessity direction of implementation. If we were to use property α and the uniqueness of maximal choice sets for the necessity direction, by Lemma 2 we would also have property γ satisfied. In this case, by using Lemma 1, we could have done a more restrictive version of our necessity using lower contour sets as in Maskin (1999). That necessity result would then be the same as in Hurwicz (1986).

To briefly summarize, Maskin (1999)'s original Nash implementation proof employing lower contour sets can be interpreted as establishing these results for choice correspondences satistfying property α and β. In a similar fashion, Barlo and Dalkiran (2009) provide necessary and sufficiency results using semiorders which can be viewed as choice correspondences satistfying property α, γ and δ [1] Moreover, Lemma 1 implies that Maskin (1999)'s original proof employing lower contour sets can be generalized to a setting with choice correspondences satisfying property α and γ. Ray (2010) also obtains necessary and sufficiency results following Maskin (1999) and Hurwicz (1986). Ray (2010)'s sufficiency direction use choice correspondences which satisfy property α and γ while the necessity direction does not impose any restrictions on choice.

Next, we explain in detail two previous results on implementation by Korpela (2012) and De Clippel (2014). We also wish to point out that both of these papers use the same Nash equilibrium and Nash implementation definitions as ours.

[1] A choice correspondence satisfy property δ if for any sets $S, T \in \mathcal{X}$, with $S \subset T$ $x, y \in C(S)$ implies that $[x] \neq C(T)$. From Theorem 10 in Sen (1971, p.315), we know that if a choice correspondence satisfy property α, γ and δ, then the strict order P we obtain from the choice correspondence is transitive.

4.2.1 Korpela (2012)

While we provide necessary and sufficient conditions for Nash implementation following Maskin (1999), Korpela (2012) uses the generalization of condition μ of Moore and Repullo (1990), condition λ, which is given as follows:

Definition 4 (Condition λ) *There is a set $Y \subset X$ and for each $i \in N$ and $\theta \in \Theta$ and $x \in F(\theta)$, there is a set $R_i(x, \theta)$ with $x \in C_i^{\theta}(R_i(x, \theta))$ such that for all $\theta' \in \Theta$, the following three conditions are satisfied:*

i. If $x \in \cap_{i \in N} C_i(R_i(x, \theta), \theta')$, then $x \in f(\theta')$.

ii. If $y \in C_i(R_i(x, \theta), \theta') \cap (\cap_{j \neq i} C_i(Y, \theta'))$, then $y \in f(\theta')$.

iii. If $z \in \cap_{i \in N} C_i(Y, \theta')$, then $z \in f(\theta')$.

Using condition λ, Korpela (2012) obtains necessity for Nash implementation by restricting attention to choice correspondences satisfying property α, while the sufficiency is attained without the need of restrictions on choice correspondences. As Moore and Repullo (1990) provides a condition for Nash implementation that is both necessary and sufficient when individuals are rational (i.e. choice correspondences respect property α and β), Korpela (2012) shows that this condition can be generalized such that we have a necessary and sufficient condition for Nash implementation when choice correspondences respect property α. We now provide Korpela's necessity result.

Theorem 4 *Let $n \geq 3$ and all choice correspondences $C : \mathcal{X} \to \mathcal{X}$ satisfy property α. If a social choice rule f is implementable in Nash equilibrium, [2] then it satisfies condition λ.*

Proof. Suppose that f is implementable in Nash equilibrium by the mechanism $\mu = (g, S)$. Define the set Y as the range of g. That is,

$$Y \equiv \{x \in X \mid g(s) = x \text{ for some } s \in S\}$$

[2] Korpela (2012) refers to this equilibrium notion as *behavioral Nash equilibrium.*

Take any state $\theta \in \Theta$, $x \in f(\theta)$ and take any Nash equilibrium, $s^*(x,\theta)$, of the game induced by the mechanism μ such that $g(s^*(x,\theta)) = x$. Define the set $R_i(x,\theta) \equiv A_i^\mu(s_{-i}^*(x,\theta))$, that is the opportunity set of player i given that other players are playing their Nash equilibrium strategies which yields an outcome of x in state θ.

Now assume that the antecedent of (i) in condition λ holds. Then, every player chooses x from their opportunity set in state θ'. By the definition of Nash equilibrium, $s(x,\theta)$ forms a Nash equilibrium at θ' as well. As f is assumed to be Nash implementable, it must be that $x \in f(\theta')$.

Next, assume that the antecedent of (ii) in condition λ holds. Take $s_i \in S_i$ such that $g(s_i, s_{-i}^*(x,\theta)) = y$. As y is chosen from the set Y by all players but i, by property α, it must be that y is chosen from any subset of Y by all $j \in N \setminus \{i\}$ whenever it is available. This implies that, $y \in C_j(A_j^\mu(m_{-j}^*(x,\theta),\theta')$ for all $j \in N \setminus \{i\}$. For player i, we already have $y \in C_i(A_i^\mu(s_{-i}^*(x,\theta),\theta')$. Hence, $(s_i, s_{-i}^*(x,\theta))$ forms a Nash equilibrium at θ'. As f is Nash implementable, it must be that $y \in f(\theta')$.

Finally, suppose that the antecedent of (iii) in condition λ holds. Then, take a strategy profile $s \in S$ with $g(s) = z$. By property α, it must be that $z \in C_i(A_i^\mu(s_{-i}^*(x,\theta),\theta')$ for all $i \in N$. Hence we have $z \in f(\theta')$. ■

Next, we supply the sufficiency theorem and proof of Korpela (2012).

Theorem 5 *If $n \geq 3$ and f satisfies condition λ, then it is implemenable in Nash equilibrium.*

Proof. Suppose that f satisties condition λ. Define the mechanism $\mu = (g, S)$, which we will show that implements f in Nash equilibrium, as follows. Each player announces a strategy profile $(\theta, x, y, n) \in \Theta \times X \times Y \times \mathbb{N}$ with $x \in f(\theta)$.

Let the associated outcome function g be defined by,

1. if $s_i = (\theta, x, y, n)$ for all $i \in N$ and $x \in f(\theta)$, then $g(s) = x$;

2. if for all $i \in N \setminus \{j\}$ and for some $j \in N$ we have $s_i = (\theta, x, y, n) \neq (\theta', x', y', n') = s_j$, then $g(s) = y'$ whenever $y' \in R_j(x,\theta)$ and $g(s) = x'$ otherwise;

3. In the remaining cases, $g(s) = y$ where y is the third component of the strategy profile announced by the player who has the highest index among those who reported the highest integer.

First, we show that, for every f-optimal alternative, there is a corresponding Nash equilibrium of the game induced by the mechanism μ. Take any $\theta \in \Theta$, $x \in f(\theta)$ and consider the strategy profile where $s_i = (\theta, x, x, 0)$ for all $i \in N$. According to what the mechanism μ permits, any player can unilaterally deviate from this strategy profile and obtain any alternative in $R_i(x, \theta)$. By definition, we have $x \in C_i(R_i(x, \theta), \theta)$. Thus, this strategy profile forms a Nash equilibrium at θ with $g(s) = x$.

Next, we show that, every Nash equilibrium of the induced by the mechanism μ yields a f-optimal outcome. There are three type of Nash equilibria that can arise from the three states of mechanism μ. Suppose that s^* is a Nash equilibrium that falls into (1) with $s_i^* = (\theta, x, y, n)$. If θ is the true state of the world, then we have $x \in f(\theta)$. Suppose the true state of the world is $\theta' \neq \theta$. Then, any agent i can deviate to rule (2) of the mechanism and obtain any $x \in R_i(x, \theta)$. As this strategy profile forms a Nash equilibrium at θ, it also forms a Nash equilibrium at θ'. Thus, we have $x \in C_i(R_i(x, \theta), \theta')$, for all $i \in N$. By (i) of condition λ, we have $x \in f(\theta')$.

Now suppose that s^* is a Nash equilibrium at θ that falls into (2). That is, $s_i = (\theta, x, y, n)$ for all $i \in N \setminus \{j\}$ and for some $j \in N$, $s_j = (\theta', x', y', n') \neq (\theta, x, y, n)$. Let the outcome of this strategy profile be $\bar{x}$. Notice that any $i \neq j$ can unilaterally deviate to case (3) and obtain any alternative in the set Y while the lone player j can obtain any alternative in $R_i(x, \theta)$. Hence, $\bar{x} \in C_i(R_i(x, \theta), \theta') \cap (\cap_{j \neq i} C_i(Y, \theta'))$. By (ii) of condition λ, we have $\bar{x} \in f(\theta')$.

Finally, suppose s^* is a Nash equilibrium at θ' that falls into (3) with the outcome of this strategy profile being $\bar{x}$. Then any player $i \in N$ can unilaterally deviate and obtain any alternative in Y. This implies that $\bar{x} \in C_i(Y, \theta')$ for all $i \in N$. Then by (iii) of condition λ we have $\bar{x} \in f(\theta')$. ■

4.2.2 De Clippel (2014)

Similar to Korpela (2012), De Clippel (2014) provides necessary and sufficient conditions for Nash implementation in a similar fashion to condition μ of Moore and Repullo (1990). However, instead of providing a condition that is both necessary and sufficient, De Clippel (2014) breaks down the three conditions in condition μ, into three different conditions for Nash implementation, namely consistency, strong consistency and unanimity. Furthermore, De Clippel (2014) shows that the existence of consistent opportunity sets is a necessary condition for Nash implementation, without any restrictions on choice correspondences. We now analyze De Clippel's notion of consistency:

Definition 5 *A collection of sets* $\mathcal{O} = \{O_i(x, \theta) : i \in N, x \in F(\theta), \theta \in \Theta\}$ *is consistent with a social choice rule* $f : \Theta \to \mathcal{X}$ *if*

i. $x \in C_i(O_i(x, \theta), \theta)$ *for all* $i \in N$, $\theta \in \Theta$, $x \in f(\theta)$,

ii. For all $\theta, \theta' \in \Theta$ *and* $x \in f(\theta)$ *if* $x \in C_i(O_i(x, \theta), \theta')$ *for all* $i \in N$, *then* $x \in f(\theta')$.

Theorem 6 *If f is Nash implementable, then there exists a collection of opportunity sets which is consistent with f.*

Proof. Suppose that f is Nash implementable by mechanism $\mu = (g, S)$. Take any state $\theta \in \Theta$, $x \in f(\theta)$ and take any Nash equilibrium, s^*, of the game induced by the mechanism μ such that $g(s^*) = x$. This strategy profile defines an opportunity set for each player, $A_i(s^*_{-i})$. We have $x \in C_i(A_i(s^*_{-i}), \theta)$ for all $i \in N$, by the definition of Nash equilibrium, so the (i) of consistency is satisfied. Next, suppose that the antecedent of (ii) of consistency is satisfied. Then s^* forms a Nash equilibrium at θ' as well. As f implementable and the outcome of this strategy profile is x, we have $x \in f(\theta')$. ■

Definition 6 *The collection of opportunity sets* $\mathcal{O}$ *is strongly consistent with f if*

i. It is consistent with f,

ii. For all x, θ for which there exists j, θ' and $x' \in f(\theta')$ such that $x \in C_i(\mathcal{X}, \theta)$ for all $i \neq j$ and for player j, $x \in C_j(O_j(x', \theta'), \theta)$, then it must be that $x \in f(\theta)$.

Definition 7 *A social choice rule respects unanimity if for any $x \in X$ and $\theta \in \Theta$, whenever $x \in C_i(X, \theta)$ for all $i \in N$, then $x \in f(\theta)$.*

Theorem 7 *If f respects unanimity and there exists a collection of opportunity sets which is strongly consistent with f, then it is implementable in Nash equilibrium.*

Proof. Suppose that f satisties unanimity and there exists a collection of opportunity sets which is strongtly consistent with f. Define the mechanism $\mu = (g, S)$, which we will show that implements f in Nash equilibrium, as follows. Each player announces a strategy profile $(\theta, x, n) \in \Theta \times X \times \mathbb{N}$. Let the associated outcome function g be defined by,

1. if $s_i = (\theta, x, n)$ for all $i \in N$ and $x \in f(\theta)$, then $g(s) = x$;

2. if for all $i \in N \setminus \{j\}$ for some $j \in N$ we have $s_i = (\theta, x, n) \neq (\theta', x', m) = s_j$ with $x \in f(\theta)$, then $g(s) = x'$ whenever $x' \in A_j(s_{-j}(x, \theta))$ and $g(s) = x$ otherwise;

3. In the remaining cases, $g(s) = x$ where x is the second component of the strategy profile announced by the player who has the highest index among those who reported the highest integer.

First, we show that, for every f-optimal alternative, there is a corresponding Nash equilibrium of the game induced by the mechanism μ. Take any $\theta \in \Theta$, $x \in f(\theta)$ and consider the strategy profile where $s_i = (\theta, x, 0)$ for all $i \in N$. According to what the mechanism μ permits, any player can unilaterally deviate from this strategy profile and obtain any alternative in $A_i(s_{-i}(x, \theta))$. By (i) of consistency, we have $x \in C_i(A_i(s_{-i}(x, \theta)), \theta)$. Thus, this strategy profile forms a Nash equilibrium at θ with $g(s) = x$.

Next, we show that, every Nash equilibrium of the induced by the mechanism μ yields a f-optimal outcome. There are three type of Nash equilibria that can arise from the three states of mechanism μ. Suppose that s^* is a Nash equilibrium that falls into (1) with

$s_i^* = (\theta, x, n)$. If θ is the true state of the world, then we have $x \in f(\theta)$. Suppose the true state of the world is $\theta' \neq \theta$. Then, any agent i can deviate to rule (2) of the mechanism and obtain any $x \in R_i(x, \theta)$. As this strategy profile forms a Nash equilibrium at θ, it also forms a Nash equilibrium at θ'. Thus, we have $x \in C_i(A_i(s_{-i}^*(x, \theta)), \theta')$, for all $i \in N$. By (ii) of consistency, we have $x \in f(\theta')$.

Now suppose that s^* is a Nash equilibrium at θ that falls into (2). That is, $s_i = (\theta, x, n)$ for all $i \in N \setminus \{j\}$ and for some $j \in N$, $s_j = (\theta', x', n') \neq (\theta, x, n)$. Let the outcome of this strategy profile be $\bar{x}$. Notice that any $i \neq j$ can unilaterally deviate to case (3) and obtain any available alternative while the lone player j can obtain any alternative in $A_i(s_{-i}^*(x, \theta))$. By (ii) of strong consistency, we have $\bar{x} \in f(\theta')$.

Finally, suppose s^* is a Nash equilibrium at θ that falls into (3) with the outcome of this strategy profile being $\bar{x}$. Then any player $i \in N$ can unilaterally deviate and obtain any alternative in X. This implies that $\bar{x} \in C_i(X, \theta)$ for all $i \in N$. Then by unanimity we have $\bar{x} \in f(\theta')$. ■

5 Concluding Remarks

This thesis studies Nash implementation under compete information with boundedly rational agents. We introduce a notion called maximal choice set and define an associated maximal choice monotonicity condition on social choice correspondences. We show that maximal choice monotonicity is necessary for Nash implementation when choice correspondences satisfy Sen's property α. Moreover, we prove that maximal choice monotonicity and no veto-power suffice for Nash implementation when agents have unique maximal choice sets.

In what follows, we will explain how the notions of maximal choice sets and opportunity sets, are related in our construction. The mechanism employed in Maskin's implementation proof considers three distinct rules. In the first, every player reports the same strategy profile including an alternative which is optimal with respect to the social choice rule and in this case the outcome function of the mechanism provides that alternative. In our formulation, the opportunity set of any agent in this situation will be equal to the maximal choice set associated with this strategy profile. In the second rule, every player but one reports the same strategy profile involving an alternative which is optimal with respect to the social choice rule. Then, Maskin's outcome function equals to the alternative chosen by all but one player if the choice of the lone player does not involve an alternative in the lower contour set of the alternative chosen by the other players. But, Maskin's outcome equals to the alternative chosen by the lone player if the choice of the lone player is an alternative in the lower contour set of the alternative chosen by the other players. In this second rule, the opportunity set of our mechanism will equal to the maximal choice set for the lone player; and the set of all alternatives for the other players. The third rule contains all the other cases and Maskin's outcome equals to what is chosen by the winner of the associated integer game. In our formulation, the opportunity set of any agent in this case equals to the set of all alternatives.

What ties our analysis to the previous work done by Korpela (2012) and De Clippel (2014) is that we define the exact structure of the set $R_i(x, \theta)$ in condition λ of Korpela (2012) or the collection of opportunity sets $\mathcal{O}$ in De Clippel (2014). The necessary and sufficient conditions derived either of them depend on the existence of these sets. However, it is not clear when do such sets exist. Therefore, it is important to stress that our definition of a maximal choice set is *independent* from the mechanism and it is solely defined via the choice correspondences of agents. This allows us to define the choice-based counterpart of Maskin monotonicity and derive sufficient conditions for implementation without the added complexity of checking for existence of opportunity sets that is present in Korpela (2012) and De Clippel (2014)'s setting.

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