ALGEBRAIC AND HOMOLOGICAL PROPERTIES OF POLYMATROIDAL IDEALS

by

ASLI T. MUSAPAŞAOĞLU

Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of
the requirements for the degree of
Master of Degree

Sabancı University
Spring 2018

ALGEBRAIC AND HOMOLOGICAL PROPERTIES OF POLYMATROIDAL IDEALS

APPROVED BY

Assist. Prof Ayesha Asloob Qureshi ....
(Thesis Supervisor)

Assoc. Prof Kağan Kurşungöz ....

Assist. Prof Zafeirakis Zafeirakopoulos ....

DATE OF APPROVAL: 30 . 07 . 2018

Reserve your right to think, for even to think wrongly is better than not to think at all.

Hypatia

ALGEBRAIC AND HOMOLOGICAL PROPERTIES OF POLYMATROIDAL IDEALS

Aslı T. Musapaşaoğlu

Mathematics, Master Thesis, 2018

Thesis Supervisor: Asst. Prof Ayesha Asloob Qureshi

Keywords: Polymatroidal ideals, Stability property, Discrete polymatroids, Monomial localization, Analytic spread

Abstract

Monomial ideals are widely studied in commutative algebra. In this thesis, we study a special class of monomial ideals called polymatroidal ideals which admit many nice algebraic and homological properties. They are distinguished by the fact that they satisfy "exchange property" and their powers have linear resolutions. Another important property of polymatroidal ideals is that their monomial localization at any monomial prime ideal is again a polymatroidal ideal. In [1], Bandari and Herzog gave a conjecture that if all monomial localizations of a monomial ideal I have linear resolution then I is polymatroidal. In chapter 4, we discuss persistence and stability properties of polymatroidal ideals and we see that their index of depth stability and the index of stability for the associated prime ideals are bounded by their analytic spread. Finally, we examine the strong persistence property of polymatroidal ideals.

POLİMATROİDAL İDEALLERİN CEBİRSEL VE HOMOLOJİK ÖZELLİKLERİ

Aslı T. Musapaşaoğlu

Matematik, Master Tezi, 2018

Tez Danışmanı: Yrd. Doç. Dr Ayesha Asloob Qureshi

Anahtar Kelimeler: Polimatroidal idealler, Kararlılık özelliği, Ayrık polimatroidler, Tek-terimli yerelleştirme, Analitik yayılım

Özet

Tek-terimli idealler, değişmeli cebirin temel konularından biridir. Bu tezde, tek-terimli ideallerin özel bir hali olan polimatroidal idealleri ve bu ideallerin sağladığı cebirsel ve homolojik özellikleri inceledik. Polimatroidal idealler ”değişme özelliği” ile ayırt edilirler. Ayrıca, polimatroidal ideallerin bütün kuvvetleri lineer çözünürlüğe sahiptir. Bu ideallerin sağladığı başka önemli bir özellik de tek terimli yerelleştirmedir. Bir polimatroidal idealin herhangi bir asal idealdeki yerelleştirmesi yine bir polimatroidal idealdir. Bandari ve Herzog [1], makalelerinde tek-terimli bir idealin bütün tek-terimli yerelleştirmeleri lineer çözünürlüğe sahipse bu idealin polimatroidal ideal olduğuna dair bir sanıda bulundu. 4.bölümde, polymatroidal ideallerin devamlılık ve kararlılık özelliklerini inceledik ve polimatroidal ideallerin bağlantılı asal ideallerinin kararlılık indeksleri ve derin kararlılık indekslerinin analitik yayılımı ile sınırlı olduğunu gördük. Son olarak, polimatroidal ideallerin güçlü devamlılık özelliğini araştırdık.

Acknowledgments

First of all, I would like to thank to the Mathematics Program of Sabancı University for supporting me and creating a warm and friendly atmosphere that I enjoyed all through my studies. I would gratefully like to thank my advisor Asst. Prof Ayesha Asloob Qureshi for her patience, encouragement and guidance at all stages of my work.

Last by no means least, I would like to thank my husband for his love and support, my family and my dear friends Çiğdem Çelik and Murat Gilasin for their support and friendship.

Contents

Contents i

Abstract v

Özet vi

Acknowledgments vii

Introduction 1

1 Preliminaries 3

1.1 Monomial ideals and their algebraic properties 3
1.2 Linear resolution 5
1.2.1 Koszul complex 6
1.3 Cohen-Macaulay rings 8
1.4 Rees rings and normality 9

2 Polymatroidal Ideals 11

2.1 Discrete polymatroids 11
2.2 Polymatroidal ideals 19
2.3 Weakly polymatroidal ideals 22

3 Monomial Localizations of Polymatroidal Ideals 23

4 Persistence and Stability Properties of Polymatroidal Ideals 30

4.1 Polymatroidal ideals and persistence property 30
4.2 Strong persistence property 34
4.3 Stability indices of polymatroidal ideals 38

Bibliography **44**

Introduction

In this thesis, we study polymatroidal ideals which arise from discrete polymatroids. Discrete polymatroids can be characterized in terms of the exchange property which is satisfied by their bases. Let $\{e_1, \dots, e_n\}$ be the standard basis of $\mathbb{Z}_+^n$ and let $B(P) \subset \mathbb{Z}_+^n$ be the base of a discrete polymatroid P on the ground set $[n]$. Then all elements of $B(P)$ have the same modulus and if $u = (u_1, \dots, u_n)$ and $v = (v_1, \dots, v_n)$ are elements of $B(P)$ with $u_i > v_i$ then there exists j with $v_j > u_j$ such that $u - e_i + e_j \in B(P)$. Discrete polymatroids like matroids and polymatroids provides a connection between algebra and combinatorics. One can associate two algebraic structures on discrete polmatroids, namely, the Ehrhart rings and polymatroidal ideals. In this survey, we focus on algebraic and homological properties of polymatroidal ideals. We give detailed proof of the results given in [1, 11, 13, 14, 19].

In Chapter 1, we give basic definitions and notation which will be used in the later chapters. In Chapter 2, we define discrete polymatroids, give their basic properties and give a detailed proof of symmetric exchange theorem, (Theorem 2.1.11). In [16], Herzog and Takayama showed that all polymatroidal ideals have linear resolutions. Moreover, in [11] Herzog and Hibi gave the complete characterization of Cohen-Macaulay polymatoridal ideals. Precisely, principal ideals, Veronese ideals and squarefree Veronese ideals are the only classes of Cohen-Macaulay polymatroidal ideals. We discuss these results in detail in Chapter 2. In [17], Hibi and Kokubo introduced weakly polymatroidal ideals as a generalization of polymatroidal ideals. These ideals are also discussed in [19] by Mohammadi and Moradi. We also give the proof of the result mentioned in [19] that weakly polymatroidal ideals have linear quotients.

In Chapter 3, we define monomial localization. Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal and P be a monomial prime ideal of I. Then we obtain a new ideal $I(P)$ by substituting variables $x_j \notin P$ by 1. The ideal $I(P)$ is called as the monomial localization of I at P. If I is a polymatroidal ideal then we see that $I(P)$ is again a polymatroidal ideal. It follows that all monomial localizations of polymatroidal ideals have linear resolution. The converse of the statement is proposed as a conjecture by Bandari and Herzog [1]. They showed this conjecture holds true under certain conditions on I.

In Chapter 4, we discuss the persistence and stability properties of polymatroidal ideals. Broadmann showed that [3] associated prime ideals of an ideal $I \subset S = K[x_1, \dots, x_n]$ stabilizes which means $\text{Ass}(I^n) = \text{Ass}(I^{n_1})$ for all $n \gg 0$. The smallest such n_1 is called *the index of stability* and denoted by $\operatorname{astab}(I)$. Also, it is known by Broadmann [2] that depth S/I^n is constant for $n \gg 0$. The smallest n_1 which satisfies depth $S/I^n =$ depth S/I^{n_1} is called *the index of depth stability* of I and denoted by $\operatorname{dstab}(I)$. Depth function can be defined as $f : \mathbb{N} \mapsto \mathbb{N}$ where $f(n) = S/I^n$. Let $I \subset S$ be an ideal. If $\operatorname{Ass}(I) \subset \operatorname{Ass}(I^2) \subset \cdots \subset \operatorname{Ass}(I^n) \subset ...$ then I satisfies the persistence property. Herzog, Rauf and Vladoiu [13] proved that polymatroidal ideals satisfy the persistence property. Later in [14], Herzog and Qureshi showed that polymatroidal ideals also satisfy the strong persistence property and their index of stability are bounded by their analytic spread. We give an overview of results presented in [13] and [14].

Chapter 1

Preliminaries

In this chapter we will give the basic definitions that will be used in the upcoming chapters.

1.1 Monomial ideals and their algebraic properties

Let K be a field and $S = K[x_1, \ldots, x_n]$ be a polynomial ring. The product of the form $\mathbf{x}^{\mathbf{b}} = x_1^{b_1} \cdots x_n^{b_n}$ where $\mathbf{b} = (b_1, \ldots, b_n) \in \mathbb{Z}_+^n$ is called a *monomial* in S. We denote the set of all the monomials in S by $\mathrm{Mon}(S)$. The set $\mathrm{Mon}(S)$ is a K-basis of S. If $p \in S$ then

$$p = \sum_{v \in \mathrm{Mon}(S)} b_v v, \; b_v \in K.$$

and we set $\mathrm{supp}(p) = \{v \in \mathrm{Mon}(S) : b_v \neq 0\}$.

The ideal $I \subset S$ is called a *monomial ideal* if it is generated by monomials. Moreover, the ideal I is called a *squarefree monomial ideal* if it is generated by squarefree monomials. Following proposition shows that monomial ideals have a unique minimal generating set.

Proposition 1.1.1. Let G be the set of monomials in the monomial ideal I which are minimal with respect to divisibility. Then $G(I)$ is the *unique minimal generating set* of I.

Proof. Let $G_1(I) = \{u_1, \ldots, u_t\}$ and $G_2(I) = \{v_1, \ldots, v_k\}$ be two minimal generating set of I. Since $u_l \in I$ there exists $v_s \in I$ such that $u_l = w_1 v_s$ for some monomial w_1. Similarly, there exists u_j such that $v_i = w_2 u_j$ for some monomial w_2. Then

$u_l = w_1 w_2 u_j$. Since $G_1(I)$ is a minimal generating set of I, it follows that $j = l$ and $w_1 w_2 = 1$. In particular, $w_2 = 1$ and $v_i = u_j \in G_1(I)$ and this shows $G_2(I) \subset G_1(I)$. By symmetry, also we have $G_1(I) \subset G_2(I)$. $\square$

We will denote the minimal generating set of I by $G(I)$. Following corollary gives a characterization for monomial ideals.

Corollary 1.1.2. Let $I \subset S$. Then the following conditions are equivalent:

(a) I is a monomial ideal;

(b) $g \in I$ if and only if $\text{supp}(g) \subset I$ for all $g \in S$.

Proof. *(a)* $\Rightarrow$ *(b)* Let I be a monomial ideal and let $g \in I$. Then there exists monomials $v_1, \ldots, v_r \in I$ and polynomials $g_1, \ldots, g_r \in S$ such that $g = \sum_{k=1}^{r} g_k v_k$. It follows that $\text{supp}(g) \subset \bigcup_{k=1}^{r} \text{supp}(g_k v_k)$. Since each $u \in \text{supp}(g_k v_k)$ is of the form $w v_k$ for some $w \in \text{Mon}(S)$, $u \in I$. Thus, $\text{supp}(g) \subset I$.

(b) $\Rightarrow$ *(a)* Let $G(I) = \{g_1, \ldots, g_r\}$. Since $\text{supp}(g_k) \subset I$ for all k, it follows that $\bigcup_{k=1}^{r} \text{supp}(g_k)$ is a monomial generating set of I. Hence, I is a monomial ideal. $\square$

Now, we will discuss the algebraic operations on monomial ideals.

Proposition 1.1.3. Let I and J be two monomial ideals. Then $I \cap J$ is again a monomial ideal. The minimal generating set for $I \cap J$ is

$$G(I \cap J) = \{lcm(v, w) : v \in G(I), w \in G(J)\}$$

Proof. Let $g \in I \cap J$. By Corollary 1.1.2, $\text{supp}(g) \subset I \cap J$. Then $I \cap J$ is a monomial ideal. Let $u \in \text{supp}(g)$. Since $\text{supp}(g) \subset I \cap J$, there exists $v \in G(I)$ and $w \in G(J)$ such that $v|u$ and $w|u$. It follows that $\text{lcm}(v, w)|u$. Since $\text{lcm}(v, w) \subset I \cap J$ for all $v \in G(I)$ and $w \in G(J)$, we obtain that the set $\{\text{lcm}(v, w) : v \in G(I), w \in G(J)\}$ is a generating set for $I \cap J$. $\square$

Definition 1.1.4. Let $I, J \subset S$ be two ideals. Then the set

$$I : J = \{g \in S : gf \in I \textit{ for all } f \in J\}$$

is called the *colon ideal* of I with respect to J.

Next proposition shows that colon ideal is a monomial ideal.

Proposition 1.1.5. Let I and J be monomial ideals. Then $I : J$ is a monomial ideal and

$$I : J = \bigcap_{w \in G(J)} I : (w).$$

Also, the set $\{v/\gcd(v, w) : v \in G(I)\}$ is a generating set of $I : (w)$.

Proof. Let $g \in I : J$. Then $gw \in I$ for all $w \in G(J)$. By Corollary 1.1.2, it follows that $\operatorname{supp}(g)w = \operatorname{supp}(gw) \subset I$. Hence, $\operatorname{supp}(g) \subset I : J$. This yields $I : J$ is a monomial ideal. It is clear that $\{v/\gcd(v, w) : v \in G(I)\} \subset I : (w)$. Now let $u \in I : (w)$. Then there exists $v \in G(I)$ such that $v|uw$. Thus, $v/\gcd(v, w)$ divides u, as desired. $\square$

Definition 1.1.6. Let $I \subset S$ be a graded monomial ideal and let $m = (x_1, \ldots, x_n)$ denote the graded maximal ideal of S. Then

$$I : m^{\infty} = \bigcup_{t=1}^{\infty} I : m^t$$

is called the saturation of I and it is again a monomial ideal.

In the end, we will discuss monomial prime ideals . Note that a monomial prime ideal is generated by set of variables in S. A squarefree monomial ideal is an intersection of monomial prime ideals.

Definition 1.1.7. Let $I \subset S$ be a monomial ideal. A prime ideal P is called an *associated prime* of I if $P = \operatorname{Ann}(u)$ for some $u \in I$. The set of all associated primes of I is denoted by $\operatorname{Ass}(I)$.

The set of associated prime ideals of a monomial ideal consists of monomial prime ideals. Let $I \subset S$ be an ideal. A prime ideal P is called a *minimal prime ideal* of I if I is contained in P and there is no prime ideal containing I which is properly contained in P. The set of minimal prime ideals of I is denoted by $\operatorname{Min}(I)$. Also, the prime ideals which contain I is denoted by $V(I)$.

1.2 Linear resolution

In this section, we will see that if an ideal has linear quotients then it has a linear resolution.

Let M be a graded R-module which is generated by homogeneous generators $m_1, \ldots, m_s$ with $\deg(m_k) = a_k$ for $k = 1, \ldots, s$. Then there exists a surjective R-module homomorphism

$$F_0 = \bigoplus_{k=1}^{s} Re_k \to M \text{with } e_k \mapsto m_k.$$

By assigning $\deg(a_k) = e_k$ for $k = 1, \ldots, k$, the map $F_0 \to M$ becomes a morphism in $\mathcal{M}(R)$ and $F_0 \cong \bigoplus_{k=1}^{s} R(-a_k)$. Hence we obtain the sequence which is exact

$$0 \longrightarrow K \longrightarrow \bigoplus_j R(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0,$$

where $\beta_{0j} = |\{i : a_i = k\}|$ and where $K = \mathrm{Ker}(\bigoplus_j R(-j)^{\beta_{0j}} \to M)$. The module K is a graded submodule of $F_0 = \bigoplus_j R(-j)^{\beta_{0j}}$. By Hilbert basis teorem, K is finitely generated. Thus we obtain again an epimorphism $\bigoplus_j R(-j)^{\beta_{ij}} \to K$. If we compose this epimorphism with the inclusion map $K \to \bigoplus_j R(-j)^{\beta_{0j}}$ we obtain the exact sequence

$$\bigoplus_j R(-j)^{\beta_{1j}} \longrightarrow \bigoplus_j R(-j)^{\beta_{0j}} \longrightarrow M \longrightarrow 0$$

of graded R-modules. Continuing in this way we obtain a long exact sequence

$$\mathbb{F} : \cdots \longrightarrow F_2 \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

of graded R-modules with $F_i = \bigoplus_j R(-j)^{\beta_{ij}}$. Such an exact sequence is called a *graded free R-resolution* of M.

Let M be a finitely generated R-module. A graded free R-resolution $\mathbb{F}$ of M is called *minimal* if for all i, the image of $F_{i+1} \mapsto F_i$ is contained in mF_i where $m = (x_1, \ldots, x_n)$. Since all free modules are projective module, the length of the minimal graded free resolution is called *projective dimension* of R. The module M has a *d-linear resolution* if the graded minimal free resolution of M is of the form

$$0 \longrightarrow R(-d-t)^{\beta_t} \longrightarrow \cdots \longrightarrow R(-d-1)^{\beta_1} \longrightarrow R(-d)^{\beta_0} \longrightarrow M \longrightarrow 0.$$

1.2.1 Koszul complex

Let R be a commutative ring with a unit and $g = g_1, \ldots, g_r$ be a sequence of elements of R. Then we define *Koszul complex* $K(g; R)$ attached to the sequence g as follows:

Let F be a free R-module with basis $e_1, \ldots, e_r$. We let $K_j(g; R)$ be the jth exterior power of F, that is, $K_j(g; R) = \bigwedge^j F$.

A basis of the free R-module $K_j(g;R)$ is given by the wedge products

$$e_F = e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j} \; if \; F = \{i_1 < i_2 < \ldots < i_j\}.$$

In particular, it follows that $\operatorname{rank} K_j(g;R) = \binom{r}{j}$. We define the differential $\partial : K_j(g;R) \mapsto K_{j-1}(g;R)$ by the formula

$$\partial(e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_j}) = \sum_{k=1}^{j} (-1)^{k+1} g_{i_k} e_{i_1} \wedge e_{i_2} \wedge \cdots \wedge e_{i_{k-1}} \wedge e_{i_{k+1}} \wedge \cdots \wedge e_{i_j}.$$

One readily verifies that $\partial \circ \partial = 0$, so that $K_\bullet(g;R)$ is indeed a complex. Now let M be an R-module. We define the complexes

$$K_\bullet(g;M) = K_\bullet(g;R) \otimes_R M \; and \; K^\bullet(g;M) = \operatorname{Hom}_R(K_\bullet(g;R), M).$$

Definition 1.2.1. Let $I \subset S$ be a graded ideal. Let $g_1, \ldots, g_r$ be the system of homogeneous generators of I. If the colon ideal $(g_1, \ldots, g_{j-1}) : g_j$ is generated by linear forms for all j then I has linear quotients.

Proposition 1.2.2. Let $I \subset S$ is a graded ideal generated in degree d and I has linear quotients. Then I has a linear resolution.

Proof. Let $I = (p_1, \ldots, p_r)$ where each p_i of degree d, and suppose that for all l $M_l = (p_1, \ldots, p_{l-1}) : p_l$ is generated by linear forms. Then $I_l = (p_1, \ldots, p_l)$ has a d-linear resolution. Indeed, we can show it by induction on l. It is obvious for $l = 1$. Assume that $l > 1$ and let $\{m_1, \ldots, m_s\}$ be the minimal set of linear forms which generates M_l. Then, one can easily see that $m_1, \ldots, m_s$ is a regular sequence. In fact, if we complete $m_1, \ldots, m_l$ to a K-basis $m_1, \ldots, m_n$ of S_1, then $f : S \to S$ with $f(x_j) = m_j$ for $j = 1, \ldots, n$ is a K-automorphism. Since $x_1, \ldots, x_l$ is a regular sequence, we obtain $m_1 = f(x_1), \ldots, m_s = f(x_s)$ is a regular sequence as well. Since $m_1, \ldots, m_s$ is a regular sequence, the Kozsul complex $K(m_1, \ldots, m_s, S)$ provides a minimal graded free resolution of S/M_l. This implies that

$$\operatorname{Tor}_i^S(S/M_l(-d), K)_{i+j} \cong \operatorname{Tor}_i^S(S/M_l, K)_{i+(j-d)} = 0$$

for $j \neq d$. Then our goal is to show

$$\operatorname{Tor}_i(I_l, K)_{i+j} = 0 \text{ for all } i \text{ and all } j \neq d.$$

Since $I_l/I_{l-1} \cong (S/I_l)(-d)$, we obtain the exact sequence

$$0 \longrightarrow I_{l-1} \longrightarrow I_l \longrightarrow (S/M_l)(-d) \longrightarrow 0.$$

This sequence yields the exact sequence

$$\operatorname{Tor}_i^S(I_{l-1}, K)_{i+j} \longrightarrow \operatorname{Tor}_i^S(I_l, K)_{i+j} \longrightarrow \operatorname{Tor}_i^S(S/m_L(-d), K)_{i+j}$$

By our induction hypothesis, one can see that both ends in this exact sequence vanish for $j \neq d$. Hence, this also holds for the middle term, as desired. $\square$

1.3 Cohen-Macaulay rings

Now we give definitions of the depth and the dimension of a ring R. In chapter 4, we will be interested in non-increasing depth functions of a local ring. Depth is first defined for Noetherian rings as a grade but our interest will be mostly restricted to the local Noetherian rings.

Definition 1.3.1. Let R be a Noetherian ring, M a finite R-module. I an ideal such that $IM \neq M$. Then the common length of the maximal M-sequences in I is called the grade of I on M and it is denoted by $\operatorname{grade}(I, M)$.

If $IM = M$ then we say $\operatorname{grade}(I, M) = \infty$.

Definition 1.3.2. Let (R, m, k) be a Noetherian local ring and M a finite R-module. Then the grade of m on M is called the depth of M and is denoted by $\operatorname{depth}(M)$.

In homological terms, one can define $\operatorname{depth}(M)$ as follows:

$$\operatorname{depth}(M) = \min\{i : \operatorname{Ext}_R^i(R/m, M) \neq 0\} = \min\{i : H_m^i(M) \neq 0\}.$$

Definition 1.3.3. Let R be a ring. Then the dimension of R is defined as follows:

$$\dim(R) = \max\{n : P_1 \subset P_2 \subset \ldots \subset P_n \text{ where all } P_i's \text{ are prime ideals}\}.$$

The dimension of R is also known as the Krull dimension. Given a prime ideal P in R, *the height* of P is defined by

$$\operatorname{height}(P) = \max\{n : P_1 \subset P_2 \subset \ldots \subset P_n \subset P \text{ where all } P_i's \text{ are prime ideals}\}.$$

The set of all prime ideals in R is denoted by $\operatorname{Spec}(R)$. Let R be a Noetherian ring and $M \neq 0$ be a finite R-module. If $\operatorname{proj\,dim} M < \infty$, then

$$\operatorname{proj\,dim}(R) + \operatorname{depth} M = \dim R.$$

This theorem is known as [9, Corollary A.4.3] *Auslander-Buchsbaum theorem*.

Now we will define Cohen-Macaulay rings and Cohen-Macaulay ideals and in chapter 4 we will give a characterization of polymatroidal ideals which are Cohen-Macaulay.

Definition 1.3.4. Let R be a Noetherian local ring. If $\operatorname{depth} R = \dim R$ then R is called a *Cohen-Macaulay ring. Let $I \subset R$ be a monomial ideal. Then I is called a* Cohen-Macaulay ideal *if the quotient ring R/I is Cohen-Macaulay.*

1.4 Rees rings and normality

In the chapter 4, we will be interested in analytic spread of an ideal I.

Definition 1.4.1. Let $I \subset S$ be a graded ideal generated by homogeneous polynomials $f_1, \ldots, f_m$ and let t be a new indeterminate over the field K. Then

$$\mathcal{R}(I) = \bigoplus_{i \geq 0} I^i t^i = S[f_1 t, \ldots, f_m t]$$

is called *the Rees ring* of I which is a graded subring of $S[t]$.

Now we give the definition of analytic spread.

Definition 1.4.2. Let $m = (x_1, \ldots, x_n)$ be a graded maximal ideal of S. Then the *analytic spread* of I is the Krull dimension of the ring $\mathcal{R}(I)/m\mathcal{R}(I)$. It is denoted by $\ell(I)$.

Let I be a monomial ideal generated in single degree. Then the analytic spread of I is the rank of the matrix with row vectors which are the exponent vectors of the minimal generators of I.

An ideal I is called integrally closed if $u^k \in I^k$ for all $u \in \operatorname{Mon}(S)$ and all k then $u \in I$. An ideal I is called a normal ideal if all powers of I are integrally closed. The Rees ring $R(I)$ is called *normal ring* if I is a normal ideal.

In general, it is not easy to see if a ring is normal. However, below we give a well-known criterion which is given by Serre, known as *Serre's Condition For Normality*: Let R be a Noetherian ring and k be a non-negative integer. Then

(R_k) R is said to satisfy (R_k) if R_p is a local ring for all $P \in \operatorname{Spec}(R)$ with $\operatorname{height}(P) \leq k$.

(S_k) R is said to satisfy (S_k) if $\mathrm{depth}(R_p) \geq \min\{k, \mathrm{height}(P)\}$ for all $P \in \mathrm{Spec}(R)$.

It is known [18, Theorem 23.8] that if R satisfies both R_1 and S_2 then R is called a normal ring.

Chapter 2

Polymatroidal Ideals

In this chapter, we will discuss the polymatroidal ideals. Polymatroidal ideals form a very special class of monomial ideals which arise from discrete polymatroids and their minimal generating set satisfies the so-called exchange property.

2.1 Discrete polymatroids

Let $\mathbb{R}_+^n$ be the set of all positive real vectors and $\{e_1, ..., e_n\}$ be the standard of $\mathbb{R}_+^n$. Let $x = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$. Then the *modulus* of x is

$$|x| = \sum_{i=1}^{n} x_i.$$

Let $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ be two vectors in $\mathbb{R}_+^n$. We call y a *subvector* of x if $x_i - y_i \geq 0$ for all i and write $y \leq x$. Also, we set

$$x \vee y = (\max\{x_1, y_1\}, ..., \max\{x_n, y_n\})$$

$$x \wedge y = (\min\{x_1, y_1\}, ..., \min\{x_n, y_n\})$$

Definition 2.1.1. Let $\mathcal{P} \subset \mathbb{R}_+^n$. Then $\mathcal{P}$ is called a polymatroid if the following conditions are satisfied:

(P1) for every $y \in \mathcal{P}$ if $x < y$ then $x \in \mathcal{P}$,

(P2) if $x, y \in \mathcal{P}$ with $|x| < |y|$ then there exists $z \in \mathcal{P}$ such that $x < z < x \vee y$.

We call the elements of $\mathcal{P}$ as *independent vectors*. Let $x \in \mathbb{R}^n_+$. Then an independent vector $y \in \mathcal{P}$ is called a maximal independent subvector of x if $y \leq x$ and $y < z \leq x$ for no $z \in \mathcal{P}$. A maximal independent subvector of $x \in \mathbb{R}^n_+$ exists since $\mathcal{P}$ is compact. A base of a polymatroid $\mathcal{P} \subset \mathbb{R}^n_+$ is a maximal independent vector of $\mathcal{P}$. Every base vector has the same modulus. This modulus is called the rank($\mathcal{P}$). If $\mathcal{M} \subset [n] = \{1, \ldots, n\}$ and $x = (x_1, ..., x_n) \in \mathbb{R}^n_+$ then we set

$$x(\mathcal{M}) = \sum_{i \in \mathcal{M}} x_i.$$

We define the ground set rank function of $\mathcal{P}$ on the ground set $[n]$ by $\rho : 2^{[n]} \mapsto \mathbb{R}_+$ and we set $\rho(\mathcal{M}) = \max\{x(\mathcal{M}) : x \in \mathcal{P}\}$ for all $\emptyset \neq \mathcal{M} \subset [n]$ and $\rho(\emptyset) = 0$. Let $x \in \mathbb{R}^n_+$. Then we define $\zeta(x) = |y|$, where $y \in \mathcal{P}$ is maximal independent subvector of x.

Lemma 2.1.2. [9, Lemma 12.1.2] Let $u, v \in \mathbb{R}^n_+$. Then

$$\zeta(u) + \zeta(v) \geq \zeta(u \vee v) + \zeta(u \wedge v).$$

Proof. Let $x \in \mathcal{P}$ be a maximal independent subvector of $u \wedge v$. Since $x \leq u \vee v$, there exists a maximal independent subvector of $u \vee v$, namely $y \in \mathcal{P}$ such that $x \leq y \leq u \vee v$. Since $y \wedge (u \wedge v) \in \mathcal{P}$ and $x \leq y \wedge (u \wedge v) \leq u \wedge v$, we have $x = y \wedge (u \wedge v)$. We claim

$$x + y = y \wedge u + y \wedge v.$$

Indeed, since $y \leq u \vee v$, we have $y(j) \leq \max\{u(j), v(j)\}$ for each $j \in [n]$. Let $u(j) \leq v(j)$. Then $x(j) = \min\{y(j), u(j)\}$ and $y(j) = \min\{y(j), v(j)\}$. Thus $x(j) + y(j) = (y \wedge u)(j) + (y \wedge v)(j)$, as required. Since $y \wedge u \in \mathcal{P}$ is a subvector of u and since $y \wedge v \in \mathcal{P}$ is a subvector of v, we obtain that $|y \wedge u| \leq \zeta(u)$ and $|y \wedge v| \leq \zeta(v)$. Hence

$$\zeta(u \wedge v) + \zeta(u \vee v) = |x| + |y| = |y \wedge u| + |y \wedge v| \leq \zeta(u) + \zeta(v)$$

□

Theorem 2.1.3. [9, Theorem 12.1.3] Let $\mathcal{P}$ be a polymatroid and ρ be its ground set rank function. If $X \subset Y \subset [n]$, then $\rho(X) \leq \rho(Y)$. Also ρ satisfies submodularity, that is,

$$\rho(X) + \rho(Y) \geq \rho(X \cup Y) + \rho(X \cap Y)$$

for all $X, Y \subset [n]$. Moreover, $\mathcal{P}$ coincides with the compact set

$$\{u \in \mathbb{R}^n_+ : u(X) \leq \rho(X), X \subset [n]\}.$$

Proof. Obviously, ρ is non-decreasing function. We set $v_A \in \mathbb{R}_+$ by

$$v_A(j) = \begin{cases} v(j), & \text{if } j \in A \\ 0, & \text{if } j \in [n] \setminus A \end{cases}$$

where $A \subset [n]$ and $v \in \mathbb{R}_+$. Let $s = \operatorname{rank} \mathcal{P}$ and $x = (s, \ldots, s) \in \mathbb{R}^n_+$. Hence $z \leq x$ for all $z \in \mathcal{P}$.

Claim: For each $A \subset [n]$, we have

$$\rho(A) = \zeta(x_A).$$

Proof of Claim: Let y be a maximal independent subvector of x_A. Then $\zeta(x_A) = |y| = y(A) \leq \rho(A)$. Also if $\rho(A) = z(A)$, for $z \in \mathcal{P}$, then, since $z_A \leq x_A$, we have $\rho(A) = z(A) = |z_A| \leq \zeta(x_A)$. Hence $\rho(A) = \zeta(x_A)$. Let $X, Y \subset [n]$. Then $\rho(X \cup Y) = \zeta(x_{X \cup Y}) = \zeta(x_X \vee x_Y)$ and $\rho(X \cap Y) = \zeta(x_{X \cap Y}) = \zeta(x_A \wedge x_Y)$. Thus, by Lemma 2.1.2, we obtain the submodularity of ρ. Let Q denote the compact set. By definition of ρ we have $\mathcal{P} \subset Q$. We claim $Q \subset \mathcal{P}$. Indeed, assume that there exists $y \in Q$ with $y \notin \mathcal{P}$. Let $x \in \mathcal{P}$ be a maximal independent subvector of y which maximizes $|M(x)|$, where

$$M(x) = \{j \in [n] : x(j) < y(j)\}.$$

Let $z = (x + y)/2 \in \mathbb{R}^n_+$ and $v \in \mathcal{P}$ with $v(M(x)) = \rho(M(x))$. Since $z \in Q$, we have

$$x(M(x)) < z(M(x)) \leq \rho(M(x)) = v(M(x)).$$

Since $|x_{M(x)}| < |v_{M(x)}|$, there is $x' \in \mathcal{P}$, with $x_{M(x)} < x' < x_{M(x)} \vee v_{M(x)}$. Hence, $x_{M(x)} < x' \wedge z_{M(x)} \leq z_{M(x)}$. Thus, $x_{M(x)}$ can not be a maximal independent subvector of $z_{M(x)}$. Let $x'' \in \mathcal{P}$ with $x_{M(x)} < x''$ be a maximal independent subvector of $z_{M(x)}$. Let $x^* \in \mathcal{P}$ be a maximal independent subvector of z with $x'' \leq x^*$. Since each of x and x^* is a maximal independent subvector of z, we have $|x| = |x^*|$. However, since $x(M(x)) < x''(M(x)) \leq x^*(M(x))$, there is $i \in [n] \setminus M(x)$ with $x^*(i) < x(i) (= y(i))$. Since $x^*(j) \leq z(j) < y(j)$ for all $j \in M(x)$, we have $|M(x^*)| > |M(x)|$. This is a contradiction since $|M(x)|$ is maximal. $\square$

Let $\mathcal{P}_1, ..., \mathcal{P}_m$ be polymatroids on the ground set $[n]$. Then the *polymatroid sum* which is denoted by $\mathcal{P}_1 \vee ... \vee \mathcal{P}_m$ is the compact subset of $\mathbb{R}^n_+$. Let $x \in \mathcal{P}_1 \vee ... \vee \mathcal{P}_m$ then it is of the form

$$x = \sum_{i=1}^{m} x_i$$

where $x_i \in \mathcal{P}_i$.

Theorem 2.1.4. [9, Theorem 12.1.5] Let $\mathcal{P}_1, ..., \mathcal{P}_m$ be polymatroids on the ground set $[n]$. Then $\mathcal{P}_1 \vee ... \vee \mathcal{P}_m$ which denotes the sum of polymatroids is again a polymatroid.

Proof. Let $\mathcal{P}_1, ..., \mathcal{P}_m$ be polymatroids on the ground set $[n]$. Since all $\mathcal{P}_i's$ are polymatroids, they contain all their subvectors. Then the sum of those vectors belongs to the polymatroid sum. Let $x, y \in \mathcal{P}_1 \vee ... \vee \mathcal{P}_m$ with $|x| < |y|$. By definition, $x = \sum_{i=1}^m x_i$ and $y = \sum_{i=1}^m y_i$ where $x_i \in \mathcal{P}$ and $y_i \in \mathcal{P}$. There exists w_i for $i = 1, ..., m$ with $x_i < w_i < x_i \vee y_i$ since all $\mathcal{P}_i's$ are polymatroids. If we take the sum of $w_i's$ then $x = \sum_{i=1}^m x_i < w = \sum_{i=1}^m w_i < x = \sum_{i=1}^m x_i \vee y = \sum_{i=1}^m y_i$. Therefore, polymatroid sum is again a polymatroid. □

Definition 2.1.5. Let $P \subset Z_+^n$ be a nonempty finite set of positive integer vectors on the ground set $[n]$. Then P is called a discrete polymatroid if it satisfies the following conditions:

(i) for all $y \in P$ and $x \in \mathbb{Z}_+^n$ with $x \leq y$ then $x \in P$,

(ii) for all $y = (y_1, \ldots, y_n) \in P$ and $x = (x_1, \ldots, x_n) \in P$ with $|x| < |y|$ there is $t \in [n]$ with $x_t < y_t$ such that $x + e_t \in P$.

Let $B(P)$ denote the base set for the discrete polymatroid P. If x is not a subvector of any $y \in P$ then it belongs to $B(P)$. It turns out if $x, y \in B(P)$ then they have the same modulus. Discrete polymatroids can be characterized by exchange property.

Example 2.1.6. (i) Let $\mathcal{M} \subset 2^{[n]}$ be a matroid. Then the set $\{z_F : F \in \mathcal{M}\}$ is a discrete polymatroid where $z_F = (z_1, \ldots, z_n)$ are vectors with $z_i = 1$ if $i \in F$ otherwise $z_i = 0$.

(ii) Let $d_1, ..., d_n$ and d be integers such that $d_1 + ... + d_n \leq d$. Let P be a discrete polymatroid consist of the vectors $x \in \mathbb{Z}_+^n$ such that $x_t \leq d_t$ for all $1 \leq t \leq n$ and $|x| \leq d$. Then P is called a discrete polymatroid of Veronese type on the ground set $[n]$.

Lemma 2.1.7. [9, Lemma 12.2.3] Let P be a discrete polymatroid. Then we have the following:

(i) Let $d \leq \operatorname{rank} P$. Then the set $P' = \{x \in P : |x| \leq d\}$ is again a discrete polymatroid of rank d.

(ii) if $\operatorname{rank} P = d$ then for each $u \in P$ the set $P_u = \{v - u : v \in P, v \geq u\}$ is a discrete polymatroid with rank $d - |u|$.

Proof. (i): Suppose that $P' = \{x \in P : |x| \leq d\}$. Let $x, y \in P$ and $d \geq |y| > |x|$. Since P is a discrete polymatroid, there exists $z \in P$ such that $x < z < x \vee y$. We have $z > x$ and since P contains all subvectors of z, there exists an integer i such that $x + e_i \leq z$. Then $x < x + e_i \leq x \vee y$ and since $|x + e_i| \leq d$, it belongs to P'. This shows that P' is a discrete polymatroid.

(ii): Let $x', y' \in P_u$ and $|x'| < |y'|$. Then, there exist $x, y \in P$ such that $x' = x - u$ and $y' = y - u$ with $|x| < |y|$. Hence, there exists $z \in P$ such that $x < z \leq x \vee y$. Take $z' = z - u$. Then clearly, $z' \in P_u$ and $x' < z' \leq x' \vee y'$. □

Theorem 2.1.8. [9, Theorem 12.2.4] Let $P \subset \mathbb{R}^n_+$ be a nonempty set of integer vectors and for every vector $x \in P$, it contains all of its integral subvectors and let $B(P)$ be its base set. Then the following are equivalent:

(i) P is a discrete polymatroid,

(ii) if $x, y \in P$ with $|x| < |y|$ then $x + e_i \in P$ and $x + e_i \leq x \vee y$,

(iii) (a) all vectors in $B(P)$ have the same modulus,

(b) if $x, y \in B(P)$ with $x_i > y_i$ for some i then there exists j with $x_j < y_j$ such that $x - e_i + e_j \in B(P)$.

Proof. (i)⇒ (ii) Let P be a discrete polymatroid and let x and y belongs to P with $|x| < |y|$. By definition, there exists z such that $x < z < x \vee y$. Since P is a discrete polymatroid, it contains all subvectors of z. Then there exists some i such that $x + e_i \leq z$. and since $z < x \vee y$, $x + e_i < x \vee y$.

(ii)⇒(i) It is obvious by definition.

(ii)⇒(iii) Let $x, y \in B(P)$ and $x_i > y_i$ for some i. Then $x_i - 1 \geq y_i$. Since x and y are in the base set, they have the same modulus. Hence if $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ then clearly, $|x| - e_i = x_1 + \ldots + x_i - 1 + \ldots + x_n = (y_1 + \ldots + y_n) - 1 = |y| - 1 < |y|$.

Then by (ii), there exists some j such that $(x - e_i) + e_j \leq (x - e_i) \vee y$. If $j = i$ then we would have $x_i = x_i - 1 + 1 = (x - e_i + e_j)(i) \leq \max\{x_i - 1, y_i\} = x_i - 1$ which is a contradiction. Hence $x_j + 1 = (x - e_i + e_j)(j) \leq \max\{x_j, y_j\} \leq y_j$ and this yields $y_j > x_j$.

(iii)⇒ (ii) Let $x, y \in P$ with $|y| > |x|$. Also, let $z' \in B(P)$ with $x < z'$. Since z' is in the base set and it has the maximum modulus, $|y| < |z'|$. Then, because every subvector of z' belongs to P, take $z \in P$ as a subvector of z' with $x \leq z$ and $|z| = |y|$. $\square$

The property (iii)(b) is called *the exchange property*. Before giving proposition, we need to define the distance between two vectors x and y in the set of bases of a discrete polymatroid P by

$$\operatorname{dist}(x, y) = \frac{1}{2} \sum_{t=1}^{n} |x_t - y_t|$$

If $x_i > y_i$ and $x_j < y_j$ then by exchange property we know that $x' = x - e_i + e_j \in B(P)$. Then, clearly $\operatorname{dist}(x, x') < \operatorname{dist}(x, y)$.

Proposition 2.1.9. [9, Proposition 12.2.6] Let P be a discrete polymatroid and $B(P)$ be its set of bases. For $x, y \in B(P)$ and $x_i < y_i$ there exists j with $x_j > y_j$ such that $x + e_i - e_j \in B(P)$.

Proof. Take i with $x_i < y_i$. If $x_{t_1} < y_{t_1}$ for some $t_1 \neq i$ then there exists s_1 with $x_{s_1} > y_{s_1}$ such that $z = y - e_{t_1} + e_{s_1} \in B(P)$. Then $z_i = y_i$ and $\operatorname{dist}(x, z) > \operatorname{dist}(x, y)$. If $x_{t_2} < z_{t_2}$ for some $t_2 \neq i$ then there exists s_2 with $x_{s_2} > z_{s_2}$ such that $z' = z - e_{t_2} + e_{s_2} \in B(P)$. Then $z'_i = y_i$ and $\operatorname{dist}(x, z') < \operatorname{dist}(x, z)$. Repeating these operations, we get $z^*_i = y_i > x_i$ where $z^* \in B(P)$ and $z^*_j \leq x_j$ for all $j \neq i$. Choose $j_1 \neq i$ with $z^*_{j_1} < x_{j_1}$. By exchange property, $x - e_{j_1} + e_i \in B(P)$. $\square$

If $P \subset \mathbb{Z}^n_+$ be a discrete polymatroid then we define $\rho_P : 2^{[n]} \to \mathbb{R}_+$ with respect to P by setting

$$\rho_P(A) = \max\{x(A) : x \in B(P)\}$$

for all nonempty subset A of $[n]$ with $\rho_P(\emptyset) = 0$.

Lemma 2.1.10. [9, Lemma 12.3.2] If $A_1 \subset A_2 \subset \ldots \subset A_t \subset [n]$ is a sequence of subsets of $[n]$, then there is $x \in B(P)$ such that $x(A_l) = \rho_P(A_l)$ for all $1 \leq l \leq t$.

Proof. By induction on t. Assume that there is $x \in B(P)$ such that $x(A_l) = \rho_P(A_l)$ for all $1 \leq l < t$. Take $y \in B(P)$ with $y(A_t) = \rho_P(A_t)$. If $x(A_t) < y(A_t)$, then there is $j \in [n]$ with $j \notin A_t$ such that $x(\{j\}) > y(\{j\})$. Then by exchange property, there is $i \in [n]$ with $x(i) < y(i)$ such that $x_1 = x - e_j + e_i \in B(P)$. Since $x(A_{t-1}) = \rho_P(A_{t-1})$, it turns out $i \notin A_{t-1}$. Thus, $x_1(A_l) = \rho_P(A_l)$ for all $1 \leq l < t$. Moreover, $x_1(A_t) \geq x(A_t)$ and $\operatorname{dist}(x, y) > \operatorname{dist}(x_1, y)$. If $x_1(A_t) = y(A_t)$ then we're done. If $x_1(A_t) < y(A_t)$, then by the above method, we obtain $x_2(A_l) = \rho_P(A_l)$

for all $1 \leq l < t$, $x_2(A_t) \geq x_1(A_t)$ and $\text{dist}(x_1, y) > \text{dist}(x_2, y)$. If we repeat this applications, it is clear that there always exists an $x_r \in B(P)$ such that $x_r(A_l) = \rho_P(A_l)$ for all $1 \leq l \leq t$. □

Theorem 2.1.11. ***(Symmetric Exchange)***[9, Theorem 12.4.1] If $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ are bases of a discrete polymatroid $P \subset Z^n_+$, then for each $i \in [n]$ with $x_i > y_i$, there is $j \in [n]$ with $x_j < y_j$ such that both $x - e_i + e_j$ and $y - e_j + e_i$ belongs to $B(P)$.

Proof. Let $B'(P) = \{z \in P : x \wedge y \leq z \leq x \vee y\}$. Then B' satisfies the exchange property and it is the base set for the discrete polymatroid $P' \subset Z^n_+$. Instead of x and y take $x' = x - x \wedge y$ and $y' = y - x \wedge y$. Suppose that $P' \subset Z^t_+$ is a discrete polymatroid, where $t \leq n$ and $x = (x_1, ..., x_k, 0...0) \in \mathbb{Z}^t_+$, $y = (0, ..., 0, y_{k+1}, ..., y_t) \in \mathbb{Z}^t_+$ where x_i and y_j are nonzero and positive. Also $|x| = |y| = \text{rank}(P')$. We need to show that for each $1 \leq i \leq k$ there is $k+1 \leq j \leq t$ such that both $x - e_i + e_j$ and $y - e_j + e_i$ are bases of P'. Let, say $i = 1$.
Case 1: Assume that $x - e_1 + e_j$ are bases of P' for all $k+1 \leq j \leq t$. By the exchange property, given k integers $x'_1, ..., x'_k$ with each $0 \leq x'_i \leq x_i$, there is a base z' of P' of the form $z' = (x'_1, ..., x'_k, y'_{k+1}, ..., y'_t)$, where each $y'_j \in Z$ with $0 \leq y'_j \leq y_j$. In particular there is $k+1 \leq j_1 \leq t$ such that $y - e_{j_1} + e_1$ is a base of P'. Since $x - e_1 + e_j$ is a base of P' for each $k+1 \leq j \leq t$ both $x - e_1 + e_{j_1}$ and $y - e_{j_1} + e_1$ are bases of P' as desired.
Case 2: Let $k \geq 2$ and $k+2 \leq t$. Then assume that there is $k+1 \leq j \leq t$ with $x - e_1 + e_j \notin P'$. Let $\mathcal{N} \subset \{k+1, ..., t\}$ denote the set of those $k+1 \leq j \leq t$ with $x - e_1 + e_j \notin P'$. We know that $\text{Conv}(P') \cap Z^t = P'$. Let $\rho = \rho_{P'}$ denote the ground set rank function of the integral polymatroid $\text{Conv}(P') \subset \mathbb{R}^t_+$. Thus $\rho(\mathcal{M}) = \max\{z(\mathcal{M}) : z \in B'\}$ for $\emptyset \neq \mathcal{M} \subset [t]$ together with $\rho(\emptyset) = 0$. In particular $\rho(\mathcal{M}) = x(\mathcal{M})$ if $\mathcal{M} \subset \{1, ..., k\}$ and $\rho(\mathcal{M}) = y(\mathcal{M})$ if $\mathcal{M} \subset \{k+1, ..., t\}$. For each $j \in \mathcal{N}$ since $x - e_1 + e_j \notin \text{Conv}(P')$, there is a subset $\mathcal{N}_j \subset \{2, 3, .., k\}$ with $\rho(\mathcal{N}_j \cup \{j\}) \leq x(\mathcal{N}_j)$. Thus,

$$\begin{aligned}
\rho(\{2,3,...,k\} \cup \{j\}) &\leq \rho(\mathcal{N}_j \cup \{j\}) + \rho(\{2,3,...,k\} \setminus \mathcal{N}_j) \\
&\leq x(\mathcal{N}_j) + x(\{2,3,...,k\} \setminus \mathcal{N}_j) \\
&\leq x(\{2,3,...,k\}) \\
&= \rho(\{2,3,...,k\}).
\end{aligned}$$

Hence, for all $j \in \mathcal{N}$, $\rho(\{2,3,...,k\} \cup \{j\}) = x(\{2,3,...,k\})$.
Claim: $\rho(\{2,3,...,k\} \cup \mathcal{N}) = x(\{2,3,...,k\})$.
Proof of Claim: We use induction on $|\mathcal{N}|$. The claim holds trivially if $|\mathcal{N}| = 1$. Let $|\mathcal{N}| > 1$ and take $j_1 \in \mathcal{N}$.Let $L = \{2,3,...,k\}$. Then by assumption,

$$\begin{aligned}
\rho(L) + \rho(L) &= \rho(L \cup (\mathcal{N} \setminus \{j_1\}) + \rho(L \cup \{j_1\}) \\
&\geq \rho((L \cup (\mathcal{N} \setminus \{j_1\})) \cup (L \cup \{j_1\})) + \rho((L \cup (\mathcal{N} \setminus \{j_1\})) \cap (L \cup \{j_1\})) \\
&= \rho(L \cup \mathcal{N}) + \rho(L) \\
&\geq \rho(L) + \rho(L).
\end{aligned}$$

By squeeze theorem, $\rho(L \cup \mathcal{N}) = \rho(L)$ and this proves our claim. By Theorem 2.1.3 we have,

$$\begin{aligned}
\rho(\{2,3,\ldots,k\} \cup \mathcal{N}) + \rho(\{1\} \cup \mathcal{N}) &\geq \rho(\mathcal{N}) + \rho(\{1,2,\ldots,k\} \cup \mathcal{N}) \\
&= y(\mathcal{N}) + \mathrm{rank}(P').
\end{aligned}$$

Thus $x(\{2,3,...,k\}) + \rho(\{1\} \cup \mathcal{N}) \geq y(\mathcal{N}) + \mathrm{rank}(P')$. Since $\mathrm{rank}(P') - x(\{2,3,...,k\}) = a_1$ and $\rho(1 \cup \mathcal{N}) \leq \rho(\{1\}) + \rho(\mathcal{N}) = a_1 + y(\mathcal{N})$, we obtain $\rho(\{1\} \cup \mathcal{N}) = a_1 + y(\mathcal{N})$. Hence, for all $\mathcal{N}' \subset \mathcal{N}$, we have

$$\begin{aligned}
a_1 + y(\mathcal{N}) &= a_1 + y(\mathcal{N}') + y(\mathcal{N} \setminus \mathcal{N}') \\
&= \rho(\{1\}) + \rho(\mathcal{N}') + \rho(\mathcal{N} \setminus \mathcal{N}') \\
&\geq \rho(\{1\} \cup \mathcal{N}') + \rho(\mathcal{N} \setminus \mathcal{N}') \\
&\geq \rho(\{1\} \cup \mathcal{N}) \\
&= a_1 + y(\mathcal{N}').
\end{aligned}$$

Thus, for all $\mathcal{N}' \subset \mathcal{N}$, $\rho(\{1\} \cup \mathcal{N}') = a_1 + x(\mathcal{N}')$. By Lemma 2.1.10, there is a base $z \in P'$ with $z(1) = a_1$ and with $z(j) = y(j)$ $(= \rho(\{j\}))$ for all $j \in \mathcal{N}$. By the exchange property (for z and y) for each $1 \leq i \leq k$ with $z_i > 0$ there is $j \in \{k+1,...,t\} \setminus \mathcal{N}$ such that $z - e_i + e_j$ is a base of P'. Thus, after repeating these procedure, we obtain a base of $z' \in P'$ of the form $z' = y - e_{j_1} + e_1$ where $j_1 \in \{k+1,...,t\} \setminus \mathcal{N}$. Thus, both $x - e_1 + e_{j_1}$ and $y - e_{j_1} + e_1$ are bases of P'. $\square$

2.2 Polymatroidal ideals

Definition 2.2.1. Let I be a monomial ideal in the polynomial ring $S = K[x_1, ..., x_n]$ and $G(I)$ be the minimal generating set of I. Then I is called *polymatroidal ideal* if the following conditions are satisfied:

(i) all the elements of the $G(I)$ have the same degree,

(ii) let $u = \mathbf{x}^{\mathbf{a}}$ with $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{Z}_+^n$ and $v = \mathbf{x}^{\mathbf{b}}$ with $\mathbf{b} = (b_1, ..., b_n) \in \mathbb{Z}_+^n$ belong to $G(I)$. If $a_i > b_i$ for some i then there exits some j with $a_j < b_j$ such that $x_j(u)/x_i$ belongs to $G(I)$.

It is shown that all powers of polymatroidal ideals have linear quotients. To give the proof, we need the following:

Theorem 2.2.2. [9, Theorem 12.6.3] Let I and J be polymatroidal ideals. Then IJ is polymatroidal.

Proof. Let I be a polymatroidal ideal with $G(I) = \{\mathbf{x}^{u_1}, ..., \mathbf{x}^{u_r}\}$ and J be a polymatroidal ideal with $G(J) = \{\mathbf{x}^{v_1}, ..., \mathbf{x}^{v_s}\}$. Clearly, $B(I) = \{u_1, ..., u_r\}$ and $B(J) = \{v_1, ..., v_s\}$ are the base sets of the discrete polymatroids with respect to I and J. Since the polymatroid sum is again a polymatroid, the base set for the product is $B(IJ) = \{u_i + v_j, u_i \in B(I), v_j \in B(J), i = 1, ..., r, j = 1, ..., s\}$ and it satisfies the symmetric exchange property. Hence IJ is polymatroidal ideal. $\square$

Theorem 2.2.3. [9, Theorem 12.6.2] A polymatroidal ideal has linear quotients.

Proof. Let I be a polymatroidal ideal with $G(I) = \{u_1, ..., u_r\}$ and $u_1 > ... > u_r$ with respect to reverse lexicographic order. We need to show that $Q = (u_1, ..., u_{q-1}) : u_q$ is generated by variables where $2 < q < s$. Since this quotient is equal to $(u_1/[u_1, u_q], ..., u_r/[u_r, u_q])$, it is enough to show that for each $2 < t < q$ there exists some $x_k \in Q$ such that x_k divides $(u_t/[u_t, u_q])$. Let $u_t = x^a$ and $u_q = x^b$ with $x^a > x^b$ where $a = (a_1, ..., a_n)$ and $b = (b_1, ..., b_n)$. Since $u_t > u_q$, and we have a reverse lexicographic order there is an integer $1 < l < n$ with $a_l < b_l$. Hence by symmetric exchange property, there exists $k \leq l < n$ with $a_k > b_k$ such that $x_k(u_t/x_l) \in G(I)$. Since $k < l$, $x_k \in Q$ and it follows that x_k divides every component of Q. $\square$

Corollary 2.2.4. [9, Corollary 12.6.4] All powers of polymatroidal ideals have linear quotients and they admit linear resolution.

Proof. Since product of polymatroidal ideals is again polymatroidal and polymatroidal ideals have linear quotients, it implies that all powers of polymatroidal ideals have linear quotients. By the Proposition 2.2.4, they have linear resolution. □

If I is a monomial ideal of S, it is known that I can be written as intersection of minimal prime ideals of I. Also we know that minimal prime ideals P are generated by subsets of variables. An ideal I is called unmixed if all minimal prime ideals of I have the same height. A Cohen-Macaulay ideal is always unmixed. If P is a monomial prime ideal and $\theta(P)$ denote the number of variables which generate P, then we set $c(I) = \min\{\theta(P) : P \in \mathrm{Min}(I)\}$. It gives

$$\dim(S/I) = n - c(I).$$

Additionally, if I is generated in one degree and has linear quotients then the colon ideal is generated by subsets of variables. Let r_j denote the number of variables which is required to generate each quotient. Then $r(I) = \max r_j$. It follows that

$$\mathrm{depth}(S/I) = n - r(I) + 1$$

Let I be a monomial ideal generated in one degree and has linear quotients. Then I is Cohen-Macaulay if and only if $c(I) = r(I) + 1$.

Example 2.2.5. (i) An ideal which is generated by all monomials of S of degree d is called Veronese ideal and it is polymatroidal and Cohen-Macaulay.

(ii) An ideal which is generated by all square-free monomials of S of degree d is called square-free Veronese ideal and it is matroidal ideal. It is also polymatroidal and Cohen-Macaulay.([11, Example 3.2])

Lemma 2.2.6. [9, Lemma 12.6.6] Let $I \subset S$ be a Cohen-Macaulay polymatroidal ideal. Then radical of I is square-free Veronese ideal.

Proof. Let I be a Cohen-Macaulay polymatroidal ideal in S. Suppose that $\cup_{v\in G(I)} \mathrm{supp}(v) = \{x_1, \dots, x_n\}$. Let $v \in \mathrm{Mon}(S)$ such that $|\mathrm{supp}(v)|$ is minimal. We can assume that $\mathrm{supp}(v) = \{x_{n-d+1}, \dots, x_n\}$. Let $u \in G(I)$ with $u >_{rev} v$ generate the monomial ideal J. Then it is known that the colon ideal $J : v$ is generated by variables. Let us call the set of these variables A which is a subset of $\{x_1, \dots, x_n\}$. Our claim is $\{x_1, \dots, x_{n-d}\}$ is a subset of A. There is $w \in G(I)$ which is divided by x_i for each $1 \le i \le n-d$. Then by Proposition 2.1.9, there exists x_j where $n-d+1 \le j \le n$ such that $w = x_i v/x_j \in G(I)$. This yields $w \in J$. Then

$x_i \in J : v$ as $x_j w = x_i v \in J$. Hence, $r(I) \geq n - d$. Since I is Cohen-Macaulay, we know that $c(I) = r(I) + 1$. It turns out $c(I) \geq n - d + 1$. Consequently, for each $M \subset \{x_1, \ldots, x_n\}$ with $|M| = d$, we have $I \subset (\{x_1, \ldots, x_n\} \setminus M)$. Hence, for each M we have a monomial $m \in G(I)$ such that $\operatorname{supp}(m) \subset M$. Since $|\operatorname{supp}(m)| \geq |\operatorname{supp}(v)| = d$, we obtain $\operatorname{supp}(m) = M$. Then the radical of I is generated by all square-free monomials which have degree d in $x_1, \ldots, x_n$. $\square$

The following theorem characterizes the Cohen-Macaulay polymatroidal ideals.

Theorem 2.2.7. [9, Theorem 12.6.7] Let I be a polymatroidal ideal. I is Cohen-Macaulay if and only if

(i) a principal ideal

(ii) a Veronese ideal

(iii) a square-free Veronese ideal.

Proof. By applying Lemma 2.2.6, suppose that $\sqrt{I}$ is generated by all square-free monomials which have degree d for $2 \leq d < n$. Then we have $c(I) = c(\sqrt{I}) = n - d + 1$. Assume that I is not square-free. This means that each $u \in G(I)$ has degree $> d$. Let $v = \prod_{k=n-d+1}^{n} x_k^{a_k} \in G(I)$ be a monomial where $\operatorname{supp}(v) = \{x_{n-d+1}, x_{n-d+2}, \ldots, x_n\}$. Claim: There is a monomial $u = \prod_{k=1}^{n} x_k^{b_k} \in G(I)$ where $b_{n-d+1} > a_{n-d+1}$.
Proof Of Claim: Let $\theta = \{x_{l_1}, \ldots, x_{l_d}\} \subset \{x_1, \ldots, x_n\}$ be subsets which have d elements. Then for each θ we have a monomial $v_\theta \in G(I)$ such that $\operatorname{supp}(v_\theta) = \theta$. If θ and β are subsets of $\{x_1, \ldots, x_n\}$ with d elements and if we take an element in the intersection of θ and β, namely x_{l_0} and $a_{l_0} < b_{l_0}$ where $x_{l_0}^{a_{l_0}} \in v_\theta$ and $x_{l_0}^{b_{l_0}} \in v_\beta$. Then after relabelling the variables, we can assume that $\theta = \{x_{n-d+1}, \ldots, x_n\}$ with $l_0 = n - d + 1$. Then our claim is satisfied. If it fails to be satisfied, then there exists a positive integer $t \geq 2$ such that $v = (x_{l_1} x_{l_2} \cdots x_{l_d})^t \in G(I)$. Let $u = x_{n-d} x_{n-d+1}^{t-1} (\prod_{l=n-d+2}^{n} x_l^t) \in G(I)$. Let J denote the monomial ideal which is generated by $w \in G(I)$ such that $w >_{rev} u$. Since $\prod_{l=n-d}^{n-1} x_l^t \in G(I)$, by applying Prop 2.1.9, we have $u_0 = x_{n-d} u / x_n \in J$ and $u_1 = x_{n-d+1} u / x_n \in J$. Hence, the colon ideal $J : u$ is generated by a subset $A \subset \{x_1, \ldots, x_n\}$ where $\{x_1, \ldots, x_{n-d}, x_{n-d+1}\} \subset A$. Thus $r(I) \geq n - d + 1$ and this yields $c(I) < r(I) + 1$ which is a contradiction.
Now, let J be a monomial ideal which generated by $u \in G(I)$ such that $u >_{rev} v$. In the proof of previous lemma, we saw that $J : v$ is generated by a subset $A \subset \{x_1, \ldots, x_n\}$ where $\{x_1, \ldots, x_{n-d}\} \subset A$. We claim that $x_{n-d+1} \in J : v$. By

using our claim and Prop 2.1.9, there is a variable x_k with $n-d+1 < k \leq n$ such that $v_0 = x_{n-d+1}v/x_k \in G(I)$. Since $v_0 \in J$ we have $x_{n-d+1} \in A$. Consequently, $r(I) \geq n-d+1$. Hence, $c(I) < r(I)+1$ and I is not Cohen-Macaulay ideal. $\square$

2.3 Weakly polymatroidal ideals

Weakly polymatroidal ideals are defined first by Hibi and Kokubo [17] as a generalization of polymatroidal ideals. Then Mohammadi and Moradi [19] showed in their paper some applications to vertex cover ideals.

Definition 2.3.1. Let I be a monomial ideal. Then I is called *weakly polymatroidal* if for all $w = \mathbf{x}^{\mathbf{a}} \in G(I)$ with $\mathbf{a} = (a_1, \ldots, a_n)$ and $v = \mathbf{x}^{\mathbf{b}} \in G(I)$ with $\mathbf{b} = (b_1, \ldots, b_n)$ if $a_1 = b_1, \ldots, a_{k-1} = b_{k-1}$ and $a_k > b_k$ for some k, there exists $t > k$ such that $x_k v/x_t \in I$.

Example 2.3.2. Let I be a monomial ideal. Then I is called stable if for any $v \in I$ and $j < \max(v)$, $x_j(v/x_{\max(v)}) \in I$ where $\max(v) = \max\{i : x_i | v\}$. Stable ideals are weakly polymatroidal.

Theorem 2.3.3. [9, Theorem 12.7.2] Let I be a weakly polymatroidal ideal. Then I has linear quotients.

Proof. Let $I = (v_1, \ldots, v_m)$ be a weakly polymatroidal ideal with $v_1 >_{lex} \ldots >_{lex} v_m$ and $x_1 > \ldots > x_n$. Let $u \in \mathrm{Mon}(S)$ where $u \in (v_1, \ldots, v_{k-1}) : v_k$. Then $uv_k \in (v_j)$ for some $j < k$. Let $v_j = x_1^{a_1} \ldots x_n^{a_n}$ and $v_k = x_1^{b_1} \ldots x_n^{b_n}$. Then there exists $s < n$ such that $a_1 = b_1 \ldots a_{s-1} = b_{s-1}$ and $a_s > b_s$. Hence, $x_s | u$ and there exists $t > s$ such that $x_s v_k/x_t \in I$. Therefore, the set $M = \{v_j : x_s v_k/x_t \in (v_j)\}$ is nonempty. Let $v_l \in M$ be the unique element such that $\deg v_j > \deg v_l$ or $\deg v_j = \deg v_l$ and $v_j <_{lex} v_l$ for any $v_j \in M$ with $j \neq l$. Then $x_s v_k/x_t = v_l w$ for some $w \in S$. If $x_s | w$ then $v_k = v_l w'$ for some $w' \in S$. This contradicts with $v_k \in G(I)$. Hence, $x_s^{b_s+1}$ divides v_l. We claim that $v_l > v_k$. Assume not which means $v_l < v_k$. Then let $v_l = x_1^{c_1} \cdots x_n^{c_n}$ where $c_1 = b_1, \ldots c_{r-1} = b_{r-1}$ and $c_r < b_r$ for some $1 \leq r \leq n$. Since $x_s^{b_s+1} | v_l$ we have $r < s$. Then, by definition of weakly polymatroidal ideals, $h = v_l x_r/x_j \in I$ for some $j > r$. Since $r < t$, $x_r | w$ and this yields $x_j w/x_r \in S$. As $h(x_j w/x_r) = x_s(v_k/x_t)$, $v_l <_{lex} h$ and $\deg h = \deg v_l$, we have $h \notin G(I)$. Let $h = v_{l'} w'$ for some l' and $w' \in S$, $w' \neq 1$. Then $\deg v_{l'} < \deg h = \deg v_l$. This is a contradiction, since $v_{l'} \in M$. Therefore, we have $v_l w \in (v_1, \ldots, v_{t-1})$, and thus $x_s v_l \in (v_1, \ldots, v_{t-1})$. Since $x_s | u$, we are done. $\square$

Chapter 3

Monomial Localizations of Polymatroidal Ideals

In this chapter, monomial localizations of polymatroidal ideals will be discussed. Let $\mathcal{P}$ be a polymatroid and $I = (\mathbf{x}^u : u \in B(\mathcal{P})) \subset S = K[x_1, \ldots, x_n]$. We set $u = (u(1), \ldots, u(n))$. If we substitute x_j by 1 in I then we obtain a new monomial ideal $I_{\{j\}} = (\mathbf{x}^{u'} : u \in B(\mathcal{P}))$ in $S_{\{j\}} = K[x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n]$ where $\mathbf{x}^{u'} = \mathbf{x}^u / x_j^{u_j}$.

Definition 3.0.1. Let $I \subset S$ be a monomial ideal and let $P \subset S$ be a monomial prime ideal. Then by substituting the variables $x_j \mapsto 1$ such that $x_j \notin P$ we obtain a new monomial ideal $I(P) \subset S(P) = K[x_i : x_i \in P]$ which is called the *monomial localization* of I with respect to the monomial prime ideal P.

One can also define the monomial localization of I with respect to monomial prime ideal P as follows:

$$I(P) = I : (\prod_{x_j \notin P} x_j)^{\infty}.$$

Indeed, for any $u \in I(P)$, it is easy to see that $u \prod_{x_j \notin P} x_j \in I$. Conversely, let $J = (\prod_{x_j \notin P} x_j)$. If $u \in I : J$ then there exists $v \in J$ such that $uv \in I$. We can write $u = u'w$, where $u' \in I(P)$ and $w \in \mathrm{Mon}(S)$. It shows that $u \in I(P)$ because $u' | u$.

Our goal is to show that if I is polymatroidal ideal then $I(P)$ is again polymatroidal. First, in the following proposition, we show that $I_{\{j\}}$ is polymatroidal. Then by repeated application of this proposition, one obtains that required result.

Proposition 3.0.2. [13, Proposition 3.1] Let I be a polymatroidal ideal in $K[x_1,\ldots,x_n]$. Then $I_{\{j\}}$ is again polymatroidal ideal.

Proof. Let I be a polymatroidal ideal generated in degree d with $G(I)=\{x^{u_1},\ldots,x^{u_r}\}$. We know that $B(P)=\{u_1,\ldots,u_r\}$ is the base set of the discrete polymatroid P with respect to I. Then $I_{\{i\}}=(x^{u_j}/x_i^{u_j(i)} : u_j\in B(P))$. Our first claim is

$$G(I_i)=\{x^{u_j}/x_i^{a_i} : \text{ if } \max u_j(i)=a_i \text{ for } u_j\in B(P)\}$$

is generated in one degree. Let $u_s\in B(P)$ with $u_s(i)\leq a_i$. We need to show that there exists $u_t\in B(P)$ with $u_t(i)=a_i$ such that $x^{u_t}/x_i^{a_i}$ divides $x^{u_s}/x_i^{a_i}$. To prove this, we use induction on $a_i-u_s(i)$. If $a_i-u_s(i)=0$ then we're done. Assume that $u_s(i)<a_i$ and let $u_k(i)=a_i$ for some $u_k\in B(P)$. By Theorem 2.1.11, there exists an integer $l\in[n]$ with $u_k(l)<u_s(l)$ such that $u_k-e_i+e_k\in B(P)$ and $u_s'=u_s-e_k+e_i\in B(P)$. Hence $x^{u_s'}/x_i^{a_i}$ divides $x^{u_s}/x_i^{a_i}$. Since $a_/i-u_s'(i)<a_i-u_s$, by induction hypothesis there exists $u_t\in B(P)$ with $u_t(i)=a_i$ such that $x^{u_t}/x_i^{a_i}$ divides $x^{u_s'}/x_i^{a_i}$. Consequently, $x^{u_t}/x_i^{a_i}$ divides $x^{u_s}/x_i^{a_i}$, as well. Secondly, we claim that the set $B'=\{u_1',\ldots,u_r' : x^{u_j}\in G(I_{\{i\}}),\ j=1,\ldots,r\}$ is the base set for a discrete polymatroid P' with rank $P'=d-a_i$ on $[n]\setminus\{i\}$. Firstly, for all $u_j'\in B'$ we have $|u_j'|=d-a_i$. Now let $u_s',u_t'\in B'$ with $u_s'(k)>u_t'(k)$. Then $k\neq i$. By applying the exchange property, for $u_s,u_t\in B(P)$, $u_s(k)=u_s'(k)>u_t'(k)=u_t(k)$ then there exists $l\in[n]$ such that $u_s(l)<u_t(l)$ and $u_m=u_s-e_k+e_l\in B(P)$. Since $u_s(i)=u_t(i)=a_i$, it follows $l\neq i$ and $u_m(i)=a_i$. Hence, we obtain $u_m'\in B'$ where $u_m'=u_s'-e_k+e_l$. □

As discussed before, one obtains the following

Corollary 3.0.3. [13, Corollary 3.2] Let I be polymatroidal ideal. Then $I(P)$ is also polymatroidal for all monomial prime ideals which contain I.

Next, we will discuss the relation between polymatroidal ideals and the ideals with the property that their all monomial localizations have linear resolutions. It appears that in case of polynomial rings in upto 3 variables, the before mentioned properties are equivalent. Below we give the proof of these equivalence.

Lemma 3.0.4. [1, Lemma 2.2] Let I be a graded ideal in S such that I has a linear resolution and $\ell(S/I)<\infty$. Then $I=(x_1,\ldots,x_n)^t$ for some t.

Proof. Since $\ell(S/I)<\infty$, it implies that $\operatorname{reg}(S/I)=\max\{i:(S/I)_i\neq 0\}$. Suppose that I has a t-linear resolution. Hence $\operatorname{reg}(S/I)=t-1$. Therefore, $(S/I)_i=0$ for $i\geq t$. Then $I=(x_1,\ldots,x_n)^t$. □

Proposition 3.0.5. [1, Lemma 2.4] Let J be a polymatroidal ideal which is generated in degree d. If J contains at least $n-1$ pure powers of variables then the following are equivalent:

(i) The ideals J and $J(P_{\{n\}})$ have a linear resolution.

(ii) $J = J_{(d;d,\dots,d,k)}$ for some k.

Proof. Let $S' = K[x_1,\dots,x_{n-1}]$ and J_i be a monomial ideal for all i in S'. Then we can write $J = J_0 + J_1x_n + \ldots + J_tx_n^t$. Let I be a monomial ideal which has a linear resolution. Let $b_1,\dots,b_n$ be positive integers. Then I' which is generated by those monomials $v \in G(I)$ where $deg_{x_j}v \le b_j$ for $j = 1,\dots,n$ has a linear resolution as well. This will be called as 'restriction lemma'.
Let us apply the restriction lemma to J. Then it turns out that J_0 has a d-linear resolution. By our assumption, $x_1^d,\dots,x_{n-1}^d \in J_0$. In particular, $\ell(S'/J_0) < \infty$. By Lemma 3.0.4, it follows that $J_0 = m^d$ where $m = (x_1,\dots,x_{n-1})$. Now we show by induction on i that $J_{t-i} = m^{d-t+i}$. For $i = 0$, we need to show $J_t = m^{d-t}$. By assumption, $J(P_{\{n\}}) = J_0 + J_1 + \ldots + J_t$ has a linear resolution. Since I_i is generated in degree $d-i$, $I(P_{\{n\}}) = J_t$ and $m^d = J_0 \subset J_t$. Hence J_t has a $d-t$ linear resolution and $\ell(S'/J_t) < \infty$. By applying Lemma 3.0.4, it follows that $J_t = m^{n-t}$.
Now, suppose that $i > 0$(and $\le t-1$). Suppose also that $J_{t-k} = m^{d-t+k}$ for $k = 0,\dots,i-1$. Then we set $I = J_0 + J_1x_n + \ldots + J_{t-i}x_n^{t-i}$ and $K = m^{d-t+i-1}x_n^{t-i+1} + \ldots + m^{d-t}x_n^t$. K is polymatroidal, and has d-linear resolution. By applying the restriction lemma to J, we have I has a d-linear resolution. Then $I \cap K = (J_0 \cap K) + (J_1x_n \cap K) + \ldots + (J_{t-i}x_nt - i \cap K) = J_0x_n^{t-i+1} + J_1x_n^{t-i+1} + \ldots + J_{t-i}x_n^{t-i+1} = (J_0 + J_1 + \ldots + J_{t-i})x_n^{t-i+1}$. Hence, $\mathrm{reg}(I \cap K) \ge d+1$. On the other hand, by the exact sequence $0 \to I \cap K \to I \oplus K \to J \to 0$, we have that $\mathrm{reg}(I \cap K) \le \max\{\mathrm{reg}(I \oplus K), \mathrm{reg}(J)+1\} = d+1$. Then $\mathrm{reg}(I \cap K) = d+1$. Therefore $I \cap K = (J_0 + J_1 + \ldots + J_{t-i})x_n^{t-i+1} = J_{t-i}x_n^{t-i+1}$. Hence J_{t-i} has a $(d-t+i)$-linear resolution and $m^d = J_0 \subset J_{t-i}$. So $\ell(S'/J_{t-i}) < \infty$. By Lemma 3.0.4, $J_{t-i} = m^{d-t+i}$. Therefore, $J = m^d + m^{d-1}x_n + \ldots + m^{d-t}x_n^t = J_{(d;d,\dots,d,k)}$. □

Corollary 3.0.6. [1, Corollary 2.5] $I \in K[x_1,x_2]$ is a polymatroidal ideal if and only if for all monomial prime ideals P the ideal $I(P)$ has a linear resolution.

Proof. (⇒) By Corollary3.0.3, it is trivial.
(⇐) Let I be a polymatroidal ideal with $G(I) = \{u_1,\dots,u_r\}$. If $\gcd(u_1,\dots,u_r) = w$, then $I = wJ$ for some ideal J. It turns out I is polymatroidal if and only if J is a

polymatroidal ideal. Hence, if we suppose that $\gcd(u_1, \ldots, u_r) = 1$ then this implies I contains pure power of x_1 or x_2. Thus by Prop 3.0.5, we are done. $\square$

Definition 3.0.7. *(Strong Exchange Property)* Let $I = (u_1, \ldots, u_r)$ be a monomial ideal in S. Then I satisfies the *strong exchange property* if:

(i) I is generated in a single degree,

(ii) For all $\deg_{x_i}(u_k) > \deg_{x_i}(u_t)$ and for all $\deg_{x_j}(u_k) < \deg_{x_j}(u_t)$ where $1 \leq k, t \leq r$, $x_j(u_k/x_i) \in G(I)$.

Proposition 3.0.8. [1, Proposition 2.7] Let $I \subset K[x_1, x_2, x_3]$ be a monomial ideal. Then the following are equivalent:

(a) I is a polymatroidal ideal,

(b) I satisfies the strong exchange property,

(c) For all monomial prime ideals P, $I(P)$ has a linear resolution.

Proof. The conditions (a)$\Leftrightarrow$ (b) is known.

(b)$\Rightarrow$ (c) Let I be a monomial ideal satisfying strong exchange property. Obviously, I is a polymatroidal ideal. Then, by Proposition 3.0.2 and Corollary 3.0.3, I(P) is polymatroidal. Since I(P) is polymatroidal, it has linear quotients. Hence, I(P) has a linear resolution.

(c)$\Rightarrow$ (b) Let $I = (u_1, \ldots, u_r)$. Then we can assume that $I = wJ$ where $\gcd(u_1, \ldots, u_r) = w$. We need to show that J is of Veronese type. Since $I(P)$ has a linear resolution for all P, $J(P)$ has also a linear resolution. Without losing of generality, we may assume that $w = 1$. Then it remains to show I is of Veronese type. Let $b_i = \max\{\deg_{x_i} u_t : u_t \in G(I), 1 \leq t \leq r, i = 1, 2, 3\}$. Then

Claim: $I = I_{(d,b_1,b_2,b_3)}$ where d is the degree of the generators of I.

First step: We claim that the set

$\mathcal{M}_i = \{v \in K[x_1, x_2, x_3] : \deg v = d,\ \deg_{x_i}(v) = b_i,\ \deg_{x_j}(v) \leq b_j \text{ if } j \neq i\} \in I.$

Since $I(P)$ has a linear resolution, $I(P_{\{i\}})$ has a linear resolution. Then $I(P_{\{i\}})$ is generated by the monomials $u \in K[x_j, x_k]$ such that $ux_i^{b_i} \in I$. Hence, by Corollary 3.0.6, $I(P_{\{i\}})$ is polymatroidal. Thus, there exists integers $0 \leq s \leq t \leq d - b_i$ such that $I(P_{\{i\}}) = (x_j^e x_k^f : e + f = d - b_i, e \leq b_j, f \leq b_k \text{ and } s \leq e \leq t)$.

Now suppose that $\mathcal{M} \notin I$ for some i. Then $s > 0$ or $t < d - b_i$. Assume $s > 0$. Thus, $x_k^{d-b_i} x_i^{b_i} \notin I$. Also, since $\gcd(u_1, \ldots, u_r) = 1$, there exists monomial

$x_k^{d-a}x_i^a \in I$ with $a < b_i$. Therefore, $x_k^{d-a} \in I(P_{\{i\}})$. This is a contradiction, since $I(P_{\{i\}})$ does not contain a pure power of x_k.

To finish the proof of our claim, we define the ideals J_{a_1,a_2,a_3} for $b_i \leq a_i \leq d$ for $i = 1, 2, 3$. The ideal J_{a_1,a_2,a_3} is generated by all generators of I and all monomials $x_1^{l_1}x_2^{l_2}x_3^{l_3}$ of degree d such that $l_j \leq a_j$ for all j and there exists $i \in [3]$ with $b_i \leq l_i \leq a_i$. Our aim is to show that J_{a_1,a_2,a_3} has a linear resolution for all a_i, in particular $J_{\{d,d,d\}}$ has a linear resolution. We will use induction on $a_1 + a_2 + a_3$:

Basis Step: $a_1 + a_2 + a_3 = b_1 + b_2 + b_3$. Then $a_i = b_i$ for all i. By the definition of the ideal J_{b_1,b_2,b_3}, we have that $J_{b_1,b_2,b_3} = I + (\mathcal{M}_i)$ for some i. Since $\mathcal{M}_i \subseteq I$, we have that $J_{b_1,b_2,b_3} = I$ and by our assumption, has a linear resolution. Suppose that $a_1 + a_2 + a_3 > b_1 + b_2 + b_3$. Then $b_i > a_i$ for some i. Let us assume $i = 1$. Then by the induction hypothesis, the ideal $L = J_{a_1-1,a_2,a_3}$ has a d-linear resolution. Let $J = J_{a_1,a_2,a_3}$. Then we have the exact sequence $0 \to L \to J/L \to 0$.

The module J/L is annihilated by x_2 and x_3. Thus, J/L is an $S/(x_2, x_3)$ module generated by the residue classes of the elements $ux_1^{a_1}$ where $u \in K[x_2, x_3]$ of degree $d - a_1$. Since no power of x_1 annihilates the generators of J/L, we obtain that J/L is a free $S/(x_2, x_3)$-module. Consequently, J/L has a d-linear resolution. Therefore, from the exact sequence above , J has a d-linear resolution. Hence, by Lemma 3.0.4, since $J_{d,d,d,}$ contains the pure powers of x_i^d, $J_{d,d,d,} = (x_1, x_2, x_3)^d$. This implies $I = I_{d,b_1,b_2,b_3}$.

Next, we show that under certain conditions on monomial ideal I in a polynomial ring with n variables, the property of admitting a linear resolution for all monomial localizations of I implies that I is a polymatroidal ideal.

Definition 3.0.9. Let M be a nonempty subset of $[n]$. Then the monomial prime ideal $P_M = (x_j : j \in M)$. I is called transversal polymatroidal ideal if it is of the form

$$I = P_{M_1}P_{M_2}\cdots P_{M_s}$$

where $M_1, \ldots, M_s$ is a collection of nonempty subsets of $[n]$ with $s \geq 1$.

Proposition 3.0.10. [1, Proposition 2.8] Let I be a monomial ideal with no embedded prime ideals and the monomial localization at P for all P has a linear resolution. Let $\mathrm{Ass}(S/I) = \{P_1, \ldots, P_s\}$ and let $m = (x_1, \ldots, x_n)$ be a graded maximal ideal of S. Then

(i) If $P_j \cap P_k = m$ for all $j \neq k$ then I is polymatroidal ideal.

(ii) If $s \leq 2$ then I is transversal polymatroidal ideal. If $s = 3$, then either I is again transversal polymatroidal ideal or I is matroidal ideal generated in degree 2 of the form $I = P_1 \cap P_2 \cap P_3$ such that $\bigcap_{j=1}^{3} G(P_j) = \emptyset$ and $G(P_j) \cup G(P_k) = \{x_1, \ldots, x_n\}$ for all $j \neq k$.

(iii) If $\text{height}(I) = n - 1$ then I is polymatroidal.

Proof. Let $P \in \text{Ass}(S/I)$. P is a minimal prime ideal of I since I is a monomial ideal with no embedded prime ideals. Hence, $\ell(S(P)/I(P)) < \infty$. Since $I(P)$ has a linear resolution, by Lemma 3.0.4, $I(P) = P^t$ for some t. Thus $I = P_1^{b_1} \cap \ldots \cap P_s^{b_s}$.

(i) Since I is generated in a single degree, by a result of Adam Van Tuyl and Francisco [7, Theorem 3.1], I is polymatroidal.

(ii) If $s = 1$ then $I = P_1^{b_1}$ is obviously transversal polymatroidal ideal.

If $s = 2$, then $I = P_1^{b_1} \cap P_2^{b_2}$. Since I is generated in a single degree, we obtain $G(P_1) \cap G(P_2) = \emptyset$. Thus, $I = P_1^{b_1} P_2^{b_2}$ and we are done.

Now, let $s = 3$ so $I = P_1^{b_1} \cap P_2^{b_2} \cap P_3^{b_3}$. Without losing of generality, suppose that I is fully supported, that is $\{x_1, \ldots, x_n\} = \bigcup_{v \in G(I)} \operatorname{supp} v$. Assume that, $P_j \nsubseteq P_k + P_i$ for all j, k, i. Since $I(P_k + P_i) = P_k^{b_k} \cap P_i^{b_i}$ is generated in a single degree, $G(P_k) \cap G(P_i) = \emptyset$ for $k \neq i$. Thus $I = P_1^{b_1} P_2^{b_2} P_3^{b_3}$ is a transversal polymatroidal ideal.

Now suppose that $P_1 \subset P_2 \cap P_3$. Since I is fully supported, $P_2 + P_3 = m$.

Claim: $P_j + P_k = m$ for all $j \neq k$. Then by part (i), we obtain that I is polymatroidal. We need to show that $P_1 + P_2 = m$ and $P_1 + P_3 = m$. Assume that $P_1 + P_2 \neq m$ and $P = P_1 + P_2$. Then $I(P) = P_1^{b_1} \cap P_2^{b_2}$. Since $I(P)$ is generated in a single degree, $G(P_1) \cap G(P_2) = \emptyset$. Then $P_1 \subset P_2 + P_3$, so we obtain $P_1 \subseteq P_3$ which is a contradiction. Hence $P_1 + P_2 = m$. Also, by similar argument, $P_1 + P_3 = m$.

Now, we have to show that $G(P_j) \cap G(P_k) \nsubseteq G(P_i)$ for distinct j, k, i. Suppose that $G(P_j) \cap G(P_k) \subseteq G(P_i)$ for some j, k, i. Let x_t be a variable. If $x_t \in G(P_j) \cap G(P_k)$, then $x_t \in G(P_i)$. Conversely, if $x_t \notin G(P_j) \cap G(P_k)$, then we can assume that $x_t \notin G(P_i)$. Hence we obtain $x_t \in G(P_i)$ since $P_j + P_i = m$. Thus, $P_i = m$, which is a contradiction.

Next, suppose that $b_1 = b_2 = b_3$. Without losing of generality, we may assume that $b_1 \geq b_2 \geq b_3$ and I is generated in degree d. Let $x_k \in G(P_1) \cap G(P_2) \setminus G(P_3)$ and $x_m \in G(P_3) \setminus G(P_1)$. Then, since $b_1 \geq b_2$, $x_k^{b_1} x_m^{b_3} \in I$. There exists integers $l \leq b_1$ and $s \leq b_3$ such that $x_k^l x_m^s \in G(I)$. Since $x_k^l x_m^s \in P_3^{b_3}$ and $x_k \notin P_3$, it follows that $x_m^s \in P_3^{b_3}$ and $s = b_3$. On the other hand, since $x_k^l x_m^s \in P_1^{b_1}$ and $x_m \notin P_1$, we obtain $x_k^l \in P_1^{b_1}$ and $l = b_1$. Thus, $x_k^{b_1} x_m^{b_3} \in G(I)$. Hence, $d = b_1 + b_3$. Now let

$x_k \in G(P_1) \cap G(P_3) \setminus G(P_2)$ and $x_m \in G(P_2) \setminus G(P_1)$. Then similarly, $x_k^{b_1} x_m^{b_2} \in G(I)$, and $d = b_1 + b_2$. Hence, $b_2 = b_3$. Now set $b = b_2 = b_3$. We need to show that $b_1 < 2b$. Suppose on the contrary, $b_1 \geq 2b$. Let $x_k \in G(P_1) \cap G(P_2)$ and $x_m \in G(P_1) \cap G(P_3)$. Then $x_k^{b_1-b} x_m^b \in I$. Thus, $b_1 = \deg(x_k^{b_1-b} x_m^b) \geq d = b_1 + b$, so $b \leq 0$, which is a contradiction.

Now, let $x_k \in G(P_1) \cap G(P_2)$ and $x_m \in G(P_1) \cap G(P_3)$ and $x_t \in G(P_3)$. Then $x_k^b x_m^{b_1-b} x_t^{2b-b_1} \in I$. Therefore, $2b = \deg(x_k^b x_m^{b_1-b} x_t^{2b-b_1}) \geq d = b_1 + b$; hence $b \geq b_1$ and then $b_1 = b$.

Now, we have $I = P_1^b \cap P_2^b \cap P_3^b$. We may assume that $b = 1$. This assumption implies $I = P_1 \cap P_2 \cap P_3$. Thus, I is generated in a single degree, we obtain $G(P_1) \cap G(P_2) \cap G(P_3) = \emptyset$.

To prove our assumption, suppose to contrary that $b > 1$. Let $x_k \in G(P_1) \cap G(P_2)$, $x_m \in G(P_1) \cap G(P_3)$ and $x_t \in G(P_2) \cap G(3)$. Then $x_k^{b-1} x_m^{b-1} x_t \in I$, since $x_k^{b-1} x_m^{b-1} \in P_1^b$, $x_k^{b-1} x_t \in P_2^b$ and $x_m^{b-1} x_t \in P_3^b$. So $2b - 1 = \deg(x_k^{b-1} x_m^{b-1} x_t) \geq d = 2b$, a contradiction.

(iii) If $s = 1$, then $I = P_1^{b_1}$ is polymatroidal, and if $s > 1$ then from (i), we are done.

Based on Proposition 3.0.5, Corollary 3.0.6, Proposition 3.0.8 and Proposition 3.0.10, Bandari and Herzog gave the following conjecture.([1, Conjecture 2.9])

Conjecture 3.0.11. A monomial ideal I is polymatroidal if and only if $I(P)$ has a linear resolution for all monomial prime ideals P.

Chapter 4

Persistence and Stability Properties of Polymatroidal Ideals

4.1 Polymatroidal ideals and persistence property

Let R be a Noetherian ring and $I \subset R$ be an ideal. If $P \in \mathrm{Ass}(I^n)$ for all $n > 0$, then P is called a persistent prime ideal. In [3], Brodmann showed that $\mathrm{Ass}(I^n) = \mathrm{Ass}(I^{n_1})$ for all $n \gg n_1$. The smallest n_1 which satisfies this equality is called *the index of stability* and denoted by $\mathrm{astab}(I)$. Moreover, $\mathrm{Ass}(I^{n_1})$ is called *the stable set of associated prime ideals of* I and denoted by $\mathrm{Ass}^\infty(I)$. Also, it is known by Broadmann that depth of R/I^n is constant for $n \gg 0$. The smallest n_1 which satisfies $\operatorname{depth} R/I^n = \operatorname{depth} R/I^{n_1}$ is called the *index of depth stability* of I and is denoted by $\mathrm{dstab}(I)$. After discussing persistence property, we will show that this holds true for polymatroidal ideals.

Definition 4.1.1. ***(Persistence Property)*** An ideal I in a ring R is said to satisfy *persistence property* if $\mathrm{Ass}(I) \subset \mathrm{Ass}(I^2) \subset \ldots \subset \mathrm{Ass}(I^n) \subset \ldots$.

Proposition 4.1.2. [13, Proposition 2.1] Let I be a graded ideal of a Noetherian ring R, and $\mathfrak{m}$ be the graded maximal ideal of R. Then we have the following:

(a) If $\operatorname{depth} R/I^n \leq \operatorname{depth} R/I^{n_1}$ for all $n_1 > n$, then $\mathfrak{m}$ is a persistent prime ideal,

(b) If $\operatorname{depth} R/I^n \leq \operatorname{depth} R/I^{n_1}$ for all $n_1 > n$, then I satisfies the persistence property,

(c) $\max_{P\in \mathrm{Ass}^\infty(I)}\{\mathrm{dstab}(IR_p)\} \leq \mathrm{astab}(I)$. Moreover, if $\operatorname{depth} R/I^n \leq \operatorname{depth} R/I^{n_1}$ for all $n_1 > n$, then $\mathrm{astab}(I) \leq \max_{P\in V(I)}\{\mathrm{dstab}(IR_p)\}$.

Proof. (a) Let $\mathfrak{m} \in \mathrm{Ass}(I^n)$, then $\operatorname{depth} R/I^n = 0$. Since $\operatorname{depth} R/I^n \leq \operatorname{depth} R/I^{n_1}$ for all $n_1 > n$, $\operatorname{depth} R/I^t = 0$ for all $t \geq n$. Thus $\mathfrak{m} \in \mathrm{Ass}(I^t)$ for all $t \geq n$. This yields the desired conclusion.

(b) We have $P \in \mathrm{Ass}(I^n)$ if and only if $PRp \in \mathrm{Ass}_{R_p}(I^nR_p)$. By part(a), one obtains that $PR_p \in \mathrm{Ass}_{R_p}(I^tR_p)$ for all $t \geq n$. Hence, $P \in \mathrm{Ass}(I^t)$ for all $t \geq n$.

(c) Let $s = \mathrm{astab}(I)$. Then, by definition, if $P \in \mathrm{Ass}^\infty(I)$, one has $P \in \mathrm{Ass}(I^t)$ for all $t \geq s$. This shows that $\operatorname{depth} R_p/I^tR_p = 0$ for all $t \geq s$. Thus $\mathrm{dstab}(IR_p) \leq s$.

Now let $r = \max_{P\in V(I)}\{\mathrm{dstab}(IR_p)\}$ and assume that $s > r$. Then there exists $P \in \mathrm{Ass}^\infty(I)$ such that $P \in \mathrm{Ass}(I^s)$. Let $P \in \mathrm{Ass}(I^r)$. Then $\operatorname{depth} R_p/I^rR_p = 0$ for all $P \in \mathrm{Ass}^\infty(I)$. Hence, by our hypothesis we obtain $\mathrm{astab}(I) \leq r < s$, a contradiction.

Thus $\operatorname{depth} R_p/I^rR_p > \operatorname{depth} R_p/I^sR_p = 0$, which contradicts with the definition of r. □

Proposition 4.1.3. [13, Proposition 2.2] Let $I \subset S = K[x_1, \ldots, x_n]$ be a graded ideal which is generated in degree d. If there exists an integer n_1 such that I^n has a linear resolution for all $n \geq n_1$ then $\operatorname{depth} I^n \geq \operatorname{depth} I^{n+1}$ for all $n \geq n_1$.

Proof. Let $f \in I$ be a homogeneous polynomial of degree d. Then fI^n is generated in degree $(n+1)d$ and $fI^n \subset I^{n+1}$. The short exact sequence

$$0 \longrightarrow fI^n \longrightarrow I^{n+1} \longrightarrow I^{n+1}/fI^n \longrightarrow 0$$

induces the long exact sequence

$$\ldots \to \mathrm{Tor}_{i+1}(K, I^{n+1}/fI^n)_{i+1+(j-1)} \to\to \mathrm{Tor}_i(K, fI^n)_{i+j} \to \mathrm{Tor}_i(K, I^{n+1})_{i+j} \to \ldots,$$

where for a graded S-module, $\mathrm{Tor}_i(K, M)_j$ denotes the jth graded component of $\mathrm{Tor}_i(K, M)$.

Both fI^n and I^{n+1} have a $(n+1)d$-linear resolution. Hence

$$\mathrm{Tor}_i(K, fI^n)_{i+j} = \mathrm{Tor}_i(K, I^{n+1})_{i+j} = 0$$

for $j \neq (n+1)d$ and all i. Moreover, $\mathrm{Tor}_{i+1}(K, I^{n+1}/fI^n)_{i+1+(j-1)} = 0$ for $j = (n+1)d$, because the module I^{n+1}/fI^n is generated in degree $(n+1)d$. This shows that the natural maps $\mathrm{Tor}_i(K, fI^n) \longrightarrow \mathrm{Tor}_i(K, I^{n+1})$ are injective for all i. It follows that $\operatorname{proj\,dim} I^n = \operatorname{proj\,dim} fI^n \leq \operatorname{proj\,dim} I^{n+1}$. Consequently, $\operatorname{depth} S/I^{n+1} \leq \operatorname{depth} S/I^n$, by Auslander-Buchsbaum theorem. □

Let $I \subset S$ be a monomial ideal. If $G(I) \subset F = K[x_{k_1}, \ldots, x_{k_r}]$ we denote by an abuse of notation the ideal $G(I)F$ by I. Then by following this notation, we obtain $\mathrm{Ass}_S(I) = \mathrm{Ass}_F(I)$. Let $v = \prod_{j \in F} x_j$ be a square-free monomial in S. Then

$$(S/I)_v \cong S'[\{x_i^{\pm} : i \in F\}]/I_F S'[\{x_i^{\pm} : i \in F\}],$$

where $S' = K[\{x_j : j \notin F]$ and where $I_F \subset S'$. We obtain the ideal I_F from I by using the K-algebra homomorphism $S \to S'$ with $x_j \mapsto 1$ for all $j \in F$. Let $I \subset S$ any monomial ideal and $P = (x_{k_1}, \ldots, x_{k_s})$. Then $I(P) \subset S(P)$ where $S(P) = K[x_{k_1}, \ldots, x_{k_s}]$ and $I(P) = I_F$ with $F = [n] \setminus \{k_1, \ldots, k_s\}$.

We need the following lemma for the later proofs.

Lemma 4.1.4. [13, Lemma 2.3] Let $I \subset S$ be a monomial ideal. Then

(a) $P \in \mathrm{Ass}(I)$ if and only if $\operatorname{depth} S(P)/I(P) = 0$;

(b) $\mathrm{Ass}(I_F) = \{P \in \mathrm{Ass}(I) : x_i \notin P \text{ for all } i \in F\}$ for all subsets $F \subset [n]$.

Proof. (a) Let $I \subset S$ be a monomial ideal. To simplify our notation let $P = (x_1, \ldots, x_r)$ with $S(P) = K[P] = K[x_1, \ldots, x_r]$. We claim that $P \in \mathrm{Ass}(S/I)$ if and only if $P \in \mathrm{Ass}(S(P)/I(P))$. This will finish our proof since P is the graded maximal ideal in $S(P)$ and this implies that $\operatorname{depth} S(P)/I(P) = 0$.

Indeed, suppose that $P \in \mathrm{Ass}(S/I)$ for some k. Then there exist a monomial u such that $I : u = P$. We can write $u = u_1 u_2$ where $u_1 \in S(P)$ and $u_2 \in \{x_{r+1}, \ldots, x_n\}$.

For any monomial v in the variables $\{x_{r+1}, \ldots, x_n\}$, we claim that $I : uv = I : u$. Let $uv \in I$. If uv belong to I, then $v \in I : u = P$, which is a contradiction since $v \notin P$. For any $x_i \in P$, $(uv)x_i = (ux_i)v \in I$ since $ux_i \in I$. Hence, $P \subset I : uv$. Finally, take any monomial $w \in S$ such that $w \in I : uv$. If w is a monomial only in the variables $\{x_{r+1}, \ldots, x_n\}$, then $(uv)w = u(vw) \in I$ implies that $vw \in P$, which is again a contradiction since neither v nor w is divisible by any of $\{x_1, \ldots, x_r\}$. Thus, $I : uv = P$.

For any $x_j \in P$, we have $ux_j \in I(P)$. So $u_1 x_j \in I(P)$. Thus, the maximal ideal $(x_1, \ldots, x_r) \subseteq I(P) : u_1$, since $u_1 \notin I(P)$, we have $I(P) : u_1 = (x_1, \ldots, x_r)$, as desired. This yields that $\operatorname{depth} S(P)/I(P) = 0$ since P is the maximal ideal.

Conversely, let $\operatorname{depth} S(P)/I(P) = 0$. This implies that $P = (x_1, \ldots, x_r) \in \mathrm{Ass}(S(P)/I(P))$. Then there exists a monomial $u \in S(P)$ with $u \notin I(P)$ such that $I(P) : u = P$. We claim that $I : u(x_{r+1} \cdots x_n) = (x_1, \ldots, x_r)$. Indeed, let

$u(x_{r+1}\cdots x_n) \in I$. Then there exists $v \in I$ such that $u(x_{r+1}\cdots x_n) = vw$ for some $w \in S$. By rewriting $v = v_1v_2$ with $v_1 \in S(P)$ and v_2 is a monomial in the variables $\{x_{r+1},\ldots,x_n\}$, we have $v_1|u$. It follows that $u \in I(P)$, a contradiction.

(b) Let $S' = K[\{x_j : j \notin F\}]$ and set $M = S'[\{x_i^{\pm} : i \in F\}]$. Then $M = S_v$ where $v = \prod_{j\in F} x_j$. Hence by extending polynomial ring and applying localization one can see that

$$\mathrm{Ass}_M(I_F M) = \mathrm{Ass}(IM) = \{PM : P \in \mathrm{Ass}_S(I), x_i \notin P \ for\ all\ i \in F\}.$$

On the contrary,

$$Ass_M(I_F M) = \{PM : P \in \mathrm{Ass}_S(I_F)\}.$$

Since the map $\varphi : P \mapsto PM$ gives a bijection between the set $\mathrm{Ass}'_S(I_F)$ and $\{PM : P \in \mathrm{Ass}_S(I_F)\}$, we have the conclusion which is required. □

Proposition 4.1.5. [13, Proposition 2.4] Let $I \subset S$ be a monomial ideal.If depth $R/I^n \leq$ depth R/I^{n_1} for all $n_1 > n$ then

$$\max_{P\in \mathrm{Ass}^{\infty}(I)}\{\mathrm{dstab}(I(P))\} \leq \mathrm{astab}(I) \leq \max_{P\in V^*(I)}\{\mathrm{dstab}(I(P))\}$$

In particular, if $\mathrm{Ass}^{\infty}(I) = V^*(I)$, one has $\mathrm{astab}(I) = \max_{P\in V^*(I)}\{dstab(I(P))\}$.

Proposition 4.1.6. [13, Proposition 3.3] Let I be a polymatroidal ideal. Then I has the persistence property.

Proof. Let $n \geq 1$ be an integer. By Lemma 4.1.4, we have $P \in \mathrm{Ass}(I^n)$ if and only if depth $S(P)/I^n(P) = 0$. Note that $I^k(P) = I(P)^k$ for all $k \geq 1$. By Corollary 3.0.3, we know that $I(P)$ is again a polymatroidal ideal. By Theorem 2.2.2 and Corollary 2.2.4, we conclude that $I(P)^k$ have a linear resolution for all $k \geq 1$. Now by Prop 4.3.3, we obtain depth $S(P)/I^k(P) = 0$ for all $k \geq n$. But this implies that $P \in \mathrm{Ass}(I^k)$ for all $k \geq n$ as desired. □

Theorem 4.1.7. [13, Theorem 3.4] Let $I \subset S$ be a polymatroidal ideal. Then $\mathcal{R}(I)$ is a normal ring.

Proof. It is a well-known fact that $\mathcal{R}(I)$ is a normal ring if and only if I is a normal ideal. By definition, I is normal if I^k is integrally closed for all $k \geq 1$. Since I is polymatroidal ideal, by Theorem 2.2.2, it is sufficient to show that I are integrally closed. Because I is in particular a monomial ideal, I is integrally closed, if and only if for $v \in \mathrm{Mon}(S)$ and $t \in \mathbb{Z}^+$ such that $v^t \in I^t$ we have $v \in I$.

Let $v \in \mathrm{Mon}(S)$ which has degree k and $t \in \mathbb{Z}^+$ such that $v^t \in I^t$. Assume I is generated in degree d. If $v^t \in I^t$ then $kt \geq dt$, that is, $k \geq d$. Let I_s be the K-subspace of I spanned by $v \in \mathrm{Mon}(S)$ of degree s. Then

$$(I^t)_{kt} = S_{kt-dt}(I^t)_{dt} = (S_{k-d})^t(I_d)^t = (S_{k-d}I_d)^t.$$

Notice that $S_{k-d}I_d = J_k$ where $J = m^{k-d}I$ is a polymatroidal ideal generated in degree k. Hence,

$$v^t \in (I^t)_{kt} = (J_k)^t.$$

Thus, v belongs to the integral closure of the base ring $K[J]$. Since $K[J]$ is normal, it follows that $v \in K[J]$. This yields that $v \in J$. Consequently, $v \in I$, as desired.
□

Corollary 4.1.8. Let $I \subset S = K[x_1, \dots, x_n]$ be a polymatroidal ideal. Then

$$\lim_{m\to\infty} \operatorname{depth} S/I^m = n - \ell(I).$$

4.2 Strong persistence property

In [14], authors gave a stronger condition for an ideal to satisfy the persistence property.

Definition 4.2.1. Let P be a prime ideal which contains the ideal I. Then I is said to satisfy *the strong persistence property* with respect to P if for all n and all $g \in (I_p^n : \mathfrak{m}_p) \setminus I_p^n$ there exists $h \in I_p$ such that $gh \notin I_p^{n+1}$. If I satisfies the strong persistence property for all prime ideals P which contain I, then we say that I satisfies the strong persistence property.

Note that the strong persistence property implies persistence property but the converse does not hold.

Proposition 4.2.2. [14, Proposition 1.2] Let I be a graded ideal of S. If I^k have a linear resolution for all k, then I satisfies the strong persistence property with respect to $\mathfrak{m}$.

Proof. Let $f \in (I^t : \mathfrak{m}) \setminus I^t$. Since $(I^t : \mathfrak{m}) \setminus I^t$ is nonzero, we obtain that $\mathrm{depth}(S/I^t) = 0$. Let I be a graded ideal generated in degree d. Then, since I^k has a linear resolution for all $k \geq 1$, we obtain that the last module in the minimal graded free resolution of I^t is of the form $S(-(dt+n-1))^{\beta_{n,dt+n-1}}$. It implies that $\mathrm{Tor}_n^S(S/\mathfrak{m}, S/I^t) \cong$

$S(-(dt+n-1))^{\beta_{n,dt+n-1}}$ where $S(-(dt+n-1))^{\beta_{n,dt+n-1}}$ is a graded K-vector space. Accordingly, we have the following isomorphisms of graded K-vector spaces

$$\mathrm{Tor}_n^S(S/\mathfrak{m}, S/I^t) \cong ((I^t : \mathfrak{m})/I^t)(-n)$$

This means that $f = f_1 + f_2$ where $f_1 \in I^t : \mathfrak{m}$ is a non-zero homogeneous element of degree $dt-1$ and $f_2 \in I^t$. Let $g \in I$ which has degree d. Consequently, $gf \notin I^{t+1}$ since degree of $\deg gf = d(t+1)-1$, as desired. □

Theorem 4.2.3. [14, Theorem 1.3] Let $I \subset R$ be an ideal. Then I is said to satisfy the strong persistence property if and only if $I^{n+1} : I = I^n$ for all n.

Proof. Suppose that I satisfies the strong persistence property, but $I^{n+1} : I \neq I^n$ for some $n \geq 1$. Then the ideal $(I^{n+1} : I)/I^n$ is non-zero. Hence, there exists $P \in \mathrm{supp}[(I^{n+1} : I)/I^n]$ where P is a minimal prime ideal. Then the ideal $(I_p^{n+1} : I_p)/I_p^n$ is nonzero. Hence there exists $g \in (I_p^{n+1} : I_p)/I_p^n$ with $\mathfrak{m}_p g \in I_p^n$. By our assumption, there exists $h \in I_p$ such that $gh \notin I_p^{n+1}$. This is a contradiction because $g \in I_p^{n+1} : I_p$.

On the other hand, let $I^{n+1} : I = I^n$ for all $n \geq 1$. Then $I_p^{n+1} : I_p = I_p^n$ for all $n \geq 1$. Hence, changing R by R_p and I by I_p, it is sufficient to show that I satisfies the strong persistence property with respect to $\mathfrak{m}$. Assume that $\mathfrak{m} \in \mathrm{Ass}(I^n)$ for some n and $f \in (I^n : \mathfrak{m}) \setminus I^n$. Conversely, suppose that $fg \in I^{n+1}$ for all $g \in I$. Then $f \in I^n$ since $I^{n+1} : I = I^n$, a contradiction.

Theorem 4.2.4. [14, Theorem 1.4] Let R be a Noetherian ring and $I \subset R$ a proper ideal of R with $\mathrm{grade}(I) > 0$. Assume that $Rp/\mathfrak{m}_p$ is an infinite field for all prime ideals P which contain I, and that $\mathcal{R}(I)$ satisfies Serre's condition S_2. Then I satisfies the strong persistence property. In particular, $I^{t+1} : I = I^t$ for all t.

Proof. Let P be a monomial prime ideal which contains I and set $L = P \oplus \bigoplus_{t\geq 1} I^t$. Then $\mathcal{R}(I)/L \cong R/P$, and thus $L \in \mathrm{Spec}(\mathcal{R}(I))$. Besides that, $\mathcal{R}(I)_L \cong \mathcal{R}(I_p)$. Because $\mathrm{grade}(I) > 0$, we obtain $\dim \mathcal{R}(I)_L = \dim R_p + 1 \geq 2$, . The S_2 condition then provides that $\mathrm{depth}\, \mathcal{R}(I_p) \geq 2$. Hence if we localize R then one can suppose that R is local with maximal ideal $\mathfrak{m}$ with $\mathrm{depth}\, \mathcal{R}(I) \geq 2$, and we need to prove that I satisfies the strong persistence property with respect to $\mathfrak{m}$. (Localization preserves our assumptions with respect to R and I) We set $\mathcal{S}(I) = \bigoplus_{t\geq 0}(I^t : \mathfrak{m})/I^t$. Clearly, $\mathcal{S}(I)$ is a graded $\overline{\mathcal{R}}(I) = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$-module. We may establish $\mathcal{S}(I)$ with the graded Kozsul homology $\overline{\mathcal{R}}(I)$-module $H_{n-1}(x_1, \ldots, x_n; \mathcal{R}(I))$. The ideal

$\mathfrak{m}$ is generated by the minimal set $\{x_1, \dots, x_n\}$. The tth graded component of $H_{n-1}(x_1, \dots, x_n; \mathcal{R}(I))$ is given as

$$H_{n-1}(x_1, \dots, x_n; \mathcal{R}(I))_t = H_{n-1}(x_1, \dots, x_n; I^t) \cong H_{n-1}(x_1, \dots, x_n; R/I^t)$$

$$\cong (I^t : \mathfrak{m})/I^t = \mathcal{S}(I)_t.$$

It means that $\mathcal{S}(I)$ is a finitely generated $\overline{\mathcal{R}}(I)$-module.

We set $\mathcal{Q}(I) = \mathcal{R}(I) : \mathfrak{m} = \bigoplus_{t \geq 0} I^t : \mathfrak{m}$ and we obtain the exact sequence

$$0 \longrightarrow \mathcal{R}(I) \longrightarrow \mathcal{Q}(I) \longrightarrow \mathcal{S}(I) \longrightarrow 0$$

of $\mathcal{R}(I)$-modules. Because $\mathcal{S}(I)$ is finitely generated $\mathcal{R}(I)$-module, as a result of this exact sequence we obtain $\mathcal{Q}(I)$ is finitely generated $\mathcal{R}(I)$-module, as well.

Because $\operatorname{depth} R > 0$, there exists a non-zero divisor $g \in \mathfrak{m}$ of R. Clearly, f is also a non-zero divisor on $\mathcal{Q}(I)$, then $\operatorname{depth} \mathcal{Q}(I) > 0$. Thus because $\operatorname{depth} \mathcal{R}(I) \geq 2$, it follows that $\operatorname{depth} \mathcal{S}(I) > 0$. Since $\overline{\mathcal{R}}(I)$ is a standard graded K-algebra, where K is the residue class field of R, and that K is an infinite field, there exists a homogeneous element $h + \mathfrak{m}I$ of degree 1 in $\overline{\mathcal{R}}(I)$ with $h \in I$, which is regular on $\mathcal{S}(I)$. Thus the multiplication map $(I^t : \mathfrak{m})/I^t \to (I^{t+1} : \mathfrak{m})/I^{t+1}$ induced by h is an injective map for all $t \geq 0$. Consequently, if $g \in (I^t : \mathfrak{m})/I^t$ for some t, then $hg \notin g^{t+1}$, as desired. $\square$

Remark 4.2.5. [14, Remark 1.5] The hypothesis of Theorem 4.2.4 implies more than just the strong persistence. Clearly, theorem proves that there exists $f \in I/\mathfrak{m}I$ such that for all $t, s \geq 0$ and all $g \in (I^t : \mathfrak{m})/I^t$ one has $gf^s \notin I^{t+l}$. The hypothesis of Theorem 4.2.4 also shows that $\dim_K (I^t : \mathfrak{m})/I^t \leq \dim_K ((I^{t+1} : \mathfrak{m})/I^{t+1}$ for all t.

Corollary 4.2.6. [14, Corollary 1.6] Let R be a Noetherian ring and let I be a proper ideal of R with $\operatorname{grade}(I) > 0$. Assume that $\mathcal{R}(I)$ is normal or Cohen-Macaulay. Then I satisfies the strong persistence property. Additionally, $P \in \operatorname{Ass}^\infty(I)$ if and only if $\ell(I_p) = \dim R_p$.

Proof. By Theorem 4.2.4, one can observe I satisfies the strong persistence property. Let P be a monomial prime ideal which contains I. Changing I by I_p and R by R_p we may suppose that $P = \mathfrak{m}$ and need to prove that $\mathfrak{m} \in \operatorname{Ass}^\infty(I)$ if and only if $\ell(I) = \dim R$.

One can notice that $\mathfrak{m} \in \operatorname{Ass}^\infty(I)$ if and only if $\mathcal{R}(I) : \mathfrak{m} \neq \mathcal{R}(I)$. Also one has $\mathcal{R}(I) : \mathfrak{m} = \mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I)$. Since $\mathcal{R}(I)$ is Cohen-Macaulay, it follows that $\operatorname{grade} \mathfrak{m}\mathcal{R}(I) = \operatorname{height} \mathfrak{m}\mathcal{R}(I)$ and $\operatorname{height} \mathfrak{m}\mathcal{R}(I) = \dim \mathcal{R}(I) - \dim \overline{\mathcal{R}}(I) \geq \dim \mathcal{R}(I) - \dim gr_I(R) = 1$. On the contrary, the exact sequence

$$0 \longrightarrow \mathfrak{m}\mathcal{R}(I) \longrightarrow \mathcal{R}(I) \longrightarrow \overline{\mathcal{R}}(I) \longrightarrow 0$$

induces the exact sequence

$$0 \longrightarrow \mathcal{R}(I) \longrightarrow \mathcal{R}(I) : \mathfrak{m}\mathcal{R}(I) \longrightarrow \operatorname{Ext}^1(\overline{\mathcal{R}}(I), \mathcal{R}(I)) \longrightarrow 0.$$

This shows that $\mathcal{R}(I) : \mathfrak{m} \neq \mathcal{R}(I)$ if and only if $\operatorname{grade} \mathfrak{m}\mathcal{R}(I) = 1$. This is equivalent to $\dim \overline{\mathcal{R}}(I) = \dim \mathcal{R}(I) - 1$. Since $\dim R = \dim \mathcal{R}(I) - 1$, the assertion follows. $\square$

In case of monomial ideals, in [14], an equivalent definition of strong persistence in terms of monomial localization is introduced. We give this definition in form of the following Lemma.

Lemma 4.2.7. [14, Lemma 2.1] Let I be a monomial ideal. Then I satisfies the strong persistence property if and only if for all monomial prime ideals which contain I and t, and all $v \in (I(P)^t : \mathfrak{m}_p) \setminus I(P)^t$ there exists a $w \in I(P)$ such that $vw \notin I(P)^{t+1}$.

Proof. Let $I \subset S$ be a monomial ideal and let $P \in \operatorname{Ass}(I)$. By Lemma 4.1.4, we know that $\operatorname{depth} S(P)/I(P) = 0$. This occurs if and only if $\mathfrak{m}_p \in \operatorname{Ass}(I(P))$, where $\mathfrak{m}_p$ is the graded maximal ideal of $S(P)$. We obtain that $P \in \operatorname{Ass}^\infty(I)$ if and only if $\mathfrak{m}_p \in \operatorname{Ass}^\infty(I(P))$. $\square$

Based on the above definition, one see the following

Proposition 4.2.8. [14, Proposition 2.4] Let $I \subset S$ be a polymatroidal ideal. Then I satisfies the strong persistence property.

Proof. Let I be a polymatroidal ideal. By Corollary 3.0.3, we know that $I(P)$ is again polymatroidal for all monomial prime ideals which contain I. By Theorem 2.2.2 and Corollary 2.2.4, we conclude that $I(P)^k$ have a linear resolution for all $k \geq 1$. Then by Proposition 4.2.2 and Lemma 4.2.7, I satisfies the strong persistence property. $\square$

4.3 Stability indices of polymatroidal ideals

In this section, we discuss the index of stability of associated prime ideals and depth for polymatroidal ideals. To do this, we need some preparations.

Definition 4.3.1. Let $G(I) = \{v_1, \dots, v_r\}$. Then Γ is called the *linear relation graph* of I if the following are the vertex and the edge set of Γ :

$$E(\Gamma) = \{\{k, l\} : \text{ there exist } v_s, v_t \in G(I) \text{ such that } x_k v_s = x_l v_t\}$$

$$V(\Gamma) = \bigcup_{\{k,l\} \in E(\Gamma)} \{k, l\}.$$

Example 4.3.2. Let I_G be the edge ideal of the finite simple graph G on the vertex set $[n]$. Then Γ of I_G has

$$E(\Gamma) = \{\{k, l\} : k, l \in V(G) \text{ and } k, l \text{ have a common neighbor} \in G\}.$$

Specifically, let G be an odd cycle with $E(G) = \{\{k, k+1\}$ for $k = 1, \dots, n\}$. Then Γ is an odd cycle with $E(\Gamma) = \{\{k, k+2\}$ for $k = 1, \dots, n\}$.

Conversely, if G is an even cycle, then Γ has two connected components Γ_1 and Γ_2 where Γ_1 is a cycle with

$$E(\Gamma_1) = \{\{2k, 2k+2\} : k = 1, \dots, n/2\}$$

and Γ_2 is a cycle with

$$E(\Gamma_2) = \{\{2k-1, 2k+2\} : k = 1, \dots, n/2\}$$

Theorem 4.3.3. [14, Theorem 3.3] Let $I \subset S = K[x_1, \dots, x_n]$ be a monomial ideal generated in a single degree and Γ of I has p vertices and q connected components. Then

$$\operatorname{depth} S/I^k \leq n - k - 1 \text{ for } k = 1, \dots, p - q.$$

Proof. It is sufficient to prove that $H_t(x_1, \dots, x_n; I^k) \neq 0$ for $k = 1, \dots, p - q$. Let $F \subset \Gamma$ be a spanning forest of Γ. This means that F is a subgraph of Γ which is a forest. Also vertices of F are equal to vertices of Γ. F has $p - q$ (distinct) edges, namely,

$$\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_{p-q}, j_{p-q}\}.$$

For the purpose of an appropriate labeling of the edges suppose that for all t, j_t is a free vertex of the forest with edges $\{i_1, j_1\}, \{i_2, j_2\}, \dots, \{i_t, j_t\}$. Specifically, we have $j_t \notin \{i_1, \dots, i_t, j_1, \dots, j_{t-1}\}$.

By the definition of Γ, this means that for each edge $\{i_t, j_t\}$ belongs to a cycle $w_t = v_{r_t} e_{j_t} - v_{s_t} e_{i_t}$ in $K(x_1, \ldots, x_n; I)$ where v_{r_t} and v_{s_t} are suitable elements in $G(I)$. Then $w = w_1 \wedge w_2 \wedge \cdots \wedge w_k$ is a non-trivial cycle in $K_k(x_1, \ldots, x_n; I^k)$. Precisely, $w \neq 0$, since in w the basis element $e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_t}$ appears in the expansion of the wedge product only once (with coefficient $v_{r_1}, v_{r_2} \cdots v_{r_k}$). The cycle w can not be a boundary of $K_k(x_1, \ldots, x_n; I^k)$ since its coefficients all belong to $I^k/\mathfrak{m}I^k$, since I is generated in a single degree. Then we obtain $[w] \neq 0$ in $H_k(x_1, \ldots, x_n; I^k)$, as desired. $\square$

Theorem 4.3.4. [14, Theorem 4.1] Let $I \subset S = K[x_1, \ldots, x_n]$ be a polymatroidal ideal. Then

$$\operatorname{astab}(I), \operatorname{dstab}(I) < \ell(I)$$

In particular, $\operatorname{astab}(I), \operatorname{dstab}(I) < n$.

In transversal polymatroidal ideals and ideals of Veronese type, $\operatorname{astab}(I) = \operatorname{dstab}(I)$. One can conjecture that if I is polymatroidal ideal then their stability indexes are equal. For the proof, we need the following.

Lemma 4.3.5. [14, Lemma 4.2] Let I be a monomial ideal and Γ be the linear relation graph of I. Suppose that Γ has p vertices and q connected components. Then

$$\ell(I) \geq p - q + 1$$

and additionally, if I is a polymatroidal ideal then equality holds.

Proof. Let $I = (x^{u_1}, \ldots, x^{u_k})$ be a monomial ideal and

$\mathcal{B} = \{u_i : 1 \leq i \leq k, u_i \text{ are the exponent vectors of } G(I)\}$. Let $U = \begin{bmatrix} u_1 \\ \vdots \\ u_k \end{bmatrix}$ be the $k \times n$ matrix where $u_1, \ldots, u_k$ are the row vectors of U. Then $\ell(I) = \operatorname{rank} U$. Let $F \subset \mathbb{Q}^n$ a $\mathbb{Q}$-vector space and let $\operatorname{span} U = \{u_r - u_s : u_r, u_s \in \mathcal{B}, u_r - u_s = \pm e_{ij} \text{ for some } i < j\}$. Moreover, let $\Gamma_1, \ldots, \Gamma_q$ be the connected components of $\Gamma(I)$. Then $F = F_1 \oplus F_2 \oplus \ldots \oplus F_q$ where F_k is $\mathbb{Q}$-subspace of F and $\operatorname{span} U$ is a generating set for all F_k and $\{i, j\} \in E(\Gamma_k)$.

Claim: $\dim U_k = |V(\Gamma_k)| - 1$.

Proof of Claim: Let $i \in E(\Gamma_k)$. Since for each vector F_k, the sum of components is zero, $e_i \notin F_k$. Hence from the identity $F_k + \mathbb{Q}e_i = \bigoplus_{j \in V(\Gamma_k)} \mathbb{Q}e_j$, we obtain the desired formula for $\dim F_k$. Now we need to show this identity holds, let $j \in V(\Gamma_k)$.

Because Γ_k is connected, there exist vertices $i = i_0, i_1, \dots, i_t = j$ in Γ_k such that $\{i_q, i_{q+1}\} \in E(\Gamma_k)$ for all q. We will proceed by induction of the length t of this path connecting i and j that $e_j \in F_k + \mathbb{Q}e_i$. If $t = 1$, then $e_{ij} = e_i - e_j \in F_k$, and thus $e_j \in F_k + \mathbb{Q}e_i$. Now let $t > 1$. By what we obtained so far, we have that $e_1 \in F_k + \mathbb{Q}e_i$. Because i_1 is connected to j by a path of length $t - 1$, by our induction hypothesis, it follows that $e_j \in F_k + \mathbb{Q}e_i$.

By reviewing, we see that $\dim F = p - q$ where q is the connected components of $\Gamma(I)$. Since all vectors in F have component sum equal to zero, it follows that F is a proper subspace of the vector space which is spanned by the column vectors of U. It follows that $p - q = \dim F \leq \operatorname{rank}(U) - 1 = \ell(I) - 1$.

Now suppose that I is polymatroidal. Then $\mathcal{B}$ is the basis of a polymatroid. We claim that $\dim F = \operatorname{rank}(U) - 1$. Before proving our claim , we first observe that $u_r - u_s \in F$ for all $u_r, u_s \in \mathcal{B}$. We will show this by induction on the distance of u_r and u_s. By definition of distance

$$\operatorname{dist}(u_r, u_s) = 1/2(\sum_{i=1}^{n} |u_r(i) - u_s(i)|).$$

If $\operatorname{dist}(u_r, u_s) = 1$, then by definition of F, we have $u_r - u_s \in F$. Assume now that $\operatorname{dist}(u_r, u_s) > 1$. Then by the exchange property, there exist i and j with $u_r(i) > u_s(i)$ and $u_r(j) < u_s(j)$ such that $u := u_r - e_{ij} \in \mathcal{B}$. Because $\operatorname{dist}(u - u_s) < \operatorname{dist}(u_r, u_s)$ by our induction hypothesis, we obtain that $u - u_s \in F$. Thus, since $u - u_r \in F$, we deduce that $u_r - u_s \in F$ as well.

Now, by using that $u_r - u_s \in F$ for all $u_r, u_s \in \mathcal{B}$, we obtain that

$$F + \mathbb{Q}u_s = \mathbb{Q}\mathcal{B}.$$

Thus, since $u_s \notin F$(because its coefficient sum is not zero), we have $\operatorname{rank}(U) = \dim \mathbb{Q}\mathcal{B} = \dim(F + \mathbb{Q}u_s) = \dim F + 1$, as desired. □

Lemma 4.3.6. [14, Lemma 4.3] Let I be a polymatroidal ideal and $P \in V^*(I)$. Then $\ell(I(P)) \leq \ell(I)$.

Proof. By Corollary 3.0.3, $I(P)$ is a polymatroidal ideal. Let $\mathcal{P}$ (respectively $\mathcal{P}'$) be the polymatroid attached to I (respectively $I(P)$). Let U be the $m \times n$ with the row vectors which comes from the bases of $\mathcal{P}$. Because $I(P)$ is obtained from I by the substitution $x_j \mapsto 1$ for $j \notin P$, the matrix U' whose row vector comes from the bases of P' is taken from U by removing the columns of U which coincide to exponents $x_j \notin P$ and removing the rows of U which do not coincide to the minimal generators of $I(P)$. Then $\operatorname{rank}(U') < \operatorname{rank}(U)$, which completes the proof. □

Proof Of Theorem 4.3.4. Applying Theorem 4.3.3 with Lemma 4.3.5, one has

$$\operatorname{depth} S/I^{\ell(I)-1} = n - \ell(I).$$

By Corollary 4.1.8, $\operatorname{depth} S/I^t = n - \ell(I)$ for all $t \gg 0$ and by Proposition 4.1.6 I satisfies the persistence property. Consequently, $\operatorname{dstab}(I) < \ell(I)$. We need to show that $\operatorname{astab}(I) < \ell(I)$. One can notice that $P \in \operatorname{Ass}^\infty(I)$ if and only if $\ell(I(P)) = \dim S(P)$. Now by Theorem 4.3.3 together with Lemma 4.3.5 we have $\operatorname{depth} S(P)/I(P)^{\ell(I(P))-1} = 0$ which implies that $P \in \operatorname{Ass}(I^{\ell(I(P))-1})$. Thus, $\operatorname{astab}(I) \leq \max\{\ell(I(P)) - 1 : P \in \operatorname{Ass}^\infty(I)\}$. Hence, $\operatorname{astab}(I) < \ell(I)$ because $\ell(I(P)) < \ell(I)$ for all $P \in \operatorname{Ass}^\infty(I)$. $\square$

In the following with the light of Theorem 4.3.4, we will examine the analytic spread of polymatroidal ideals that are attached to polymatroids.

Example 4.3.7. (a) Graphic Matroids: Let G be a graph with $E(G) = \{e_1, \ldots, e_n\}$. Then the set of bases $\mathcal{B}$ of graphic matroid G is defined as:

$$\mathcal{B} = \{u_F : u_F = \sum_{k \in E(F)} e_k \text{ where F is a spanning forest of } G\}.$$

(b) Transversal polymatroids: Let $\mathcal{M}$ be a collection of subsets $M_1, \ldots, M_r$ of $[n]$ and let P be a transversal polymatroid with respect to $\mathcal{M}$. Then

$$\mathcal{B}(P) = \{e_{j_1} + \ldots + e_{j_d} : j_r \in M_r,\ 1 \leq r \leq d\}.$$

If in addition $j_r \neq j_s$ for all $1 \leq j, s \leq d$ with r, s, then $\mathcal{B}(P)$ is a generating set of a transversal matroid.

(c) Polymatroids of Veronese type: Given an integer d and a vector $a = (a_1, \ldots, a_n)$ with $a_i \geq 0$. Let P be a polymatroid of Veronese type (d, a). Then

$$\mathcal{B}(P) = \{b = (b_1, \ldots, b_n) \in \mathbb{Z}_+^n : |b| = d,\ 0 \leq b_i \leq a_i\}$$

Now, we will discuss the analytic spread of I which is attached to a graphic matroid. We begin with some information from graph theory:

(a) Let G be a finite simple graph with vertex set $V(G)$ and edge set $E(G)$. If $F \subset V(G)$ then G_F is the graph with $V(G_F) = F$. If $v \in V(G)$ then we set $G/v = G_{V(G)\setminus\{v\}}$. If number of connected components of G is less than the number of connected components of $G \setminus v$ then v is called a cutpoint. We call a graph G biconnected if it is connected and has no cutpoints.

(b) We call a biconnected subgraph of G biconnected component of G if it is maximal. Let G be a graph. Then G is the union of biconnected components. Any two distinct biconnected components intersect at most in a cutpoint.

(c) Suppose G is biconnected. Then any two distinct edges belong to a cycle.

Proposition 4.3.8. [14] [Proposition 4.4] Let G be a graph and I be a matroidal ideal attached to the graphic matroid of G. Let $G_1, \dots, G_q$ be the biconnected components of G which contain more than one edge. Then $\ell(I) = |E(\bigcup_{j=1}^{q} G_j)| - q + 1$.

Proof. Let Γ be the linear relation graph of I. By Lemma 4.3.5, it is sufficient to prove that $|V(\Gamma)| = |E(\bigcup_{j=1}^{q} G_j)|$ and p connected components. Then $I \subset K[x_j | e_j \in E(G)]$ is the matroidal ideal where $I = (v_T = \prod_{e_j \in E(T)} x_j, T$ is a spanning forest of $G)$. Let r be the number of biconnected components of G. To begin with notice that each spanning forest T of G can be written as $\bigcup_{j=1}^{r} F_j$ where $F_1, \dots, F_r$ are spanning trees of the distinct biconnected components of G. Since Γ is a linear relation graph, it follows that $\{i, j\}$ is an edge of Γ if there exists a spanning forest T with $e_j \in E(T)$ such that $(E(T) \setminus \{e_j\}) \cup \{e_i\}$ is also a spanning forest of G. If e_j and e_i are edges in different biconnected components of G, then any spanning forest T with $e_j \in T$, the subgraph with edge set $(E(T) \setminus \{e_j\}) \cup \{e_i\}$ contains a cycle. This cycle is absolutely contained in the biconnected component which contains the edge e_i. Thus if $G_1, \dots, G_m$, $m \geq q$, are all the biconnected components of G, then $\Gamma = \bigsqcup_{l=1}^{m} G_l$. Clearly, if G_l contains only one edge, then $\Gamma_l = \emptyset$. Thus $\Gamma = \bigsqcup_{l=1}^{p} \Gamma_l$.

Now we claim that each Γ_j for $j = 1, \dots, q$ is a complete graph, which will end our proof. Indeed, if e_j and e_i belong to same biconnected component G_l of G, there is a cycle C in G which passes through e_j and e_i. We can produce a spanning forest T of G such that $E(C) \setminus \{e_i\} \subset E(T)$. Then $(E(T)\ \{e_j\}) \cup \{e_i\}$ is again a spanning tree of G_L. Then we have $\{i, j\} \in E(\Gamma)$, as required. □

Example 4.3.9. In Figure 1, G is the graph with 3 biconnected components. Γ shows the linear relation graph of the matroidal ideal I attached to G. By Proposition 4.3.8, one can see that $\ell(I) = 6$.

Now, we will discuss the analytic spread of transversal polymatroids. Answer is not known for transversal matroids. If I be a transversal polymatroidal ideal, then $I = \prod_{k=1}^{r} P_k$. If we bring together the monomial prime ideals P_k which are generated by varibles we obtain that $I = vJ$ where $v \in \mathrm{Mon}(S)$ and J is product of remaining

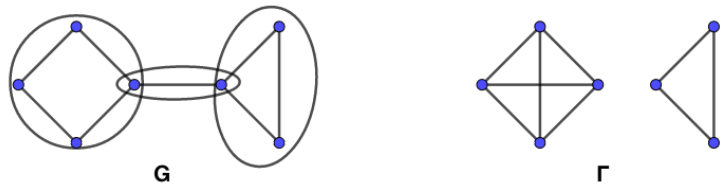

Figure 4.1: figure 1

P_k. Thus J is also a transversal polymatroidal ideal. Because $\ell(I) = \ell(J)$, we suppose that $v = 1$. Thus if in the beginning we assume that none of the P_k is a principal ideal then for $k = 1, \ldots, r$, we set

$$T_k = \{i | x_i \in P_k\}.$$

A simplicial complex $\triangle$ is the collection of subsets of vertex set $[n]$ such that if $F \in \triangle$ and $F_1 \subset F$ then $F_1 \in \triangle$. Elements of a simplicial complex is called a face. $\triangle$ is generated by the maximal faces. So if $F_1, \ldots, F_r$ are the maximal faces of $\triangle$ then $\triangle =< F_1, \ldots, Fr >$.

Proposition 4.3.10. [14, Proposition 4.6] Let Γ be the linear relation graph of the transversal polymatroidal ideal $I = \prod_{k=1}^{r} P_k$, and let $\triangle_1, \ldots, \triangle_q$ be the connected components of the simplicial complex $\triangle =< F_1, \ldots, F_r >$. Then Γ has q connected components $\Gamma_1, \ldots, \Gamma_q$, and for $m = 1, \ldots, q$, the connected component Γ_m is the complete graph on the vertex set of $\triangle_m$.

Proof. It is sufficient to prove that $\{k, l\} \in E(\Gamma)$ if and only if k and l belong to the vertex set of $\triangle_m$ for some m. Let k and l be two vertices of $\triangle_m$. Suppose that $\triangle_m =< F_1, \ldots, F_s >$ with $s \leq r$. For $\triangle_m$, we define the so-called intersection graph G with $V(G) = F_k$ and $E(G) = \{\{F_k, F_l\}, F_k \cap F_l \neq \emptyset\}$. The graph G is connected since $\triangle_m$ is connected. In particular, we may choose a spanning tree H of G . Then $|E(H)| = s - 1$. Hence, there exist $s - 1$ distinct pairs (F_x, F_y) such that $I_{xy} = F_x \cap F_y \neq \emptyset$ for $x, y \in \{1, \ldots, s\}$. We set $v \in \mathrm{Mon}(S)$ where $degv = r - 1$ whose support consists of $s - 1$ variables chosen with indices from each of I_{xy} and the remaining variables chosen with indices from each of F_l with $s + 1 \leq i \leq r$. Then $x_k v, x_l v \in G(I)$ and give $\{k, l\} \in E(\Gamma)$, as required. On the other hand, let $\{k, l\}$ be an edge of Γ. Let $V(\triangle) = \{1, \ldots, n\}$ be the vertex set of $\triangle$. Then $[n]$ is the disjoint union of $V(\triangle_m)$, $m = 1, \ldots, q$. Hence any monomial has a unique presentation $z = z_1 \cdots z_q$ with $\mathrm{supp}(z_m) \in V(\triangle_m)$. Furthermore, if $z \in G(I)$ then

$\deg z_m$ is the number of F_k's which belong to $\triangle_m$. Because $\{k,l\} \in E(\Gamma)$, there exist $v, w \in G(I)$ such that $x_k v = x_l w$. Assume that $k \in V(\triangle_m)$ and $l \in V(\triangle_i)$ with $m \neq l$. Then $x_k v_m = w_m$, which is impossible by degree reasons. $\square$

Next we search for the analytic spread of polymatroidal ideals of Veronese type. Let I be a polymatroidal ideal of Veronese type generated in degree d. Let $a = (a_1, \ldots, a_n)$ be an integer vector with $a_i > 0$. Then the bases of the polymatroid of Veronese type are the integer vectors $z = (z_1 \ldots, z_n)$ with $0 \leq z_i \leq a_i$. Without lose of generality, suppose that $\sum_{j=1}^{n} a_j > d$. Obviously, Γ has only one connected component therefore $V(\Gamma) = \mathrm{supp}(I)$. We define $\mathrm{supp}(I) = \{j \in \mathbb{Z}^+ : x_j | u, \text{ where } u \in G(I)\}$. Indeed, let $k, l \in \mathrm{supp}(I)$ with $k \neq l$ and suppose that $k = 1$ and $l = 2$. Let $v \in G(I)$ with $v = x_1^{a_1} x_2^{a_2-1} v$ where $w \in \mathrm{Mon}(S)$ and has degree $d - a_1 - a_2 + 1$ with $1, 2 \notin \mathrm{supp}(w)$, and whose exponent vector is componentvise bounded by $(a_3, \ldots, a_n)$. Such a monomial exist because $\sum_{i=3}^{n} a_i \geq d - a_1 - a_2 + 1$. Hence the monomial $x_2 v / x_1 \in G(I)$, it follows that $\{1, 2\} \in E(\Gamma)$. Specifically, $\ell(I) = \mathrm{supp}(I)$.

Bibliography

[1] S. Bandari, J. Herzog, *Monomial localizations of polymatroidal ideals*, European J. Comb. 34(4)(2013)752-763.

[2] M. Broadmann, *The asymptotic nature of the analytic spread*, Math. Proc. Cambridge Philos. Soc. 86(1979)35-39.

[3] M. Broadmann, *Asymptotic stability of* $\mathrm{Ass}(M/I^nM)$, Proc. Am. Math. Soc. 74(1979)16-18.

[4] W. Bruns, J. Herzog, *Cohen-Macaulay rings*, Cambridge University Press, London, Cambridge, New York, 1993.

[5] A. Conca, J. Herzog, *Castelnuovo-Mumford regularity of products of ideals*, Collect. Math. 54(2)(2003)137-152.

[6] D. Eisenbud, C. Huneke, *Cohen-Macaulay Rees algebras and their specialization*, J. Algebra 81(1983)202-224.

[7] C. Francisco, A. Van Tuyl, *Some families of componentwise linear monomial ideals*, Nagoya Math. J. 187(2007)115-156.

[8] C. Francisco, H. Tai Ha, A. Van Tuyl, *Coloring of hypergraphs, perfect graphs and associated primes of powers of monomial ideals*, J. Algebra 331(2011)224-242.

[9] J. Herzog, T. Hibi, *Monomial ideals*, GTM 260, Springer, 2010.

[10] J. Herzog, T. Hibi, *Discrete polymatroids*, J. Algebr. Comb. 16(2)(2002)239-268.

[11] J. Herzog, T. Hibi, *Cohen-Macaulay polymatroidal ideals*, European. J. Comb. 27(2002)239-268

[12] J. Herzog, T. Hibi, *The depth of powers of an ideal*, J. Algebra 291(2005)534-550.

[13] J. Herzog, A. Rauf, M. Vladoiu, *The stable set of associated prime ideals of a polymatroidal ideal*, J. Algebr. Comb. DOI 10.1007/s10801-012-0367-z(2012).

[14] J. Herzog, A. A. Qureshi, *Persistence and stability properties of powers of ideals*, J. Pure Appl. Algebra 219(3)(2015)530-542.

[15] J. Herzog, T. Hibi, X. Zheng, *Monomial ideals whose power have a linear resolution*, Math. Scand. 95(1)(2004)23-32.

[16] J. Herzog, Y. Takayama, *Resolutions by mapping cones*, Homology, Homotopy Appl. 4(2)(2002)277-294.

[17] M. Kokubo, T. Hibi, *Weakly polymatroidal ideals*, Algebra Colloq. 13(2006) 711-720.

[18] H. Matsumura, *Commutative ring theory*, Cambridge University Press,(1986).

[19] F. Mohammadi, S. Moradi, *Weakly polymatroidal ideals with applications to vertex cover ideals*, Osaka J. Math. 47(2010)627-636.

[20] M. Vladoiu, *Discrete polymatroids*, An.St. Univ Ovidius, Constanta 14(2)(2006)97-120.