ON THE END-OF-LIFE INVENTORY PROBLEM

by

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On the End-of-Life Inventory Problem

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ÜRÜN ÖMRÜ SONU ENVANTERİ PROBLEMİ ÜZERİNE

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Özet

Bu çalışmada, ürünleri servis yaşam döngüsünün son aşamasında olan bir üretici için, ömür sonu envanteri problemini ele aldık. Bu aşama parça üretimi sonlandırıldığında başlar ve son hizmet sözleşmesi sona erinceye kadar devam eder. Bu problemi çözmek için kullanılan taktiklerden en popüleri, son sipariş miktarı olarak adlandırılan, son aşamanın başında yeterli miktarda yedek parça üretmek ya da yerleştirmektir. Bunu takiben tamir-değişim politikası defolu ürünleri tamir ederek veya değiştirerek müşterilere hizmet vermektedir. Diğer taraftan günümüzde, ürünlerin fiyatları hızlıca düşerken tamir ve hizmet maliyetleri zaman içinde genelde sabit kalmaktadır. Böyle bir durumda, müşterilerin hizmet taleplerini karşılamak için alternatif bir politika uygulamak mali bakımdan daha etkili bir seçim olabilir. Bu politika müşterilere yeni nesil ürünlerde fiyat indirimi veya benzer tipte yeni bir ürün önerme şekilinde olabilir. Bu çerçevede amaç, en iyi son sipariş miktarı ve alternatif politikaya geçiş zamanı ikilisini beklenen toplam maliyeti minimuma indirecek şekilde bulmaktır. Bu tezde bu problemi farklı matematiksel teknikler gerektiren statik ve dinamik yaklaşımlar kullanarak incelemekteyiz.

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Abstract

We consider the so-called *End-of-Life inventory* problem for a manufacturer of spare parts in the final phase of the service life cycle. The final phase starts when the part production is terminated and continues until the last service contract expires. One of the most popular tactics to cope with this problem is to place a sufficient volume of spare parts at the beginning of the final phase which is called the *final order* quantity. Then the *repair-replacement* policy serves the costumers by repairing or replacing the defective items. On the other hand, nowadays, a considerable price erosion happens for the products while repair and service costs stay steady over time. If so, it is more cost effective to consider an *alternative policy* to meet the service demands after some time. This policy may offer the costumers a new product of similar type or a discount on a next generation product. In this setup, the purpose is to find an optimal pair of final order quantity and switching time to an alternative policy which minimizes the total expected discounted costs. We study this problem under the static and dynamic approaches which require different mathematical techniques.

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Chapter 1

Introduction

1.1 End-of-Life Inventory Management

Today our lives are surrounded by a huge variety of goods. Most of our individual and social needs are nourished by many different brands and commodities. The rapid improvement of technology has increased the competition for companies to produce new goods. In fact, companies need huge number of satisfied customers to quench their thirst to earn more profits. One way of obtaining satisfied customers for companies is to offer tempting service options. Hence, calling the recent decades the "golden age" of services is not far from the fact. In a recent benchmark study covering more than 120 companies from different sectors including aerospace and defense, automotive, and consumer goods, Deloitte Research Glueck et al. [2007] shows that business units related to service provide on average 75% higher profitability compared with the overall business profitability. Although the revenues of these units amount to only a quarter of total revenues, they yield almost 50% of the total profit.

From operational and managerial perspectives, providing an efficient service to customers is challenging. This is due to demand variability and service part inventories over service period. The main challenge is to fulfill service obligations and at the same time to avoid a huge number of obsolete service parts at the end of the service phase. Service parts may be associated with capital products that call for rapid service in the case of failure, in particular, telecommunications, healthcare, utilities, or consumable products which the customer uses recurrently, i.e. items which get used up or discard such as office supplies and electronic items. The original equipment manufacturers (OEMs) are dealing with the inventory management of service parts.

It would be worthwhile to introduce some of the terminology of service part inventory management. It is a primary concern to identify in which phase of the service life cycle the part is. Phases are identified according to the demand pattern the part is following. The life cycle of a spare part does not mimic the product life cycle necessarily. In general, there are three phases of the service life cycle of spare parts, namely, the initial, normal, and final phases. In the initial phase, the production of spare parts starts and the first demand for service arrives. However, demand in this phase is low, adaption for demand fluctuations is allowed by changing the production rates. During the normal phase the production of service parts is up and running which provides management with the ability to adjust production rate to meet demand. Final phase starts when the part production is terminated and ends when the last service (or warranty) contract expires. In general, the final phase is the longest period within the life cycle of a service part. For instance, in the electronic industry this phase may last four up to thirty years, while the production of electronic appliances is normally terminated after less than two years as pointed out in Teunter and Haneveld [2002]. On the other hand, increasing rate of innovation, especially in the electronics market, makes a very short life cycle of production. As a consequence, the final order of service parts is typically placed within a year after final production. The main challenge of this phase, for the manufacturer, is the acquisition of parts with a huge functional demand. Basically, the manufacturer tries to avoid a massive number of obsolete units at the end of this phase while its primary aim is to meet all customer's requests. Various strategies have been applied in recent decades to cope with the final phase inventory problem, for instance substituting another part for the obsolete one, obtaining the discontinued part from another manufacturer, redesigning the product, and purchasing a sufficient volume of the obsolete part to sustain production. To satisfy product replacement during the final phase, the manufacturer needs to procure a certain amount of service parts at once to cover the demand during the remaining period. This is called a *last-time* or a *life-time* buy; see Bradley and Guerrero [2008].

In the literature of service parts inventory management, the inventory control of service parts in the final phase of the service life cycle is known as the *end-of-life* (EOL) inventory problem, the *final buy problem* (FBP), or the *end of production* problem (EOP). Another important concept in the literature of service parts inventory management is the *repair-replacement* policy. Under this policy, the defective product is either repaired or replaced by a functioning part depending on its condition. This part may either be a new part or a repaired returned item. In a recent study, Pourakbar et al. [2012] propose a new methodology which introduces the possibility of switching to an alternative policy, such as offering a discount on a new model of the product, giving credit to customers, or swapping the defective product with the same or a similar one. They call this policy an *alternative policy*. This policy has recently received an extensive attention in the literature as a compelling policy to the meet demand.

The term of contract, or warranty, is also a crucial concept in this field. The warranty may be considered as either one-dimensional or multi-dimensional. Under a one-dimensional policy, the warranty will expire when a single attribute threshold, like age, is passed while in a multi-dimensional policy, the warranty will expire if the first criteria will be passed.

Research on the end-of-life inventory management is rich and extensive. Researchers have considered various strategies and assumptions to cope with this problem. In general, the research can be divided into three main categories: service-driven, cost-driven and forecasting based approaches. In a service-driven approach, a service level is optimized regardless of the cost incurred by the system. A cost-driven approach gives a monetary value to different operations related to service and then tries to minimize the total cost. A forecasting based view, focuses only on mimicking the demand behavior during the final phase to meet the demand. The cost-driven approach is the most relevant one to our study and as a result, we will go into the detail of this policy in the next chapter.

1.2 Outline

In this study, we consider the end-of-life inventory problem for the manufacturer of service parts in its final phase of the service life cycle. Following Pourakbar et al. [2012], the manufacturer may switch to an alternative policy during the final phase which is a more cost effective policy. In this setup, the objective is to find an optimal pair of final order quantity and switching time to an alternative policy which minimizes the total expected discounted cost. In fact, the switching time is a stopping time based on the realization of the arrival process of defective items where the arrival process is given by a non-homogenous Poisson process. Mathematically, we formulate the problem much more generally by considering the class of all possible stopping times. This means that our decision time to switch to an alternative policy at a certain point, also depends on the realization of the demand process up to that time. As such the approach of Pourakbar et al. [2012] considering only deterministic switching time is a very special case of our model. Four optimization problems are introduced based on different strategies. In each problem, we analyze rigorously the properties of the objective function to propose an exact or ϵ-optimal algorithm to solve. Finally we give some numerical examples to understand how sensitive the polices are on different parameters.

In the next part of the study, we study this problem under a general continuous switching time structure. In fact, we consider the end-of-life inventory problem as an optimal stopping problem. This gives a solution which is optimal within the class of all static and dynamic policies. To approximate the optimal stopping problem, there are different techniques in the literature, among which we consider the standard tool of discrete time Markov dynamic programming. To apply this technique appropriately, we assume that the stopping times take values on some pre-determined discrete set. Indeed, we approximate the continuous stopping times set with a discrete one by introducing an ϵ-error level mesh. Then the Bellman optimality equations are constructed to find optimal final order quantity and stopping region. Finally numerical results are given to compare the performance of optimal dynamic policy with other policies.

Chapter 2

Literature Review

2.1 Introduction

“ Business absolutely devoted to service will have only one worry about profits. They will be embarrassingly large. Henry Ford, founder of one of the world’s largest manufacturing companies, once said. Decades later, however, companies are still struggling to heed this advice. Manufacturers are looking for growth and profits in all corners of the globe, but they often neglect the very large opportunities much closer to home in their own service businesses ” Glueck et al. [2007]. However, a major task in service management is the timely and cost efficient provision of spare parts. The traditional strategy of spare parts acquisition is to place a large amount of final orders at the initial phase, causing major holding costs and a high level of obsolescence risk. There are different strategies from different perspectives to solve the problem. In fact, there is an extensive pool of researches related to those strategies. In general, research on the end-of-life inventory problem can be divided into three groups: service-driven, cost-driven and forecasting based approaches. More recent papers take into account other sources of meeting the demand and also there are other researches which consider the different types of warranty as their assumptions. In this chapter, the pioneer and most recent papers of the different approaches will be discussed in detail, and a short review of other papers on the service inventory management literature also is given. In Table 1, we classify the papers by their research focus.

2.2 Service-Driven Approach

Many researches on the service inventory management literature belong to the service-driven approach. Basically, in the service-driven approach, the purpose of research is to optimize some service measures such as the proportion of customers receiving spare parts and the filling rate -probability of running out of the stock- to meet the demand in the final phase. In other words, in this approach, a service level is optimized regardless of the cost incurred by the system. The leading papers on this approach (Fortuin [1980], Fortuin [1981]) describe a service level approach and address non-repairable items or consumable spare parts. The latter is refereed those parts which leave the system permanently after satisfying demand. He drives a number of curves by which the optimal final order quantity for a given service level can be obtained. He considers an exponentially decreasing demand pattern and applies a normal approximation to derive expression for several service levels. In another study, Hill et al. [1999] address the problem of determining stock replacement policies to meet the demand for spare parts in the final phase of service life cycle. The authors solve this problem under assumptions that the number of items still in use is decreasing and the parts fail randomly according to a Poisson process with an underlying rate decreasing exponentially. They use the dynamic programming approach in continuous time to derive optimal policies which minimize the mean total discounted cost of set-up order, production, unsatisfied demand, and left spare parts over the final phase. In fact, they propose a newsvendor approach to determine the optimal replenishment size if there is only one option to place a final order.

Another remarkable paper in this approach is given by Van Kooten and Tan [2009]. They consider a final ordering situation for a single spare part that does not interact with other parts, specifically taking the effect of condemnation into account. They model the problem under a continuous-time Markov chain which the failures of a spare part occur due to a Poisson process and the repair lead times are distributed exponentially. A defective spare part is immediately attempted to be repaired upon its arrival to a repair shop. After designing the model as a transient Markov chain, they define the actual service level that the customers receive and also calculate the first and second moment of the time until absorption. Accordingly, a final order size is obtained that guarantees a certain service level during the final phase. They compare the final order quantity that are obtained by the Markovian model and the approximated model on the other hand, and the optimal one which is obtained through simulation. They observe that in most cases the Markovian results are close to the simulation ones. They apply this methodology for a manufacturer of complex technological machines in the Netherlands.

Inderfurth and Mukherjee [2008] develop another service-driven approach. They consider three options to satisfy demand in the final phase of life cycle or as they call *post product life cycle* period. They assume the option of setting up a single large order within the final lot of regular production, performing extra production runs until the end of service and using remanufacturing to gain spare parts from used products. Obtaining the optimal combination of these three options is the main challenge of this paper. To overcome this difficulty, they use the decision tree and stochastic dynamic programming methods simultaneously and propose a heuristic method. The decision tree approach is a suitable tool in the case of limited size, while their heuristic method reduces the problem's complexity to a simple two-parameter order-up-to policy.

There are other papers indirectly related to the service-driven approach. Indeed, they address production planning and control of remanufacturing products. They suggest the idea that returned items may be provided as spare parts for the original equipment manufacturers (OEMs). One of these papers which considers the remanufacturing or recycling returned items is Souza et al. [2002]. As reported in the paper, remanufacturing has been characterized as "... an industrial process in which worn-out products are restored to like-new condition. Through a series of industrial processes in a factory environment, a discarded product is completely disassembled. Useable parts are cleaned, refurbished, and put into inventory. Then the new product is reassembled from the old and, where necessary, new parts to produce a fully equivalent -and sometimes superior- in performance and expected lifetime to the original new product". Lund [1983] develops an analytical model to maximize profits and minimize average flow time and as-well-as a simulation method. In particular, his model is a decision support tool for a manager to make decision for mixed products.

2.3 Cost-Driven Approach

A cost-driven approach gives a monetary value to different service-related operations, and then adopts to a policy to minimize the total cost. In other words, all the costs associated with serving customer during the final phase of spare parts, holding inventory, scrapping spare parts, procurement cots, etc. are taken into account. The purpose is to find an optimal final order quantity which will minimize the total cost. Basically, a cost-driven approach decides on the quantity purchased by weighting the cost of ordering too many against the cost of buying too few, or in other words, a newsvendor type approach. Research on the cost-driven approach is much more extensive than the previous approach. There are other classifications inside of this category, like product's type, merely consumable and capital, or the sourcing options to satisfy the demand.

Over this category of research, the most pertinent to ours is Fortuin and Martin

[1999]. They discuss extensively how to control the service parts in the final phase of the product's life cycle. They start by emphasizing the service parts importance in the maintenance of industrial systems and consumer products. Continuing that the control of service parts is a complex matter due to the difficulty of forecasting the demand and logistic of service parts. They try to answer the main questions of managing service parts, such as which items are needed as service parts? which service parts have to be stocked? when do we need to (re)order? how much do we need to (re)order?. They give some suggestion, from a management point of view, to answer the questions, however, they admit that to answer those questions mathematically is not easy. In the paper, Teunter and Haneveld [1998] address the final order for the spare parts of an expansive machine. This machine contains a number of the so-called critical components. A failure of such a component causes the machine to break down. During the first part of life cycle of the machine, before the service contact expires, spare parts can be bought at any time, while after this time the supplier offers the customer a final chance to order spare components. That is the customer is allowed to place one final order. Their purpose in this paper is to minimize the total expected discounted costs including holding costs, procurement costs and out-of-order costs in case of a shortage. They assume that the customer is arriving according to a Poisson process. They show that a multi-component final order problem can be approximately decomposed into single component final order problem. After that, they derive a simple optimality condition for calculating optimal final order. To implement their approach, they use a real life example, a company which sells Gas Turbines, Reciprocation Compressors and Centrifugal Compressors and this company allows their customers to place one final order when it stops supplying spare parts.

Another cost-driven approach is developed by Teunter and Fortuin [1998]. In this paper, they introduce Philips company and its productions. The service period depends on the type of product involved in this company. Like other companies, the main problem of Philips to meet the demand at this period is the long duration of service period in comparison to the production period. In this paper, for the first time, they introduce the terminology of *end-of-life* period and define as the part of the service period after the product has been taken out of production. In Philips, *Logistics Operations Philips Consumer Service* (LOPCS) is in charge of supplying the spare parts. According to the company, the products are classified into two types: professional and non-professional which the first one is refereed to the most expensive equipments while the second one indicates that the equipment is sold, for the larger part, to private customers. They apply their method to find a near optimal final order in a cost minimization problem for the non-professional equipments in Philips. Actually, this paper is a case-study of their previous paper to understand how successful this method is in reality as well. To predict the demand distribution, in this paper, they develop a method based on the demand history of a component at the moment of final ordering. Then, using the expected cost calculations, the optimal shortage probability, i.e. the optimal probability the final order is smaller then the EOL demand, is been calculating and they give three examples to depict the accuracy of their method.

Next, Teunter and Fortuin [1999] consider two types of policies in the end-of-life period, the so-called *simple and remove* policies. A simple policy places a final order at the beginning of the end-of-life period and removes all remaining stock at the end. A remove policy adds the feature of a remove-down-to levels at the end of each month. These levels are used to reduce cost by removing stock before all the service contracts have expired. Their purpose is finding optimal and close to optimal final orders using a minimal cost approach. Given the production, holding, removing and shortage cost parameters, by applying a dynamic programming technique, they try to find those order quantities. They seek the final orders that minimize the accumulate cost functions over the entire EOL by considering a discounted cost criteria. In sensitivity analysis section, they show that in all cases the expected discounted cost associated with the optimal remove strategy is at most the cost associated with the optimal simple strategy. They contribute also that the simple policy suffices since it has low administration cost.

Besides these authors, Teunter and Haneveld [2002] consider an appliance manufacturer's problem of controlling the inventory of a service part in the final phase. They assume that if the part is not ordered at the beginning of the final phase, its price will be higher in the later stages. They propose an ordering policy consisting of an initial order-up to level at time zero like the beginning of the final phase, and a subsequent series of decreasing order-up-to levels for various intervals of the planning horizon. Also, Cattani and Souza [2003] develop another cost-driven approach that studies the effect of delaying a last-time buy. In fact, if the decision can be delayed, the expected overage and underage costs can be reduced. They build a model to understand the relation. Their results provide an insight on the effect of the final order quantity under various scenarios of demand. They observe that benefits of a delay to the manufacturer of last time buy are non-decreasing and concave in the delay time. A longer delay is always as good as or better than a shorter delay. They illustrate that it is necessary for the manufacturer to compensate the supplier for the losses incurred!

Bradley and Guerrero [2008] address the life-cycle mismatch problem when the life cycles of parts end before the life cycles of the products in which those parts are used. Their contribution in this paper is to extend the research on the life-time buys to the more complex and realistic circumstance with one product having multiple parts that become obsolete over its lifetime. They prove the existence and uniqueness of the optimal solution for this problem and drive an implicit analytical solution. They claim that there is not any closed-form expression for the optimal solution and instead they drive simple closed-form heuristic policies which one of them is lower bound and the other is upper bound on the optimal solution. They evaluate the accuracy of the heuristic performances by using simulation of demand behavior while observing which heuristics perform best in different scenarios. Indeed, they develop an accurate metastatic by observing the heuristics performances. Managerially, they find out that while lifetime buys can be an effective tactic for sequential obsolete parts when demand is stationary, their effectiveness is greatly diminished in some scenarios with non-stationary life cycle demand patterns.

In more recent works, Inderfurth and Kleber [2013] address multiple-options like extra production and re-manufacturing to meet the demand during the final phase, however, this problem yields a complicated stochastic dynamic decision. They suggest a heuristic procedure for parameter determination which accounts the main stochastic and dynamic interactions in decision making. They develop two steps to build their heuristic models: firstly they select a simple policy for period-period decision making and secondly they propose a heuristic procure to determine all policy parameters such that they are close-optimal. They apply their method on an automotive sector.

Leifker et al. [2014] investigate a contract extension on a regular strategy to meet the demand during the final phase, while the advantages and disadvantages of this decision need to be considered by both parties: the customer and supplier. As a result, the company should answer these two questions: under which conditions will both the manufacturer and customer prefer a contract extension? and what is the value of extending the contact? They assume that there is a probability that the customer may request a contract extension at the end of the contract period and this probability depends on the number of active products in operations at the end of the initial contract period. They also consider some other assumptions to construct their model, the manufacturer knows how many units of the products are in operation at any time, the length of any potential contract extension is known at the beginning and the period under examination by the end-of-life problem does not necessarily end with the manufacturers required to supply replacement parts. They examine two potential models for solving the problem: a dynamic programming model in which the possibility of salvage is taken into account and a simple two-stage stochastic model in which salvage is not allowed. They investigate their models from managerial insights and explain that the two-stage algorithm for the final order quantity is a useful tool for the managers in increasing their profits in the case where the possibility of contract extension occurs. An increase in the initial contract life results in an increase in the optimal order quantity as well as a corresponding decrease in the expected profit.

Another study is by Behfard et al. [2015], they develop a heuristic method to find the near-optimal last time buy quantity in presence of an imperfect repair option of the failed parts that can be returned from the filed. The supplier is for advanced capital goods, for instance, mainframe computer systems, aircraft, chemical plans and medical systems, they collaborate with two industrial partners (computer machinery and printing machines). To construct their model, they make trade-offs between one alternative supply option, namely repair of the filed parts that are returned from the filed. Since stochastic dynamic programming can not solve the large scale problems efficiently, they propose an efficient heuristic method assuming a base stock policy for the repair decisions. A numerical experiment to test the performance in terms the accuracy of the method is given and according to their results, alternative policy is worth considering even if it is expensive and also they indicate that reduction of the demand variability significantly reduces the last time buy quantity.

As mentioned before, there are some papers in this section which indirectly are related to the cost-driven approach. Most of them take other aspects of service management into account. A short summary of those are given to understand the importance of this field.

As Iskandar and Murthy [2003] define "a warranty is a contractual agreement between the manufacturer and customer, which requires the manufacturer to rectify all item failures either through repair or replacement should failure occur within the period specified in the warranty. Warranty serves a dual role it protects the buyer from being sold defective items and at the same time, restricts unreasonable claims on the manufacture by buyers. Over the last few years, manufacturers have used warranty as an effective advertising tool to promote their product." There are two types of warranty: one - and two- dimensional policies. A one-dimensional warranty policy is characterized by a one-dimensional time line called the warranty period, while two-dimensional warranty policy is indicated by a region in a two-dimensional plane with one dimension representing time and the other representing usage. The origin time corresponds to the time of a sale. A typical example is an automobile warrantied for three years or 100,000 kilometers for travel. In this paper, they consider the repair-replacement strategies for products sold with two-dimensional failure free warranty policies. Under this policy, the manufacturer may either repair the failed item or replace it with a new one. Their strategy divides the warranty region into two sub-regains and they study for every sub regions, different repair-replacment strategies by assuming a constant cost to repair failed item over the warranty region.

Another paper, Atasu and Cetinkaya [2006] focus on the reverse supply chain process used for product returns to recover value by re-processing them via re-manufacturing operations. They try to develop analytical models for the efficient use of the returns in making production, inventory, and re-manufacturing decisions during the active market, which refers to the sale's period of the product. This model considers a stylistic setting where a collector collects used product returns and ships them to the manufacturer who, in turn, recovers value by re-manufacturing and supplies products. They investigate the impact of timing and quality of the collector shipments of used product returns. They indicate that the fasted reverse supply chain many not always be the most efficient one.

Samatlı-Paç and Taner [2009] study and investigate different repair strategies for one- and two-denominational warranties with the objective of minimizing manufacture expected warranty cost. They propose static, improved and dynamic repair strategies. Actually, the quaso-renewal processes are used to model the product failures along with the associated repair actions. It is worth reminding that a two-dimensional warranty is a natural extension where the warranty period is characterized by a region defined simultaneously by time and usage. And the quasi-renewal process is characterized by a scaling parameter that alters the random variable corresponding to time until next failure after each renewal. They generalize the univariate quasi-renewal process to multivariate distributions to model two-dimensional warranties on a cost warranty function. They draw a conclusion that according to the computation results the dynamic policy generally outperforms both static and improved policies on highly reliable products whereas the improved policy is the best for products with the low reliability.

Kleber et al. [2012] propose a buy-back broken products strategy in order to improve control of both the demand for spare parts and supply of recoverable parts. This strategy specifically target dysfunctional products. They introduce a dynamic approach and consider a strategy which includes re-manufacturing complemented by a final order as a benchmark in their work. A numerical example is given to compare the potential gains of both strategies and it shows that both strategies can be beneficial for the OEM. This paper is the first attempt to investigate the value of buying back for the broken products for spare parts management.

2.4 Forecasting Based Approach

Forecasting-based approach focuses on forecasting the demand for a discontinued service part instead of dealing with the production or inventory cost. The major aim of this area is to provide the probabilistic tools to estimate the customer demand in the final phase of spare parts to meet the demand.

The approach was first developed by Moore Jr [1971] who tries to forecast the demand for the past-model replacement parts during the life-cycle of the products. For controlling manufacturing, inventory and obsolescence costs of past-model replacement parts (he calls the spare parts as past-model parts), all-time requirements forecasting is suggested. He introduces a new forecasting technique based on the principle of estimating sales requirements for all-time into the future along with a dynamic inventory model to meet the demand during the final phase. He also introduces the concept of an *all-time requirement*, i.e., the cumulative demand for a part from the present for all time into the future, in the development of generating long range forecasts of replacement part demand, then he transforms these foretastes into manufacturing schedules by using a dynamic inventory model. His idea to forecast the demand after the first peak demand is a transformation of sales data from an arithmetic scale to a logarithmic scale. He obtains the year of peak demand according to the actual annual sales data, then for parts which indicate sales decay, a plot of sale after the peak year against the index number of the year of those sales is obtained on a fully logarithmic scale. Consecutively, he determines the ellipse, parabola and starlight line which fit best the transformed sales data. Finally, he transforms the curve from the logarithmic to an arithmetic sale to provide yearly sales forecasts. For implementation, he applies his technique to an American auto manufacture, and shows that for 100 complete parts histories, the average error in cumulative demand estimates for the last four years of sales activity is less than 6 percent of the actual sales.

Following Moore Jr [1971], Ritchie and Wilcox [1977] try to forecast the spare demand, this time, by using the renewal theory. They claim that there must be a relationship between machine sales and demand for spare parts of the machine. They find out that if one part is less essential to the functioning of a machine the quicker demand for it declines. They use these arguments to count the number of effective machines in a month, i.e. those machines which give rise to a demand for spares. Then, they estimate the expected demand for spares in month n by two parameters, one is the rate at which a component fails per unit time per machine, and the other is the effective machine numbers per month. As they indicate in the conclusion section, the main drawback of this method is the computational burden to determine the model parameters for each item and the costs of providing and updating the records needed for this purpose. Solomon et al. [2000] address a methodology to forecast life cycles of an electronic part in which both years of obsolescence and life cycle stages are predictable. This methodology embeds both market and technology factors according to the dynamic assessment of the sales data. This paper also introduces a new concept as the *life cycle mismatch* problem for the first time, which is defined as a lack of synchronization between the part and product life cycles.

One of the more mathematical and technical papers of this category is Iida [2002]. In this paper, he considers a non-stationary periodic dynamic production-inventory model with an uncertain production capacity and uncertain demand. The production capacity varies stochastically according to the uncertainties in the production process, for instance, unexpected breakdowns and unplanned maintenance. To minimize the total discounted expected costs, he obtains the upper and lower bounds on optimal policies for infinite horizon problems which are derived by considering some finite horizon problems.

Hong et al. [2008] estimate and forecast the demand for a service part on the final phase by considering three factors: the failure rate of a part, the replacement of a failed part and the number of the units of a product population which are operational during the final phase. They estimate the demand by using these factors in a stochastic model. They give the prediction interval of the number of effective part demand, as well as the expected value of the part demand, and closed-from solutions in the case of a constant failure rate are provided. Their numerical results show the capabilities of their approach in comparison with the Ritchie-Wilcox model.

Most recent paper in this topic is Kim et al. [2017]. In this research, they try to forecast the spare part demand for the consumer goods using the so-called *installed base of the product*, that is, the number of products still in use. This type of information is retentively easily available in the case of maintenance contracts. They propose a set of installed base concepts with associated simple empirical forecasting mythologies that can be applied in practice.

Table 2.1: Overview of the existing literature on the end-of-life inventory management

	Approach			
Literature	cost-driven	service-driven	forecasting	other
Moore Jr [1971]			✓	
Ritchie and Wilcox [1977]			✓	
Fortuin [1980]		✓		
Fortuin [1981]		✓		
Lund [1983]		✓		
Teunter and Haneveld [1998]	✓			
Teunter and Fortuin [1998]	✓			
Teunter and Fortuin [1999]	✓			
Hill et al. [1999]		✓		
Fortuin and Martin [1999]	✓			
Solomon et al. [2000]			✓	
Souza et al. [2002]		✓		
Teunter and Haneveld [2002]	✓			
Iida [2002]			✓	
Cattani and Souza [2003]		✓		
Iskandar and Murthy [2003]				✓
Atasu and Cetinkaya [2006]				✓
Bradley and Guerrero [2008]	✓			
Hong et al. [2008]			✓	
Inderfurth and Mukherjee [2008]		✓		
Samatlı-Paç and Taner [2009]				✓
Van Kooten and Tan [2009]		✓		
Bradley and Guerrero [2009]	✓			
Van Kooten and Tan [2009]		✓		
Pourakbar et al. [2012]	✓			
Kleber et al. [2012]				✓
Leifker et al. [2012]	✓			
Inderfurth and Kleber [2013]	✓			
Leifker et al. [2014]				✓
Leifker et al. [2014]	✓			
Behfard et al. [2015]	✓			
Kim et al. [2017]			✓	

Chapter 3

The End-of-Life Inventory Problem

In this chapter we introduce the end-of-life inventory problem of a consumer electronics manufacturer as discussed in (Pourakbar et al. [2012]) . In the first section we give a description of the problem under the repair-replacement and alternative policies, and we introduce all costs which the manufacturer incurs over the final phase. In the second section a detailed derivation of the objective function and the corresponding optimization problem are provided. In the same section we also derive some additional useful properties for the analysis of this problem in the following chapters.

3.1 Introduction

In the end-of-life inventory problem, the defective products arrive according to a Poisson point process to a repair or replacement. To introduce this arrival process let $(\Omega, \mathcal{H}, \mathbb{P})$ be a probability space hosting the point process $(T_i, R_i)_{i\in\mathbb{N}}$. The random variable T_i, $i \in \mathbb{N}$ denotes the arrival time of the ith customer having a defective product and requesting repair. The counting process of defective products $N \equiv \{N(t) : t \geqslant 0\}$ defined by

$$N(t) := \sum\nolimits_{i=1}^{\infty} 1_{\{T_i \leqslant t\}}, \qquad t \geqslant 0, \tag{3.1}$$

is assumed to be a non-homogeneous Poisson process with a bounded Borel arrival intensity function λ. The random variables R_i, $i \in \mathbb{N}$, on the other hand, are independent and identically distributed Bernoulli random variables indicating the condition of the defective items. They are defined as

$$R_i = \begin{cases} 1 & \text{if i'th product can be repaired} \\ 0 & \text{if i'th product cannot be repaired} \end{cases} \tag{3.2}$$

with probability $q \in [0, 1]$ of being one. The thinned arrival processes

$$N_0(t) := \sum\nolimits_{i=1}^{\infty} (1 - R_i) 1_{\{T_i \leqslant t\}} \qquad \text{and} \qquad N_1(t) := \sum\nolimits_{i=1}^{\infty} R_i 1_{\{T_i \leqslant t\}}, \tag{3.3}$$

count the number of non-repairable and repairable products over time and it is well known Çınlar [2011] that the arrival processes N_0 and N_1 are independent non-homogeneous Poisson processes having intensity functions $\lambda_0 = (1 - q)\lambda$, $\lambda_1 = q\lambda$ respectively. In the sequel, we let $\mathbb{F} \equiv (\mathcal{F}_t)_{t\geqslant 0} \subseteq \mathcal{H}$ denote the filtration of the point process $(T_i, R_i)_{i\in\mathbb{N}}$; that is, the flow of information associated with both the arrival times of the products and their conditions. Next to the arrival process of defective products let T denote the time at which all service obligations with respect to this product of the supplier expires, and $x \in \mathbb{Z}_+ = \{0, 1, 2, \ldots\}$ be the initial inventory level available at the repair facility. It is assumed that keeping inventory is costly and so we incur inventory holding cost of $h > 0$ per spare part per unit of time. At the same time, we use in our model both the total cost and the net present value approach with discount rate $\delta > 0$. At time zero we need to order an inventory of spare parts and the cost of obtaining/producing the final batch of spare parts is given by a so-called *procurement cost* function $c : \mathbb{Z}_+ \mapsto R_+$. It is assumed that the function c is non-decreasing and satisfies $c(0) = 0$ and $\lim_{x\to\infty} c(x) = \infty$. Starting with x units, the supplier uses the *repair-replacement* policy until some (possibly random) time $\tau \leqslant T$. In the most general case τ is a stopping time with respect to the filtration $\mathbb{F}$. The set of all bounded stopping times with respect to this filtration is also denoted by $\mathbb{F}$.

Under the considered policy, if an arriving item is repairable, it is repaired at some repair cost c_{re} plus some service cost c_{se}. If the item is non-repairable and the inventory level of spare parts is non-zero, the item is replaced with a spare one from the inventory at service cost c_{se} only. However, if no spare part is available in inventory, then the defective item is replaced using an alternative policy and the cost of this alternative policy is given by the function c_a. An example of an alternative policy is the possibility to replace the defective item by a substitutable product. If this happens at time u the total cost is given by $c_a(u)$ plus some additional penalty cost $p(u)$. This penalty cost is added to penalize the nonavailability of a spare part during the repair replacement policy. In practice the penalty cost can for example represent the additional cost of an emergency order for this substitutable product to be transported from a different location. Due to the availability of an alternative policy with known cost function c_a during the operation interval $[0, T]$, it might become more cost effective to abandon the repair-replacement policy at a certain moment in time and start using from that moment on the alternative policy. Due to this, we incorporate the possibility that at a (possibly stochastic) time $\tau \leqslant T$, the supplier completely switches to the *alternative policy* and discards the existing inventory (if there is any) at a scrapping cost of c_{scr} per item. If a service request arrives at time u after the switching time τ, then the alternative policy is used at cost $c_a(u)$. Here, the functions c_a and p are both non-increasing. Namely, the alternative method (e.g., a substitutable product) as well as the penalty associated with unscheduled use of it become cheaper over time. Unless stated otherwise this condition will always hold in this thesis.

For ease of notation, we denote the total cost of applying the alternative policy before the switching time as

$$c_{ap}(u) := c_a(u) + p(u), \qquad u \geqslant 0. \tag{3.4}$$

In the above model, a final order order quantity of size x of spare parts and eventually switching at time $\tau \leqslant T$ to an alternative policy are decisions to be determined by the decision maker. Such a policy is called an (x, τ)-policy. Clearly, the first variable x is static, and its value is determined at time zero. The switching decision, on the other hand, can be dynamic, and in the general formulation of the problem τ is a stopping time of the filtration $\mathbb{F}$. The set of all these stopping times is also denoted by $\mathbb{F}$.

In our formulation, only the scrapping cost can be negative, all other cost terms are positive. If there is a net revenue associated with scrapped parts, we have $c_{scr} < 0$, otherwise it is non-negative. To avoid pathological cases where ordering is profitable because of scrapping, we assume that the function $x \mapsto c(x) - c_{scr}^{-}x$ is increasing where

$$c_{scr}^{-} := -\min\{c_{scr}, 0\} \tag{3.5}$$

and the limit value as $x \to \infty$ is infinity. Also for the scrapping action to be economically justifiable, we must have $h - \delta c_{scr} \geqslant 0$. If this condition fails to hold, instead of scrapping an item at some time τ, we can keep it indefinitely in the inventory at a total cost of $h \int_{\tau}^{\infty} e^{-\delta u} du = (h/\delta) e^{-\delta\tau}$ which would be less than $c_{scr} e^{-\delta\tau}$. Since the actions repair or replacement by an non-defective spare part from inventory within the repair-replacement policy cost at least c_{se} it is natural to assume that the penalty cost of using the alternative policy within the repair-replacement period has also at least at cost of c_{se}. This means by the positive cost of the alternative policy that $c_{ap}(u) > c_{se}$ for every $0 \leqslant u \leqslant T$. These three conditions on the cost functions and the parameters always hold in this study unless stated otherwise.

In the next table we list for completeness the main cost components of the end-of-life problem.

Table 3.1: Notation summary

Notation	Definition
$c(.)$	Procurement cost function
$c_a(.)$	Cost function of using the alternative policy
$p(.)$	Penalty cost function of using alternative policy before switching time τ
$c_{ap}(.)$	Cost function $c_a(.) + p(.)$ of using alternative policy before switching time τ
h	Holding cost per item per unit of time
c_{se}	Service cost per item
c_{re}	Repair cost per repairable item
c_{scr}	Scrapping cost per item
δ	Discount rate of net present value
q	Repair probability of a defective product

3.2 The Objective Function For (x, τ) Policies

In this section we derive the expected discounted cost of any (x, τ)-policy, $x \in \mathbb{Z}_+$, $\tau \in \mathbb{F}$ and introduce the optimization problem to be solved. To make it easier to the reader to distinguish the different cost components and the structure of a (x, τ)-policy we list in Figure 3.1 the timing of the different actions and their costs. In this figure the random variable σ_x denotes the (random) time of inventory depletion in case we order x spare parts at time 0. This is given by the stopping time

$$\sigma_x := \inf\{t > 0 : N_0(t) \geqslant x\}. \tag{3.6}$$

As shown in the figure, the total expected discounted cost is the sum of the procurement cost and expected discounted operation costs. The procurement cost of ordering a final batch of x spare parts at time 0 is given by $c(x)$. The expected discounted operation costs, of any (x, τ)-policy, on the other hand, consist of the following components:

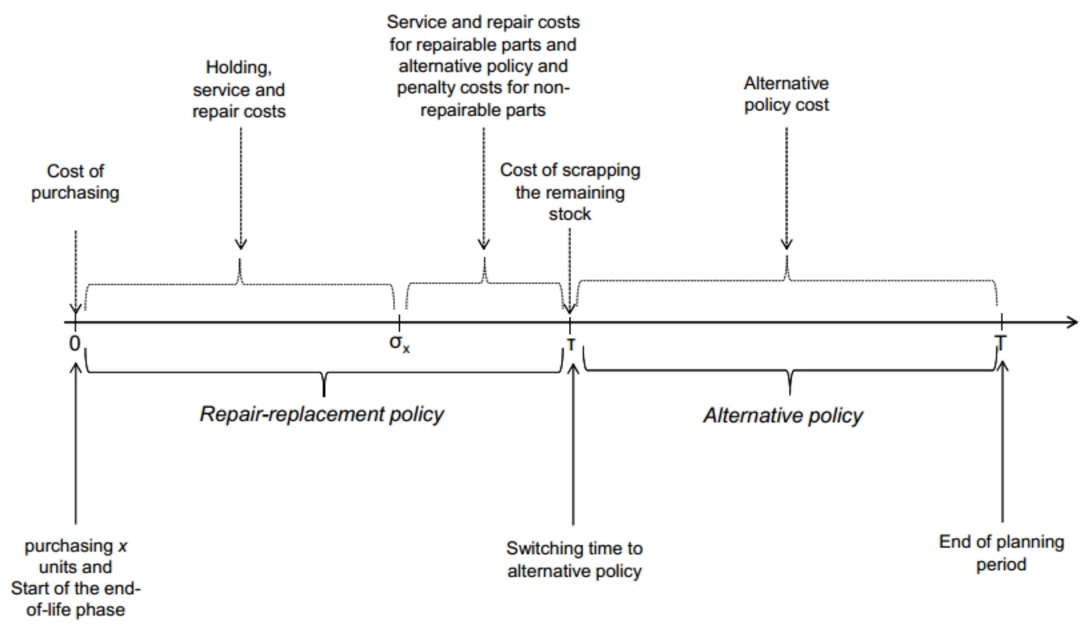


Figure 3.1: Decisions and costs over the time line.

- **Inventory holding costs:** As shown in Figure 3.1 we switch to the alternative policy at time $\tau \leqslant T$ and scrap at that time (possibly) leftover inventory of spare parts. Hence it is clear that the random discounted inventory holding costs are given by

$$h \int_0^{\tau} e^{-\delta u}(x - N_0(u))^+ du \tag{3.7}$$

with $(z)^+ := \max(z, 0)$. This shows that the expected discounted holding costs equal

$$h\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x - N_0(u))^+ du\right). \tag{3.8}$$

- **Service costs:** As shown in Figure 3.1 the service costs consist of service costs for both repairable and non-repairable products. For repairable products service costs only occur during the repair-replacement phase from time 0 to time τ in any (x, τ)-policy. For non-repairable products service costs additionally only occur if at the time of arrival of a defective product the stock of spare parts is positive. Hence for these non-repairable products these service costs only occur from time 0 to time $\tau \wedge \sigma_x$ with

$$\tau \wedge \sigma_x := \min\{\tau, \sigma_x\} \tag{3.9}$$

and σ_x given in relation (3.6). This shows that the random discounted service costs are given by

$$c_{se} \int_0^{\tau} e^{-\delta u} dN_1(u) + c_{se} \int_0^{\tau \wedge \sigma_x} e^{-\delta u} dN_0(u).$$

Since it is well know for any bounded Borel measurable function k that the stochastic processes $M_i = \{M_i(t) : t \geqslant 0\}, i = 0, 1$ given by

$$M_i(t) = \int_0^t k(u) dN_i(u) - \int_0^t k(u)\lambda_i(u) du$$

are $\mathbb{F}$-martingales (Çınlar [2011]) and $\tau \leqslant T$ is a bounded stopping time it follows by Doob's stopping theorem (Çınlar [2011]) that the expected discounted service costs equal

$$\begin{aligned} & c_{se}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda_1(u)du\right) + c_{se}\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda_0(u)du\right) \\ & = qc_{se}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)du\right) + (1-q)c_{se}\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right). \end{aligned} \tag{3.10}$$

- **Repair costs:** As shown in Figure 3.1 we only incur repair costs during the repair-replacement phase from time 0 to time τ. Hence the random discounted repair costs are given by
$$c_{re}\int_0^\tau e^{-\delta u}dN_1(u). \tag{3.11}$$
By a similar martingale argument as used for the service cost case and applying Doob's stopping theorem, the expected discounted repair costs equal
$$c_{re}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda_1(u)du\right) = qc_{re}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)du\right). \tag{3.12}$$

- **Alternative policy costs:** As shown in Figure 3.1 the random discounted costs of applying the alternative policy consist of the cost of applying the alternative policy before time τ due to the nonavailability of spare parts before the switching time τ and the cost of applying the alternative policy after the switching time. Hence the random discounted cost of using the alternative policy are given by
$$\int_{\tau\wedge\sigma_x}^\tau e^{-\delta u}c_{ap}(u)dN_0(u) + \int_\tau^T e^{-\delta u}c_a(u)dN(u). \tag{3.13}$$
Again by a similar martingale argument as used for the service costs, applying Doob's stopping theorem, and using $c_{ap}(u) = c_a(u) + p(u)$, the expected discounted costs of the alternative policy equal
$$(1-q)\mathbb{E}\left(\int_{\tau\wedge\sigma_x}^\tau e^{-\delta u}c_{ap}(u)\lambda(u)du\right) + \mathbb{E}\left(\int_\tau^T e^{-\delta u}c_a(u)\lambda(u)du\right)$$

$$= \begin{cases} (1-q)\mathbb{E}(\int_0^\tau e^{-\delta u} c_{ap}(u)\lambda(u)du) - (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u} c_{ap}(u)\lambda(u)du\right) \\ +\mathbb{E}\left(\int_\tau^T e^{-\delta u} c_a(u)\lambda(u)du\right) \end{cases}$$

$$= \begin{cases} \int_0^T e^{-\delta u} c_a(u)\lambda(u)du + \mathbb{E}\left(\int_0^\tau e^{-\delta u}[(1-q)p(u) - qc_a(u)]\lambda(u)du\right) \\ -(1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u} c_{ap}(u)\lambda(u)du\right). \end{cases} \tag{3.14}$$

- **Scrapping costs:** As shown in Figure 3.1 the random discounted scrapping costs at time τ are given by

$$c_{scr}e^{-\delta\tau}(x - N_0(\tau))^+.$$

This shows that the expected discounted scrapping costs in a (x, τ) policy equal

$$c_{scr}\mathbb{E}(e^{-\delta\tau}(x - N_0(\tau))^+). \tag{3.15}$$

Adding up the separate operation cost components in relations (3.8), (3.10), (3.12),(3.14) and (3.15) the expected discounted operation cost $C(x, \tau)$ of any (x, τ)-policy is given by

$$C(x,\tau) := \begin{cases} \underbrace{h\mathbb{E}\left(\int_0^\tau e^{-\delta u}(x-N_0(u))^+du\right)}_{\text{expected holding cost}} + \underbrace{qc_{re}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)du\right)}_{\text{expected repair cost}} \\ +\underbrace{qc_{se}\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)du\right) + (1-q)c_{se}\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right)}_{\text{expected service cost}} \\ +\underbrace{(1-q)\mathbb{E}\left(\int_{\tau\wedge\sigma_x}^\tau e^{-\delta u}c_{ap}(u)\lambda(u)du\right) + \mathbb{E}\left(\int_\tau^T e^{-\delta u}c_a(u)\lambda(u)du\right)}_{\text{expected alternative policy cost}} \\ +\underbrace{c_{scr}\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+)}_{\text{expected scrapping cost}} \end{cases} \tag{3.16}$$

This implies using relation (3.14) that

$$C(x,\tau) = \begin{cases} h\mathbb{E}(\int_0^\tau e^{-\delta u}(x-N_0(u))^+du + c_{scr}\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+) \\ +\mathbb{E}\left(\int_0^\tau e^{-\delta u}[q(c_{re}+c_{se}-c_a(u)) + (1-q)p(u)]\lambda(u)du\right) \\ +(1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se}-c_{ap}(u)]du\right) + \int_0^T e^{-\delta u}c_a(u)\lambda(u)du. \end{cases} \tag{3.17}$$

To rewrite the expression in relation (3.17) in a more suitable form we first observe by the chain rule for the stochastic process $t \mapsto e^{-\delta t}(x-N_0(t))$ that

$$\begin{aligned} e^{-\delta\tau}(x-N_0(\tau))^+ &= e^{-\delta(\tau\wedge\sigma_x)}(x-N_0(\tau\wedge\sigma_x)) \\ &= x-\delta\int_0^{\tau\wedge\sigma_x} e^{-\delta u}(x-N_0(u))du - \int_0^{\tau\wedge\sigma_x} e^{-\delta u}dN_0(u) \\ &= x-\delta\int_0^\tau e^{-\delta u}(x-N_0(u))^+du - \int_0^{\tau\wedge\sigma_x} e^{-\delta u}dN_0(u). \end{aligned} \tag{3.18}$$

This implies by Doob's stopping theorem

$$\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+) = x-\delta\mathbb{E}\left(\int_0^\tau e^{-\delta u}(x-N_0(u))^+du\right) - (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right). \tag{3.19}$$

Replacing now the expectation for the scrapping value in (3.16) with the expression in (3.19), we obtain after some simple re-arrangement of the terms the following more suitable alternative representation of the expected discounted operation costs

$$C(x,\tau) = \begin{cases} c_{scr}x + (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]du\right) \\ +\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du\right) \\ +(h - \delta c_{scr})\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x - N_0(u))^+)du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \end{cases} \tag{3.20}$$

Together with the procurement cost component, the optimization problem associated with the end-of-life inventory problem is therefore given by

$$\upsilon(P_{\mathbb{F}}) = \inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{F},0\leqslant\tau\leqslant T}\{c(x) + C(x,\tau)\}, \tag{$P_{\mathbb{F}}$}$$

and one needs to find a (x,τ)-policy, if it exists, attaining the infimum above. Note by relation (3.17) or (3.20) that

$$C(0,0) = \int_0^T e^{-\delta u}\lambda(u)c_a(u)du$$

and this is the cost of the policy of not ordering and at time 0 immediately applying the alternative policy. Since the cost $\int_0^T e^{-\delta u}c_a(u)\lambda(u)du$ in relation (3.20) of this policy is independent of x and τ one can also solve the optimization problem

$$\upsilon(\widetilde{P}_{\mathbb{F}}) = \inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{F},0\leqslant\tau\leqslant T}\{c(x) + \widetilde{C}(x,\tau)\} \tag{$\widetilde{P}_{\mathbb{F}}$}$$

with

$$\widetilde{C}(x,\tau) := C(x,\tau) - \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{3.21}$$

This means using relation (3.20) that the objective function is given by

$$\widetilde{C}(x,\tau) = \begin{cases} c_{scr}x + (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]du\right) \\ +\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du\right) \\ +(h - \delta c_{scr})\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x - N_0(u))^+ du\right). \end{cases} \tag{3.22}$$

Also it is obvious that

$$\upsilon(P_{\mathbb{F}}) = \upsilon(\widetilde{P}_{\mathbb{F}}) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{3.23}$$

Note by using $\widetilde{C}$ as our operation cost function we actually measure the difference in cost between any (x,τ)-policy and the policy of not ordering and using immediately at time 0 the alternative policy. In the next chapter we will also consider $(x, \tau\wedge\sigma_x)$ policies with

$$\tau\wedge\sigma_x := min\{\tau,\sigma_x\}. \tag{3.24}$$

It is well known that this is also a stopping time with respect to the filtration $\mathbb{F}$. This means we consider the subclass of policies where we switch to the alternative policy at the stopping time τ or at the time the inventory level hits 0 whichever occurs first. This class of policies is considered since under these policies we will never incur the (possibly high) penalty costs of using the alternative policy during the repair-replacement phase. Before writing down an expression for the objective function for this class of policies we observe that for any stopping time $\tau \in \mathbb{F}$

$$\int_0^{\tau} e^{-\delta u}(x - N_0(u))^+ du = \int_0^{\tau\wedge\sigma_x} e^{-\delta u}(x - N_0(u))^+ du.$$

This shows

$$\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x-N_0(u))^+du\right) = \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}(x-N_0(u))^+du\right). \tag{3.25}$$

Also it is easy to check that

$$e^{-\delta\tau}(x-N_0(\tau))^+ = e^{-\delta(\tau\wedge\sigma_x)}(x-N_0(\tau\wedge\sigma_x))^+$$

and so

$$\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+) = \mathbb{E}(e^{-\delta(\tau\wedge\sigma_x)}(x-N_0(\tau\wedge\sigma_x))^+). \tag{3.26}$$

Replacing now τ by $\tau\wedge\sigma_x$ in relation (3.17) and applying relations (3.25) and (3.26) we obtain for every $\tau \in \mathbb{F}$

$$C(x,\tau\wedge\sigma_x) = \begin{cases} h\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x-N_0(u))^+du\right) + c_{scr}\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+) \\ +\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du\right) \\ +\int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \end{cases} \tag{3.27}$$

An alternative expression for $C(x,\tau\wedge\sigma_x)$ applying relation (3.20) and replacing τ by $\tau\wedge\sigma_x$ is given by

$$\begin{aligned} C(x,\tau\wedge\sigma_x) &= \begin{cases} c_{scr}x + (1-q)\mathbb{E}(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se}-c_{scr}-c_{ap}(u)]du \\ +\mathbb{E}(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[q(c_{se}+c_{re}-c_a(u))+(1-q)p(u)]du \\ +(h-\delta c_{scr})\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x-N_0(u))^+)du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du \end{cases} \\ &= \begin{cases} c_{scr}x + (1-q)\mathbb{E}(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-(1-q)c_{scr}-c_a(u)]du \\ +(h-\delta c_{scr})\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}(x-N_0(u))^+du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du \end{cases} \end{aligned} \tag{3.28}$$

Hence, if we like to determine the optimal policy among all $(x, \tau \wedge \sigma_x)$-policies, we need to solve the optimization problem

$$\upsilon(P_{\mathbb{F}\wedge\sigma}) = \inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{F},0\leqslant\tau\leqslant T}\{c(x) + C(x, \tau \wedge \sigma_x)\}. \tag{$P_{\mathbb{F}\wedge\sigma}$}$$

As for the previous problem we rescale the above optimization problem using relation (3.21) and so we need to solve the equivalent optimization problem

$$\upsilon(\widetilde{P}_{\mathbb{F}\wedge\sigma}) = \inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{F},0\leqslant\tau\leqslant T}\{c(x) + \widetilde{C}(x, \tau \wedge \sigma_x)\}. \tag{$\widetilde{P}_{\mathbb{F}\wedge\sigma}$}$$

Again we obtain as before

$$\upsilon(P_{\mathbb{F}\wedge\sigma}) = \upsilon(\widetilde{P}_{\mathbb{F}\wedge\sigma}) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{3.29}$$

To compare the expressions for $C(x, \tau \wedge \sigma_x)$ and $C(x, \tau)$ it follows replacing τ by $\tau \wedge \sigma_x$ in relation (3.17) that

$$C(x,\tau) - C(x, \tau \wedge \sigma_x) = \mathbb{E}\int_{\tau\wedge\sigma_x}^{\tau} e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du \tag{3.30}$$

Hence in case

$$q(c_{se} + c_{re} - c_a(u)) + (1 - q)p(u) \geqslant 0$$

for every $0 \leqslant u \leqslant T$ there exists an optimal solution of optimization problem $(P_{\mathbb{F}})$ among the set of policies $\tau \wedge \sigma_x$, $\tau \in \mathbb{F}$ and $x \in \mathbb{Z}_+$. Observe the integral in relation (3.30) can have a positive or negative value. To explain the formula in relation (3.30) we observe the following. If we apply the repair-replacement policy until time τ instead of time $\tau \wedge \sigma_x$ we will still have the repair option within the time interval $[\tau \wedge \sigma_x, \tau]$ and we need to pay for any item arriving at u on average the cost $q(c_{re} + c_{se}) + (1 - q)c_{ap}(u)$. In case we already started with the alternative policy at time

$\tau \wedge \sigma_x$ we need to pay for any arriving item at time u the cost $c_a(u)$ and this explains the formula in relation (3.30).

Since it is also clear that in some pathological cases we will not order anything due to the low cost of the alternative policy at time 0 we mention the following result.

Lemma 1. *If $c_{ap}(0) \leqslant c_{se} + c_{scr}^-$ with c_{scr}^- listed in relation (3.5) then it is optimal not to order in optimization problem ($P_{\mathbb{F}}$). If this holds then the optimal switching time $\tau^*_{P_{\mathbb{F}}}$ belongs to $\mathbb{D}$, the set of deterministic stopping times in $[0, T]$, and is a solution of the optimization problem*

$$\inf_{\tau \in \mathbb{D}} \left\{ \int_0^\tau e^{-\delta u} \lambda(u)[q(c_{re} + c_{se} - c_a(u)) + (1-q)p(u)]du \right\}. \tag{3.31}$$

Proof. To verify the result it is sufficient to show for every $\tau \in \mathbb{F}$ that the function $x \mapsto c(x) + C(x, T)$ is non-decreasing. If $c_{scr} \geqslant 0$ then by our assumption $c_{ap}(0) \leqslant c_{se}$. Since the function c_{ap} is non-increasing we obtain $c_{ap}(u) \leqslant c_{se}$ for all $u \in [0, T]$. This implies that the function $x \mapsto \mathbb{E}(\int_0^{\tau \wedge \sigma_x} e^{-\delta u} \lambda(u)[c_{se} - c_{ap}(u)]du$ is non-decreasing in x and we obtain by relation (3.17) that for every $\tau \in \mathbb{F}$ the function $x \mapsto C(x, \tau)$ is non-decreasing. Since the procurement cost function c is by assumption also non-decreasing we obtain that the function $x \mapsto c(x) + C(x, \tau)$ is non-decreasing and we have verified the monotonicity for $c_{scr} \geqslant 0$. If $c_{scr} \leqslant 0$ we obtain by our assumption and c_{ap} non-increasing that $c_{ap}(u) \leqslant c_{se} - c_{scr}$ for every $0 \leqslant u \leqslant T$. This implies that the function $x \mapsto \mathbb{E}(\int_0^{\tau \wedge \sigma_x} e^{-\delta u} \lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]du$ is non-decreasing. Hence by relation (3.20) the function $x \mapsto C(x, \tau) - c_{scr}x$ is also non-decreasing. Since for $c_{scr} \leqslant 0$ the function $x \mapsto c(x) + c_{scr}x$ is non-decreasing it follows by adding that the function $x \mapsto c(x) + C(x, \tau)$ is non-decreasing. Hence we may conclude

$$v(P_{\mathbb{F}}) = \inf_{\tau \in \mathbb{F}} \left\{ \mathbb{E}\left(\int_0^\tau e^{-\delta u} \lambda(u)[q(c_{re} + c_{se} - c_a(u)) + (1-q)p(u)]du \right) \right\}.$$

Since the integrand in the above integral does not depend on the stopping time it follows easily that an optimal stopping time of the above optimization problem is given by an deterministic stopping time and the result is shown. □

Since the cost of using the alternative policy is always positive it is obvious that under the natural condition $p(u) \geqslant c_{se}$ the (sufficient) condition in Lemma 1 does not hold. A similar result can be derived for optimization problem $(P_{\mathbb{F}\wedge\sigma})$. Also in this optimization problem it is clear that in some pathological cases we will not order due to the low cost of the alternative policy at time 0. Using a similar proof as in Lemma 1 one can verify the following result.

Lemma 2. *If $c_a(0) \leqslant c_{se} + qc_{re} + (1-q)c^-_{scr}$ with c^-_{scr} listed in relation (3.5) then it is optimal not to order in optimization problem $(P_{\mathbb{F}\wedge\sigma})$ and to start immediately at time 0 with the alternative policy.*

Proof. It is sufficient to show that the function $x \mapsto c(x) + C(x, \tau \wedge \sigma_x)$ is non-decreasing. If $c_{scr} \geqslant 0$ then $c_a(0) \leqslant c_{se} + qc_{re}$. Since c_a is non-increasing it follows that $c_a(u) \leqslant c_{se} + qc_{re}$ for every $0 \leqslant u \leqslant T$. This shows that the function $x \mapsto \mathbb{E}\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - c_a(u)]du$ is non-decreasing and by relation (3.27) it follows for every $\tau \in \mathbb{F}$ that the function $x \mapsto C(x, \tau \wedge \sigma_x)$ is non-decreasing. Since by assumption c is non-decreasing, the function $x \mapsto c(x) + C(x, \tau \wedge \sigma_x)$ is non-decreasing and we have shown the result for $c_{scr} \geqslant 0$. If $c_{scr} \leqslant 0$ then $c_a(0) \leqslant c_{se} + qc_{re} - (1-q)c_{scr}$ and hence $c_a(u) \leqslant c_{se} + qc_{re} - (1-q)c_{scr}$ for every $0 \leqslant u \leqslant T$. This shows that the function $x \mapsto \mathbb{E}\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)]du$ is non-decreasing Applying now relation (3.28) yields $x \rightarrow C(x, \tau \wedge \sigma_x) - c_{scr}x$ is nondecreasing. Since $x \mapsto c(x) + c_{scr}x$ is nondecreasing, we conclude that $x \mapsto c(x) + C(x, \tau \wedge \sigma_x)$ is non-decreasing and we have shown the result for $c_{scr} \leqslant 0$. □

In the next section we investigate the global properties of the objective function.

3.3 On the Global Behavior of the Objective Function

In this section we investigate under which sufficient conditions on the cost functions and the parameters the objective functions $x \mapsto c(x)+C(x,\tau)$ and $x \mapsto c(x)+C(x,\tau \wedge \sigma_x)$ are discrete convex for every $\tau \in \mathbb{F}$. This property is useful in solving optimization problems $(P_{\mathbb{F}})$ or $(P_{\mathbb{F}\wedge\sigma})$ for some special subset of policies to be considered in the next chapter. Before mentioning the next result observe a function $f : \mathbb{Z}_+ \to \mathbb{R}$ is called discrete convex on $\mathbb{Z}_+$ if its first order difference

$$\Delta_x f(x) := f(x+1) - f(x), x \in \mathbb{Z}_+$$

is a non-decreasing function on $\mathbb{Z}_+$. The function f is called discrete concave if the function $-f$ is discrete convex.

Lemma 3. *If the Borel measurable function f is non-decreasing and non-positive on $[0,T)$, then for every $\tau \in \mathbb{F}$,$0 \leqslant \tau \leqslant T$ the function*

$$x \mapsto F(x) := \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(v)\lambda_0(v)dv\right) \tag{3.32}$$

is non-increasing and discrete convex on $\mathbb{Z}_+$. If the function f is non-increasing and non-negative on $[0,T]$, then this function is non-decreasing and discrete concave on $\mathbb{Z}_+$.

Proof. It is sufficient to give the proof of the first result only. The second claim follows replacing f by $-f$. Since f is non-positive it is obvious that the function F is non-increasing. To show that the function F is discrete convex, we note by Doob's stopping theorem

$$\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(v)\lambda_0(v)dv\right) = \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(v)dN_0(v)\right) = \mathbb{E}\left(\sum_{k=1}^{x} f(\sigma_k)1_{\{\sigma_k\leqslant\tau\}}\right).$$

Hence for every $x \in \mathbb{Z}_+$ it follows

$$\Delta_x F(x) := F(x+1) - F(x) = \mathbb{E}\big[f(\sigma_{x+1})1_{\{\sigma_{x+1}\leqslant\tau\}}\big]. \tag{3.33}$$

This shows using $\sigma_{x+1} \leqslant \sigma_{x+2}$ implying $1_{\{\sigma_{x+1}\leqslant\tau\}} \geqslant 1_{\{\sigma_{x+2}\leqslant\tau\}}$ and f non-decreasing and non-positive on [0,T] and $\tau \leqslant T$ that

$$f(\sigma_{x+1})1_{\{\sigma_{x+1}\leqslant\tau\}} \leqslant f(\sigma_{x+2})1_{\{\sigma_{x+2}\leqslant\tau\}}.$$

This shows applying relation (3.33) that for every $x \in \mathbb{Z}_+$

$$\Delta F(x) = \mathbb{E}(f(\sigma_{x+1})1_{\{1_{\{\sigma_{x+1}\leqslant\tau\}}}) \leqslant \mathbb{E}(f(\sigma_{x+2})1_{\{1_{\{\sigma_{k+2}\leqslant\tau\}}}) = \Delta F(x+1),$$

and we have verified the discrete convexity property. □

If the stopping time τ is deterministic then by the same proof for convexity it is easy to verify that we only need to assume that the function f is non-positive and non-decreasing on $[0,\tau]$. Applying the above lemma one can show under some general (sufficient) conditions on the cost function c_{ap} and the cost parameters c_{se} and c_{scr} that both functions $x \mapsto c(x) + \widetilde{C}(x,\tau)$ and $x \mapsto c(x) + C(x,\tau)$ are discrete convex on $\mathbb{Z}_+$.

Lemma 4. *If the procurement cost function c is discrete convex on $\mathbb{Z}_+$ and $c_{ap}(T) \geqslant c_{se} - c_{scr}$, then for every $\tau \in \mathbb{F}$, $0 \leqslant \tau \leqslant T$ the functions $x \mapsto c(x) + C(x,\tau)$and $x \mapsto c(x) + \widetilde{C}(x,\tau)$ are discrete convex on $\mathbb{Z}_+$.*

Proof. By relation (3.21) it is sufficient to show the result for the function with C.

Since c_{ap} is non-increasing and $c_{ap}(T) \geqslant c_{se} - c_{scr}$, it follows that $c_{ap}(u) \geqslant c_{se} - c_{scr}$ for every $u \leqslant T$. This implies that the function $u \mapsto e^{-\delta u}(c_{se} - c_{scr} - c_{ap}(u))$ is non-positive and non-decreasing. Hence, by Lemma 3 the function

$$x \mapsto \mathbb{E}\left(\int_0^{T \wedge \sigma_x} e^{-\delta u}\lambda_0(u)(c_{se} - c_{scr} - c_{ap}(u))du\right)$$

is discrete convex. Since the random function $x \mapsto ((x - N_0(u))^+)$ is also discrete convex and $h - \delta c_{scr} \geqslant 0$ it follows from relation (3.20) that the function $x \mapsto C(x, T)$ is discrete convex again. Finally, the discrete convexity of c completes the proof. $\square$

Since we always assume (unless stated otherwise) that $p(u) \geqslant c_{se}$ and c_{ap} is non-increasing, it follows for $c_{scr} \geqslant 0$ that the condition $c_{ap}(T) \geqslant c_{se} - c_{scr}$ is always satisfied. Note also for a deterministic stopping time $0 \leqslant \tau \leqslant T$ one only need to assume in the above lemma that

$$c_{ap}(\tau) \geqslant c_{se} - c_{scr}. \tag{3.34}$$

We will now investigate under which conditions on the cost function c_a and the parameters c_{se}, c_{re} and c_{scr}, the function $x \mapsto c(x) + C(x, \tau \wedge \sigma_x)$ is discrete convex.

Lemma 5. *If the procurement cost function c is discrete convex on $\mathbb{Z}_+$ and $c_a(T) \geqslant c_{se} + qc_{re} - (1-q)c_{scr}$, then for every $\tau \in \mathbb{F}$, $0 \leqslant \tau \leqslant T$, the functions $x \mapsto c(x) + C(x, \tau \wedge \sigma_x)$ and $x \mapsto c(x) + \widetilde{C}(x, \tau \wedge \sigma_x)$ are discrete convex on $\mathbb{Z}_+$.*

Proof. Since c_a is non-increasing and $c_a(T) \geqslant c_{se} + qc_{re} - (1-q)c_{scr}$, it follows that $c_a(u) \geqslant c_{se} + qc_{re} - (1-q)c_{scr}$ for every $0 \leqslant u \leqslant T$. This implies that the function $u \mapsto e^{-\delta u}(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u))$ is non-positive and non-decreasing and by Lemma 3 the function $x \mapsto \mathbb{E}\int_0^{\tau \wedge \sigma_x} e^{-\delta u}\lambda(u)[(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u))]du$ is discrete convex on $\mathbb{Z}_+$. Applying relation (3.28) and using $h - \delta c_{scr} \geqslant 0$ yields the desired result. $\square$

As for Lemma 4 the above result also hold for a deterministic stopping time $\tau \in \mathbb{D}$ if

$$c_a(\tau) \geqslant c_{se} + qc_{re} - (1-q)c_{scr}. \tag{3.35}$$

In the last section of this chapter we construct a lower bound on the optimal minimal cost. This lower bound will be useful in solving the End-of-Life problem by replacing the stopping times by stopping times only attaining values in a discrete finite subset of $[0, T]$ to be determined beforehand.

3.4 A Lower Bound on the Optimal Objective Value.

To solve the optimization problem $(P_{\mathbb{F}})$ by replacing the stopping times by stopping times having only a finite number of values and controlling the relative error by doing so, we need a positive lower bound on the optimal objective value of optimization problem $(P_{\mathbb{F}})$. To construct such a lower bound, we introduce the function $g : \mathbb{Z}_+ \times \mathbb{F} \mapsto \mathbb{R}$ given by

$$g(x,\tau) = \begin{cases} c(x) + c_{scr}x - c_{scr}^{+}(1-q)\mathbb{E}(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du) \\ \\ +\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - c_a(u)]du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du \end{cases} \tag{3.36}$$

with $c_{scr}^{+} = \max\{c_{scr}, 0\}$. In the next lemma we derive some useful properties of the function g and at the same time show that $g(x,\tau)$ is a lower bound on the cost of any (x,τ) policy $\tau \in \mathbb{F}$.

Lemma 6. *For every $\tau \in \mathbb{F}$, $\tau \leqslant T$ the function $x \mapsto g(x,\tau)$ listed in relation (3.36) is non-decreasing and* $\lim_{x\uparrow\infty} g(x,\tau) = \infty$. *Also for every $x \in \mathbb{Z}_+$ and $\tau \in \mathbb{F}$, $\tau \leqslant T$ it* *follows in case* $p(u) \geqslant c_{se}$ *for every* $0 \leqslant u \leqslant T$ *that*

$$c(x) + C(x,\tau) \geqslant g(x,\tau). \tag{3.37}$$

Proof. By the monotone convergence theorem (Çınlar [2011])

$$\lim_{x\uparrow\infty} \mathbb{E}\left(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right) = \int_0^T e^{-\delta u}\lambda(u)du.$$

This shows by the definition of g in relation (3.36) that for every $\tau \in \mathbb{F}$

$$\lim_{x\uparrow\infty} g(x,\tau) - c(x) - c_{scr}x < \infty.$$

Since by assumption $\lim_{x\uparrow\infty} c(x) - c_{scr}^{-}x = \infty$ we obtain $\lim_{x\uparrow\infty} c(x) + c_{scr}x = \infty$ and this implies

$$g(\infty,\tau) = \lim_{x\uparrow\infty} g(x,\tau) = \infty.$$

To show that the function $x \mapsto g(x,\tau)$ is non-decreasing for every $\tau \in \mathbb{F}$ we observe for $c_{scr} \geqslant 0$ that by relation (3.19)

$$x-(1-q)\mathbb{E}\left(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right) = e^{-\delta T}\mathbb{E}((x-N_0(T))^+)+\delta\int_0^T e^{-\delta u}\mathbb{E}((x-N_0(u))^+)du.$$

This shows for $c_{scr} \geqslant 0$ that the function $x \mapsto c_{scr}x - c_{scr}(1-q)\mathbb{E}\left(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right)$ is non-decreasing. Using that the procurement cost function c is non-decreasing we obtain by relation (3.36) the result for $c_{scr} \geqslant 0$. For $c_{scr} \leqslant 0$ the result is obvious since by assumption the function $x \mapsto c(x) + c_{scr}x$ is non-decreasing. To show for every $\tau \in \mathbb{F}$ that $c(x) + C(x,\tau) \geqslant g(x,\tau)$ we first observe using $c_{ap}(u) \geqslant c_{se}$ that the cost $(1-q)\mathbb{E}(\int_{\tau\wedge\sigma_x}^{\tau} e^{-\delta u}\lambda(u)c_{ap}(u)du)$ of applying the alternative policy before the switching time satisfies

$$(1-q)\mathbb{E}\left(\int_{\tau\wedge\sigma_x}^{\tau} e^{-\delta u}\lambda(u)c_{ap}(u)du\right) \geqslant (1-q)c_{se}\mathbb{E}\left(\int_{\tau\wedge\sigma_x}^{\tau} e^{-\delta u}\lambda(u)du\right)$$

This implies by relation (3.16) that

$$C(x,\tau) \geqslant \begin{cases} h\mathbb{E}\left(\int_0^\tau e^{-\delta u}(x-N_0(u))^+du\right) + c_{scr}\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+) \\ +(c_{se}+qc_{re})\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)du\right) + \mathbb{E}\left(\int_\tau^T e^{-\delta u}\lambda(u)c_a(u)du\right) \end{cases}$$

$$\geqslant \begin{cases} h\mathbb{E}\left(\int_0^\tau e^{-\delta u}(x-N_0(u))^+du\right) + c_{scr}\mathbb{E}(e^{-\delta\tau}(x-N_0(\tau))^+) \\ +\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \end{cases}$$

Using relation (3.19) and $h-\delta c_{scr} \geqslant 0$ we obtain

$$C(x,\tau) \geqslant \begin{cases} c_{scr}x + (h-\delta c_{scr})\mathbb{E}\left(\int_0^\tau e^{-\delta u}(x-N_0(u))^+du\right) \\ -c_{scr}(1-q)\mathbb{E}(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)du) \\ +\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du \end{cases}$$

$$\geqslant \begin{cases} c_{scr}x - c_{scr}^+(1-q)\mathbb{E}(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du) \\ +\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \end{cases}$$

This shows the desired result. □

To construct a computable positive lower bound on the optimal objective value $\upsilon(P_{\mathbb{F}})$ we introduce

$$\zeta = \inf\{0 \leqslant u \leqslant T : c_{se} + qc_{re} - c_a(u) \geqslant 0\} \tag{3.38}$$

with the convention $\inf\varnothing = T$. Since the function c_a is non-increasing the function

$u \mapsto c_{se} + qc_{re} - c_a(u)$ is non-decreasing and so in the general case the constant ζ is easy to determine numerically by bisection. Using Lemma 6 one can show the following result.

Lemma 7. *If $c_{ap}(u) \geqslant c_{se}$ for every $0 \leqslant u \leqslant T$ then the optimal objective value $\upsilon(P_{\mathbb{F}})$ of optimization problem ($P_{\mathbb{F}}$) satisfies*

$$\upsilon(P_{\mathbb{F}}) \geqslant (c_{se} + qc_{re}) \int_0^{\varsigma} e^{-\delta u} \lambda(u) du + \int_{\varsigma}^{T} e^{-\delta u} \lambda(u) c_a(u) du \tag{3.39}$$

with ς listed in relation (3.38).

Proof. By Lemma 6 it follows that

$$\begin{aligned}
\upsilon(P_{\mathbb{F}}) &= \inf_{x \in \mathbb{Z}_+, \tau \in \mathbb{F}, 0 \leqslant \tau \leqslant T} \{c(x) + C(x, \tau)\} \\
&\geqslant \inf_{x \in \mathbb{Z}_+, \tau \in \mathbb{F}, 0 \leqslant \tau \leqslant T} \{g(x, \tau)\} \\
&\geqslant \inf_{\tau \in \mathbb{F}, 0 \leqslant \tau \leqslant T} \{g(0, \tau)\} \\
&= \inf_{\tau \in \mathbb{F}, 0 \leqslant \tau \leqslant T} \left\{ \mathbb{E} \left(\int_0^{\tau} e^{-\delta u} \lambda(u) [c_{se} + qc_{re} - c_a(u) du) \right) \right\} + \int_0^{T} e^{-\delta u} \lambda(u) c_a(u) du.
\end{aligned} \tag{3.40}$$

Since the integrand in the above optimization problem does not depend on the stopping time τ it is easy to see that an optimal stopping time of this optimization problem is given by some $\tau \in \mathbb{D}$, $0 \leqslant \tau \leqslant T$ and so

$$\begin{aligned}
& \inf_{\tau \in \mathbb{F}, 0 \leqslant \tau \leqslant T} \left\{ \mathbb{E} \left(\int_0^{\tau} e^{-\delta u} \lambda(u) [c_{se} + qc_{re} - c_a(u)] du \right) \right\} \\
= \; & \inf_{\tau \in \mathbb{D}, 0 \leqslant \tau \leqslant T} \left\{ \int_0^{\tau} e^{-\delta u} \lambda(u) [c_{se} + qc_{re} - c_a(u)] du \right\}.
\end{aligned}$$

Using that the function c_a is non-decreasing it follows that the function $\tau \mapsto \int_0^{\tau} e^{-\delta u} \lambda(u) [c_{se} + qc_{re} - c_a(u)] du$ is convex and by standard first order arguments the solution of this problem is given by ς defined in relation (3.38). Hence we obtain

$$v(P_{\mathbb{F}}) \geqslant (c_{se} + qc_{re}) \int_0^{\varsigma} e^{-\delta u} \lambda(u) du + \int_{\varsigma}^{T} e^{-\delta u} \lambda(u) c_a(u) du$$

and we have shown the result. □

Since

$$(c_{se}+qc_{re}) \int_0^{\varsigma} e^{-\delta u} \lambda(u) du + \int_{\varsigma}^{T} e^{-\delta u} \lambda(u) c_a(u) du \geqslant \int_0^{T} e^{-\delta u} \lambda(u) \min\{c_{se}+qc_{re}, c_a(u)\} du$$

the result in Lemma 7 also implies the weaker lower bound

$$v(P_{\mathbb{F}}) \geqslant L := \int_0^{T} e^{-\delta u} \min\{c_{se} + qc_{re}, c_a(u)\} du > 0 \tag{3.41}$$

In the next chapters, we will consider in more detail different subclasses of (x, τ) policies. We will start in the following chapter with the subclass of static and related policies and in the last chapter we will consider the class of all stopping times.

Chapter 4

On Static and Related Policies

In this chapter we study the end-of-life inventory problem under a general framework where both the final order quantity and switching time are static and they are determined initially at the beginning of the final phase. To cover all possible policies, we consider four subclasses of (x, τ)-policies. These policies are differentiated from each other by their restrictions on selecting the switching time. In each problem, we study conditions under which the objective function is convex in the final order quantity. This enables us to determine for a given fixed switching time the optimal final order quantity using the first order conditions. If convexity does not hold, we provide an upper bound for the final order quantity to devise an enumeration based search algorithm. Having determined for each switching time how to compute the optimal order quantity we then discuss how a near optimal switching time can be obtained by selecting a cleverly chosen finite set of deterministic switching times and evaluating for each of these switching times the objective value and selecting the best one. Since this induces an error we also derive an upperbound on this error and this bound is used to predetermine beforehand the error in objective value of the selected near optimal switching time. Finally some numerical examples are given to study how sensitive the policies and expected costs are with respect to different values of parameters.

4.1 Introduction

In this section we first introduce the set of (x, τ)-policies with τ being a deterministic time. For notational convenience the set of deterministic switching times is denoted by $\mathbb{D}$. Any (x, τ)-policy with $\tau \in \mathbb{D}$ is called as a *static* policy. Static policies are simple to use in practice since at time zero it is known at which time we will discard possible inventory of spare parts and replace after that time the repair-replacement policy with the alternative policy. The optimal policy within this class of policies is called an *optimal static* policy. Note that an optimal static policy only depends on the probability law of the arrival process and it does not depend on the realizations of the arrival process as is the case for stopping times $\tau \in \mathbb{F}$. This means if we decide at time 0 to switch to the alternative policy at the fixed time τ we apply this rule irrespective of the realizations of the arrival process up to that time. The more general class of dynamic policies depend on the realization of the arrival process and these policies will be discussed in the next chapter. Restricting ourselves to a static (x, τ)-policy we have the following options with respect to the selection of the switching time τ.

I. The most primitive class of (x, τ) policies is that policy in which we never switch to the alternative policy; that is, we set $\tau = T$ and simply apply the repair-replacement policy to all the defective items during the whole final phase and select the minimal expected discounted cost policy among all (x, T) policies. In case a zero inventory level occurs before the end of the period (i.e. the event $\sigma_x < T$ happens), we incur a cost of $c_{ap}(u)$ serving a request of a non-repairable item arriving at time $\sigma_x \leqslant u \leqslant T$ via the alternative policy. In this formulation, the final order quantity x is the only decision variable, and the problem of finding the minimal expected discounted cost reduces to solving the optimization problem

$$\upsilon(P_T) := \inf_{x \in \mathbb{Z}_+} \{c(x) + C(x,\tau)\} \tag{P_T}$$

This one-dimensional problem is considered in Teunter and Fortuin [1999] in a discrete-time setting. To find the optimal order quantity, the authors use a dynamic programming approach to formulate the problem but propose a heuristic approach based on marginal analysis to obtain a "near" optimal order quantity. Using now the definiton of the function $\widetilde{C}$ in relation (3.21) for $\tau = T$ it follows that the set of optimal solution of optimization problem (P_T) is the same as the set of optimal solutions of the rescaled optimization problem

$$\upsilon(\widetilde{P}_T) = \inf_{x \in \mathbb{Z}_+} \{c(x) + \widetilde{C}(x,\tau)\}. \tag{$\widetilde{P}_T$}$$

Clearly by relation (3.21) we also obtain

$$\upsilon(P_T) = \upsilon(\widetilde{P}_T) + \int_0^T e^{-\delta u} \lambda(u) c_a(u) du. \tag{4.1}$$

In Section 4.3 we will discuss in detail the optimization problem $(\widetilde{P}_T)$ under the extended discounted cost structure of Pourakbar et al. [2012].

II. A natural extension of optimization problem (P_T) is to allow at any deterministic time between 0 and T to switch to the alternative policy. In other words, we select the switching time from the set $[0,T]$. This set of deterministic stopping times is denoted by $\mathbb{D}$. In this case we consider the subclass of all static (x,τ) policies, $x \in \mathbb{Z}_+$,$\tau \in \mathbb{D}$, $0 \leqslant \tau \leqslant T$. Among this class we now select a static (x,τ)-policy with minimal expected discounted cost. In this formulation both x

and τ are decision variables and we need to solve the optimization problem

$$\upsilon(P_{\mathbb{D}}) := \inf_{x \in \mathbb{Z}_+, 0 \leqslant \tau \leqslant T, \tau \in \mathbb{D}} \{c(x) + C(x, \tau)\}. \qquad (P_{\mathbb{D}})$$

This problem is discussed in Pourakbar et al. [2012] and a heuristic method is proposed without any rigorous analysis on how well the proposed policy performs compared to the optimal static policy. In Frenk et al. [2018] a rigorous analysis of this problem is given. Using again the definition of the function $\widetilde{C}(x, \tau)$ in relation (3.21) for any $0 \leqslant \tau \leqslant T$, it follows that the set of optimal solutions of optimization problem $(P_{\mathbb{D}})$ is the same as the set of optimal solutions of the rescaled optimization problem

$$\upsilon(\widetilde{P}_{\mathbb{D}}) = \inf_{x \in \mathbb{Z}_+, \tau \in \mathbb{D}, 0 \leqslant \tau \leqslant T} \{c(x) + \widetilde{C}(x, \tau)\}. \qquad (\widetilde{P}_{\mathbb{D}})$$

Clearly by relation (3.21) we also obtain

$$\upsilon(P_{\mathbb{D}}) = \upsilon(\widetilde{P}_{\mathbb{D}}) + \int_0^T e^{-\delta u} \lambda(u) c_a(u) du. \qquad (4.2)$$

In Section 4.4 we will discuss in detail the optimization problem $(\widetilde{P}_{\mathbb{D}})$ under the extended discounted cost structure of Pourakbar et al. [2012].

III. The next two classes of policies to be considered in this chapter are the so-called *pseudo-static* policies. We start with the simplest one and assume that we only switch to the alternative policy before time T if the inventory level drops to zero. This means we consider the class of policies $T \wedge \sigma_x$, $\tau \in \mathbb{D}$ with $x \wedge y := \min\{x, y\}$ introduced in relation (3.9). Using these type of policies we avoid the (possibly high) penalty costs of applying the alternative policy during the repair-replacement phase. The price we have to pay for this is that the switching time is not known in advance and depends on future realization of the arrival process. The final order quantity x is now the only decision variable, but this time the minimal cost is obtained by solving

$$\upsilon(P_{T\wedge\sigma}) = \inf_{x\in\mathbb{Z}_+}\{c(x) + C(x, T\wedge\sigma_x)\} \qquad (P_{T\wedge\sigma})$$

This one-dimensional optimization problem is considered in Pourakbar et al. [2012] as an approximation of the original optimization problem $(P_{\mathbb{D}})$. Pourakbar et al. [2012] provide a heuristic procedure to solve optimization problem $(P_{T\wedge\sigma})$ and the paper does not contain any argument indicating how close the objective value of the solution generated by this heuristic is to the objective value of the optimal static policy within the class of (x,τ)-policies, $\tau\in\mathbb{D}$. Using now the definition of the function $\widetilde{C}(x,\tau)$ in relation (3.21) replacing τ by $T\wedge\sigma_x$ we know that all optimal solutions of the optimization problem $(P_{T\wedge\sigma})$ are the same as the optimal solutions of the optimization problem

$$\upsilon(\widetilde{P}_{T\wedge\sigma}) := \inf_{x\in\mathbb{Z}_+}\{c(x) + \widetilde{C}(x, T\wedge\sigma_x)\}. \qquad (\widetilde{P}_{T\wedge\sigma})$$

Clearly by relation (3.21) we also obtain

$$\upsilon(P_{T\wedge\sigma}) = \upsilon(\widetilde{P}_{T\wedge\sigma}) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{4.3}$$

In Section 4.5 we will discuss in detail the optimization problem $(\widetilde{P}_{T\wedge\sigma})$ under the extended discounted cost structure of Pourakbar et al. [2012].

IV. A natural generalization of the above policy is given by the class $\tau\wedge\sigma_x$ with $0\leqslant\tau\leqslant T$, $\tau\in\mathbb{D}$. In this case we apply the alternative policy after the deterministic switching time τ or after we hit inventory level zero, which ever occurs first. To determine the optimal policy within this class of policies we need to solve the optimization problem

$$\upsilon(P_{\mathbb{D}\wedge\sigma}) := \inf_{x\in\mathbb{Z}_+, 0\leqslant\tau\leqslant T, \tau\in\mathbb{D}}\{c(x) + C(x, \tau\wedge\sigma_x)\}. \qquad (P_{\mathbb{D}\wedge\sigma})$$

This problem is discussed in Pourakbar et al. [2012] and a heuristic method is proposed without any rigorous analysis on how the proposed heuristic performs in comparison to the optimal static policy. Using the definition of the function $\widetilde{C}(x,\tau)$ in relation (3.21) replacing τ by $\tau\wedge\sigma_x$ we know that the set of optimal solutions of optimization problem $(P_{\mathbb{D}\wedge\sigma})$ is the same as the set of optimal solutions of the rescaled optimization problem

$$\upsilon(\widetilde{P}_{\mathbb{D}\wedge\sigma}) := \inf_{x\in\mathbb{Z}_+, 0\leqslant\tau\leqslant T, \tau\in\mathbb{D}}\{c(x) + \widetilde{C}(x, \tau\wedge\sigma_x)\}. \qquad (\widetilde{P}_{\mathbb{D}\wedge\sigma})$$

Clearly by relation (3.21) we also obtain

$$\upsilon(P_{\mathbb{D}\wedge\sigma}) = \upsilon(\widetilde{P}_{\mathbb{D}\wedge\sigma}) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \qquad (4.4)$$

In Section 4.6 we will discuss in detail the optimization problem $(\widetilde{P}_{\mathbb{D}\wedge\sigma})$ under the extended discounted cost structure of Pourakbar et al. [2012].

It is obvious from the relations between the different optimization problems that the optimal objective values satisfy

$$\upsilon(P_{\mathbb{D}}) \leqslant \upsilon(P_T) \qquad \text{and} \qquad \upsilon(P_{\mathbb{D}\wedge\sigma}) \leqslant \upsilon(P_{T\wedge\sigma}). \qquad (4.5)$$

Also the policies $(x,\tau) = (0,0)$ and $(x,\tau\wedge\sigma_x) = (0,\tau\wedge 0) = (0,0)$ for every $0\leqslant\tau\leqslant T$ are feasible policies for the optimization problems $(P_{\mathbb{D}})$ and $(P_{\mathbb{D}\wedge\sigma})$. These policies represent not ordering and at time 0 starting with the alternative policy, and by the interpretation of $\widetilde{C}$ this implies that $\upsilon(\widetilde{P}_{\mathbb{D}}) \leqslant 0$ and $\upsilon(\widetilde{P}_{\mathbb{D}\wedge\sigma}) \leqslant \upsilon(\widetilde{P}_{T\wedge\sigma}) \leqslant 0$.

4.2 The Objective Function for Static Policies.

To simplify the formula for the objective function for the class of static (x,τ)-policies, $x \in \mathbb{Z}_+, \tau \in \mathbb{D}$ we observe by relation (3.22) and Fubini theorem that

$$\widetilde{C}(x,\tau) = \begin{cases} c_{scr}x + (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]du\right) \\ \\ + \int_0^\tau e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du \\ \\ +(h - \delta c_{scr}) \int_0^\tau e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.6}$$

Hence for $x = 0$ it follows

$$\widetilde{C}(0,\tau) = \int_0^\tau e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du \tag{4.7}$$

and we obtain from relation (4.6) the alternative expression

$$\widetilde{C}(x,\tau) = \begin{cases} c_{scr}x + (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]du\right) \\ \\ +\widetilde{C}(0,\tau) + (h - \delta c_{scr}) \int_0^\tau e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.8}$$

To rewrite the integral $\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]du\right)$ we observe for any $\tau \in \mathbb{D}$ and Lebesgue integrable Borel measurable function f that by Fubini theorem

$$\begin{aligned} \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} f(u)du\right) &= \mathbb{E}\left(\int_0^\tau f(u)1_{\{\sigma_x > u\}}du\right) \\ &= \int_0^\tau f(u)\mathbb{E}(1_{\{\sigma_x > u\}})du \\ &= \int_0^\tau f(u)\mathbb{P}(N_0(u) < x)du. \end{aligned} \tag{4.9}$$

This shows using relation (4.6) that for any static (x, τ) policy

$$\widetilde{C}(x,\tau) = \begin{cases} c_{scr}x + (1-q)\int_0^\tau e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]\mathbb{P}(N_0(u) < x)du \\ \\ +\int_0^\tau e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du \\ \\ +(h - \delta c_{scr})\int_0^\tau e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du \end{cases} \tag{4.10}$$

or

$$\widetilde{C}(x,\tau) = \begin{cases} c_{scr}x + (1-q)\int_0^\tau e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]\mathbb{P}(N_0(u) < x)du \\ \\ +\widetilde{C}(0,\tau) + (h - \delta c_{scr})\int_0^\tau e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.11}$$

Since we are also interested in the global behavior of the function $x \to \widetilde{C}(x, \tau)$ for any $\tau \in \mathbb{D}$, we introduce the first order difference operator

$$\Delta_x\widetilde{C}(x,\tau) := \widetilde{C}(x+1,\tau) - \widetilde{C}(x,\tau), x \in \mathbb{Z}_+. \tag{4.12}$$

The next lemma is easy to show.

Lemma 8. *For every $\tau \in \mathbb{D}$ and $x \in \mathbb{Z}_+$*

$$\Delta_x\widetilde{C}(x,\tau) = \begin{cases} c_{scr} + (1-q)\int_0^\tau e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]\mathbb{P}(N_0(u) = x)du \\ \\ +(h - \delta c_{scr})\int_0^\tau \mathbb{P}(N_0(u) \leqslant x)du. \end{cases} \tag{4.13}$$

Proof. It follows that

$$\begin{aligned} \mathbb{E}((x+1-N_0(u))^+) - \mathbb{E}((x-N_0(u))^+) &= \mathbb{E}((x+1-N_0(u))^+ - \\ &\quad (x-N_0(u))^+ 1_{\{N_0(u)\leqslant x\}}) \\ &= \mathbb{P}(N_0(u) \leqslant x). \end{aligned} \tag{4.14}$$

Applying now relation (4.10) yields the desired result. □

To simplify the objective function for $(\tau \wedge \sigma_x)$ policies, $\tau \in \mathbb{D}$ we observe by relation (3.28) and the definition of for every stopping time $\tau \in \mathbb{D}$ that

$$\widetilde{C}(x, \tau \wedge \sigma_x) = \begin{cases} c_{scr}x + (1-q)\mathbb{E}(\int_0^{\tau \wedge \sigma_x} e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]du \\ +\mathbb{E}(\int_0^{\tau \wedge \sigma_x} e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du \\ +(h - \delta c_{scr}) \int_0^{\tau} e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du \end{cases}$$

$$= \begin{cases} c_{scr}x + (1-q)\mathbb{E}(\int_0^{\tau \wedge \sigma_x} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)]du \\ +(h - \delta c_{scr}) \int_0^{\tau} e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.15}$$

By relation (4.9) and (4.15) this yields

$$\widetilde{C}(x, \tau \wedge \sigma_x) = \begin{cases} c_{scr}x + (1-q) \int_0^{\tau} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)]\mathbb{P}(N_0(u) < x)du \\ +(h - \delta c_{scr}) \int_0^{\tau} e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.16}$$

We are also interested in the global behavior of the function $x \to \widetilde{C}(x, \tau \wedge \sigma_x)$ for any $\tau \in \mathbb{D}$. Again we introduce the first order difference operator

$$\Delta_x \widetilde{C}(x, \tau \wedge \sigma_x) := \widetilde{C}(x+1, \tau \wedge \sigma_{x+1}) - \widetilde{C}(x, \tau \wedge \sigma_x), x \in \mathbb{Z}_+. \tag{4.17}$$

One can now show the following result for $\Delta_x \widetilde{C}(x, \tau \wedge \sigma_x)$.

Lemma 9. *For every $\tau \in \mathbb{D}$ and $x \in \mathbb{Z}_+$ it follows*

$$\Delta_x \widetilde{C}(x, \tau \wedge \sigma_x) = \begin{cases} c_{scr} + (1-q) \int_0^{\tau} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)] \\ \mathbb{P}(N_0(u) = x)du + (h - \delta c_{scr}) \int_0^{\tau} \mathbb{P}(N_0(u) \leqslant x)du \end{cases} \tag{4.18}$$

Proof. Applying relation (4.16) and using relation (4.14) we obtain the desired result.

□

In the next subsections we will analyze for the considered classes of static and pseudo-static policies the corresponding optimization problems of selecting the optimal one within these classes.

4.3 Analysis of Optimization Problem (P_T)

In this section we propose an algorithm to select among the class of all (x, T)-policies the optimal one having minimal expected discounted cost and so we need to solve optimization problem (P_T) given by

$$\upsilon(P_T) := \inf_{x \in \mathbb{Z}_+}\{c(x) + C(x, \tau)\}. \tag{P_T}$$

As shown in section 4.1 the optimal solutions of optimization problem (P_T) are the same as the optimal solutions of

$$\upsilon(\widetilde{P}_T) := \inf_{x \in \mathbb{Z}_+}\{c(x) + \widetilde{C}(x, T)\}. \tag{$\widetilde{P}_T$}$$

with $\widetilde{C}(x, T)$ given by (see relation (4.11))

$$\widetilde{C}(x,T) = \begin{cases} c_{scr}x + (1-q)\int_0^T e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]\mathbb{P}(N_0(u) < x)du \\ \\ +\widetilde{C}(0,T) + (h - \delta c_{scr})\int_0^T e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.19}$$

Also by relation (4.1) we know

$$\upsilon(P_T) = \upsilon(\widetilde{P}_T) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{4.20}$$

Note that the value $\widetilde{C}(0,T)$ in relation (4.19) is simply a constant and hence we need to find that value of x which minimizes the remaining expressions. Due to the previous remarks we will analyze in this section optimization problem $(\widetilde{P}_T)$. Since immediately switching to the alternative policy can be optimal if the cost of using the alternative policy in comparison to the repair-replacement option is already low at time 0 we first show under which sufficient conditions on the parameters this can be concluded. The next result is a special case of Lemma 1.

Lemma 10. *If $c_{ap}(0) \leqslant c_{se} + c_{scr}^{-}$ with c_{scr}^{-} listed in relation (3.5) then it is optimal not to order in optimization problem (P_T) and the optimal objective value of the optimization problem (P_T) equals*

$$\upsilon(P_T) = \int_0^T e^{-\delta u}\lambda(u)[q(c_{se} + c_{rer}) + (1-q)c_{ap}(u)]du. \tag{4.21}$$

In the general case, the optimization problem $(\widetilde{P}_T)$ does not seem to exhibit a convenient discrete convexity structure (in x) for a fast identification of an optimal solution. Hence we need to do a complete enumeration over the decision variable x and so it is convenient to compute beforehand a computable upper bound on the optimal order quantity. Observe that such a upper bound should exists since in the case of revenue of leftover spare parts at the end of the horizon this revenue will be less than the procurement cost and the cost of holding these spare parts in inventory. Hence it is never profitable to order at time 0 a large number of spare parts knowing the probability law of the demand arrival process. For numerical implementations, we next discuss how we can construct a finite upper bound for this optimal order quantity. To that end, we introduce for $x \in \mathbb{Z}_+$ and $0 \leqslant \tau \leqslant T$ the functions

$$k(x,\tau) := c(x) + c_{scr}x + (h - \delta c_{scr})\int_0^{\tau} e^{-\delta u}(x - \Lambda_0(u))^+ du \tag{4.22}$$

$$g_1(x,\tau) := (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)(c_{ap}(u) + c_{scr} - c_{se})^+ du\right), \tag{4.23}$$

where $\Lambda_0(u) = \int_0^u \lambda_0(s)\, ds = \mathbb{E}(N_0(u))$, for $u \geqslant 0$. Clearly, for fixed $\tau \leqslant T$ the function $x \mapsto g_1(x, \tau)$ is non-decreasing, and due to the monotone convergence theorem we have

$$g_1(\infty, \tau) = (1-q) \int_0^\tau e^{-\delta u} \lambda(u)(c_{ap}(u) + c_{scr} - c_{se})^+ du < +\infty.$$

Lemma 11. *There exists an optimal solution of problem ($\tilde{P}_T$) and any optimal solution $x^*_{\tilde{P}_T}$ of optimization problem ($\tilde{P}_T$) satisfies $x^*_{\tilde{P}_T} \leqslant u_{\tilde{P}_T}$ with*

$$u_{\tilde{P}_T} := \min\{x \in \mathbb{Z}_+ : k(x, T) > g_1(\infty, T)\} \; < +\infty. \tag{4.24}$$

Proof. Since by our standard assumption the function $x \mapsto c(x) - c^-_{scr} x$ is non-decreasing with limit value ∞ at infinity it follows that the function $x \mapsto c(x) + c_{scr} x$ is also non-decreasing with limit value ∞ at infinity. By this observation and $h - \delta c_{scr} \geqslant 0$ it follows that the function $x \mapsto k(x, T)$ in relation (4.22) is non-decreasing and satisfies $k(0, T) = 0$ and $k(\infty, T) := \lim_{x \to \infty} k(x, T) = +\infty$. This shows that the value $u_{\tilde{P}_T}$ in relation (4.55) is finite and hence well-defined. By Jensen's inequality, we have $\mathbb{E}((x - N_0(u))^+ \geqslant (x - \Lambda_0(u))^+$ for every $u \geqslant 0$. Using this inequality in relation (4.19) yields

$$\begin{aligned} c(x) + \tilde{C}(x, T) \;&\geqslant\; \left\{ \begin{array}{l} c(x) + \tilde{C}(0, T) + c_{scr} x + (h - \delta c_{scr}) \int_0^T e^{-\delta u} (x - \Lambda_0(u))^+ du \\ \\ -(1-q) \int_0^T e^{-\delta u} \lambda(u)(c_{ap}(u) + c_{scr} - c_{se})^+ \mathbb{P}(N_0(u) \leqslant x - 1) du \end{array} \right. \\ &\geqslant\; \left\{ \begin{array}{l} c(x) + \tilde{C}(0, T) + c_{scr} x + (h - \delta c_{scr}) \int_0^T e^{-\delta u} (x - \Lambda_0(u))^+ du \\ \\ -(1-q) \int_0^T e^{-\delta u} \lambda(u)(c_{ap}(u) + c_{scr} - c_{se})^+ du \end{array} \right. \end{aligned} \tag{4.25}$$

and so the lower-bound in relation (4.25) is equal to $\tilde{C}(0, T) + k(x, T) - g_1(\infty, T)$.

This shows for every $x \geqslant u_{\tilde{P}_T}$ that

$$\begin{aligned} c(x) + \widetilde{C}(x,T) &\geqslant \widetilde{C}(0,T) + k(x,T) - g_1(\infty,T) \\ &> \widetilde{C}(0,T) \\ &= c(0) + \widetilde{C}(0,T). \end{aligned}$$

Since no ordering at time zero and using the repair-replacement up to time T is feasible in optimization problem $(\widetilde{P}_T)$ this proves the existence of an optimal solution and the constructed upperbound. □

By Lemma 29 it follows immediately that solving problem $(\widetilde{P}_T)$ is the same as solving

$$v(\widetilde{P}_T) = \min_{x \epsilon \mathbb{Z}_+, x \leqslant u_{P_T}} \left\{ c(x) + \widetilde{C}(x,T) \right\}. \tag{4.26}$$

By relation 4.26 we may now solve optimization problem $(\widetilde{P}_T)$ by complete enumeration over the finite feasible set. However, the constructed upperbound $u_{\tilde{P}_T}$ might be large and so we need to evaluate the objective function in a lot of feasible points. Since the function $x \mapsto g_1(x,T)$ is non-decreasing, there is a possibility to improve the upper bound in Lemma 29 iteratively. As already observed this might reduce the number of possible function evaluations in our complete enumeration.

Lemma 12. *If $u_{\tilde{P}_T,0} := u_{\tilde{P}_T}$ and for every $n \in \mathbb{Z}_+$*

$$u_{\tilde{P}_T,n+1} := \min\{x \in \mathbb{Z}_+ : k(x,T) > g_1(u_{\tilde{P}_T,n}, T)\}, \tag{4.27}$$

then the sequence $u_{\tilde{P}_T,n}, n \in \mathbb{Z}_+$ is non-increasing and any optimal solution $x^*_{\tilde{P}_T}$ of optimization problem $(\widetilde{P}_T)$ satisfies $x^*_{\tilde{P}_T} \leqslant u_{\tilde{P}_T,n}$ for every $n \in \mathbb{Z}_+$.

Proof. By Lemma 29 the result holds for $n = 0$. Suppose now by induction that the sequence $u_{\tilde{P}_T,n}$, $n \leqslant m$ is non-increasing and there exists an optimal solution $x^*_{\tilde{P}_T}$ of optimization problem $(\widetilde{P}_T)$ satisfying $x^*_{\tilde{P}_T} \leqslant u_{\tilde{P}_T,m}$. Since the function $x \mapsto g_1(x,T)$ is non-decreasing, we obtain by our induction hypothesis and relation (4.27) that

$$k(u_{\tilde{P}_T,m},T) \geqslant g_1(u_{\tilde{P}_T,m-1},T) \geqslant g_1(u_{\tilde{P}_T,m},T).$$

This shows again by relation (4.27) that $u_{\tilde{P}_T,m+1} \leqslant u_{\tilde{P}_T,m}$. As in the proof of Lemma 29 it follows for every $x \in \{u_{\tilde{P}_T,m+1},...,u_{\tilde{P}_T,m}\}$ that

$$\begin{aligned}
c(x)+\widetilde{C}(x,T) &\geqslant \left\{\begin{array}{l} c(x)+\widetilde{C}(0,T)+c_{scr}x+(h-\delta c_{scr})\int_0^T e^{-\delta u}(x-\Lambda_0(u))^+du \\ \\ -(1-q)\int_0^T e^{-\delta u}\lambda(u)(c_{ap}(u)+c_{scr}-c_{se})^+\,\mathbb{P}(N_0(u)<x)\,du \end{array}\right. \\
&\geqslant \left\{\begin{array}{l} c(x)+\widetilde{C}(0,T)+c_{scr}x+(h-\delta c_{scr})\int_0^T e^{-\delta u}(x-\Lambda_0(u))^+du \\ \\ -(1-q)\int_0^T e^{-\delta u}\lambda(u)(c_{ap}(u)+c_{scr}-c_{se})^+\,\mathbb{P}(N_0(u)<u_{\tilde{P}_T,m})\,du, \end{array}\right.
\end{aligned}$$

which is equal to $\widetilde{C}(0,T)+k(x,T)-g_1(u_{\tilde{P}_T,m},T)$. This implies for any $x \in \{u_{\tilde{P}_T,m+1},...,u_{\tilde{P}_T,m}\}$ that

$$\begin{aligned}
c(x)+\widetilde{C}(x,T) &\geqslant \widetilde{C}(0,T)+k(x,T)-g_1(u_{\tilde{P}_T,m},T) \\
&> \widetilde{C}(0,T) \qquad (4.28) \\
&= c(0)+\widetilde{C}(0,T)
\end{aligned}$$

and so by our induction hypothesis and relation (4.28) the result follows. □

For any given functions c_a, c, p, λ and the constants c_{se}, c_{scr}, q, h, δ it is possible to compute the sequence $u_{\tilde{P}_T,0}, u_{\tilde{P}_T,1},\ldots$ of non-increasing upper bounds. The procedure will stop if at a certain iteration step the new computed upper bound is the same as the previous upper bound, and since the upperbounds are integers this takes at most $u_{\tilde{P}_T,0}$ iteration steps. After having stopped at some upper bound, call it $\underline{\mathrm{u}}_{\tilde{P}_T}$, we can carry out a complete enumeration and evaluate for x=0, ..., $\underline{u}_{\widetilde{P}_T}$ the function values

$$c(x) + \widetilde{C}(x,T)$$

to determine an optimal solution of this optimization problem. To compute the value $\widetilde{C}(x,T)$ at a given point x we consider the first order difference operator for $\tau = T$ given by (see Lemma 8)

$$\begin{aligned} \Delta_x \widetilde{C}(x,T) &= \widetilde{C}(x+1,T) - \widetilde{C}(x,T) \\ &= \begin{cases} c_{scr} + (h - \delta c_{scr}) \int_0^T e^{-\delta u} \mathbb{P}(N_0(u) \leqslant x) du \\ \\ +(1-q) \int_0^T e^{-\delta u} \lambda(u)[c_{se} - c_{scr} - c_{ap}(u)] \mathbb{P}(N_0(u) = x) du \end{cases} \end{aligned} \tag{4.29}$$

and observe

$$\widetilde{C}(x,T) = \sum_{k=0}^{x-1} \Delta_x \widetilde{C}(k,T) + \widetilde{C}(0,T). \tag{4.30}$$

To simplify for a differentiable function c_{ap} the numerical computation of $\Delta_x \widetilde{C}(x,T)$ listed in relation (4.46) we first list the following useful result of non-homogeneous Poisson processes.

Lemma 13. *Let N be a non-homogeneous Poisson process with arrival rate function β which β is piecewise continuous and ψ a differentiable function and ψ' its derivative function. Then for every $x \in \mathbb{Z}_+$ and $\tau \leqslant T, \tau \in \mathbb{D}$ we have*

$$\int_0^\tau \psi(u)\beta(u)\mathbb{P}(N(u) = x)du = \int_0^\tau \psi'(u)\mathbb{P}(N(u) \leqslant x)du + \psi(0) - \psi(\tau)\mathbb{P}(N(\tau) \leqslant x). \tag{4.31}$$

Proof. It is well known (Ross [2014]) for a non-homogeneous Poisson process with an intensity function β that, for every $k \in \mathbb{Z}_+$, the function $\varphi(u) := \mathbb{P}(N(u) \leqslant k)$, for

$u \geqslant 0$, is differentiable and satisfies

$$\varphi'(u) = -\beta(u)\,\mathbb{P}(N(u) = k)$$

with the initial condition $\varphi(0) = \mathbb{P}(N(0) \leqslant k) = 1$. Applying now the chain rule of differentiation yields

$$\begin{aligned} \psi(\tau)\varphi(\tau) - \psi(0) &= \textstyle\int_0^\tau \psi'(u)\varphi(u)du + \int_0^\tau \psi(u)\varphi'(u)du \\ &= \textstyle\int_0^\tau \psi'(u)\varphi(u)du - \int_0^\tau \psi(u)\beta(u)\mathbb{P}(N(u) \leqslant k)du \end{aligned} \tag{4.32}$$

from which (4.31) follows after re-arranging the terms. ◻

Using the above lemma it is now possible to give the following formula for the first order difference $\Delta_x \widetilde{C}(x,T)$ and this formula is more stable for numerical calculation.

Lemma 14. *For every $x \in \mathbb{Z}_+$ we have*

$$\Delta_x \widetilde{C}(x,T) = \begin{cases} c_{se} - c_{ap}(0) + e^{-\delta T}(c_{ap}(T) + c_{scr} - c_{se})\mathbb{P}(N_0(T) \leqslant x) \\ + \int_0^T e^{-\delta u}[h - c'_{ap}(u) + \delta(c_{ap}(u) - c_{se})]\mathbb{P}(N_0(u) \leqslant x)du. \end{cases} \tag{4.33}$$

Proof. Applying Lemma 13 with $\psi(u) = e^{-\delta u}[c_{se} - c_{scr} - c_{ap}(u)]$ gives

$$\begin{aligned} &\textstyle\int_0^T e^{-\delta u}\lambda_0(u)[c_{se} - c_{scr} - c_{ap}(u)]\mathbb{P}(\mathbb{N}_0(u) = x)du \\ &= \begin{cases} -\int_0^T e^{-\delta u}c'_{ap}(u)P(N_0(u) \leqslant x)du \\ -\delta\int_0^T e^{-\delta u}[c_{se} - c_{scr} - c_{ap}(u)]P(N_0(u) \leqslant x)du \\ +[c_{se} - c_{scr} - c_{ap}(0)] - e^{-\delta T}[c_{se} - c_{scr} - c_{ap}(T)]P(N_0(T) \leqslant x) \end{cases} \end{aligned} \tag{4.34}$$

and finally using (4.34) in (4.13) for $\tau = T$ yields the desired result after straightforward simplifications. ◻

By Lemma 14 it is obvious that one can calculate the first order difference $\Delta_x \widetilde{C}(x,T)$ by evaluating numerically the integral

$$\int_0^T e^{-\delta u}[h - c'_{ap}(u) + \delta(c_{ap}(u) - c_{se})]\mathbb{P}(N_0(u) = k)du$$

for different values of k. In case the function c_{ap} is an elementary function and our arrival intensity function is given by a piecewise constant arrival intensity function one can write this integral as a sum of elementary functions avoiding numerical integration.

Under some additional sufficient conditions on the function c_{ap} and the parameters c_{se}, c_{scr}, one can show that the function $x \mapsto c(x) + \widetilde{C}(x,T)$ is discrete convex. In general one only needs to use the upper bounding technique discussed above in combination with complete enumeration in case the procurement cost function c is not discrete convex or if the function c is discrete convex and the function c_{ap} and the parameters c_{se}, c_{scr} satisfy the inequality

$$c_{ap}(0) > c_{se} + c_{scr}^{-} \text{ and } c_{ap}(T) < c_{se} - c_{scr}.$$

If $c_{ap}(T) \geqslant c_{se} - c_{scr}$ and the procurement cost function is discrete convex, one can show the following result and this is a special case of Lemma 4. Observe a different proof using directly the first order differences is given in (Frenk et al. [2018]).

Lemma 15. *If the procurement cost function c is discrete convex on $\mathbb{Z}_+$ and $c_{ap}(T) \geqslant c_{se} - c_{scr}$, then the function $x \mapsto c(x) + \widetilde{C}(x,T)$ is discrete convex on $\mathbb{Z}_+$.*

If it is costly to scrap the inventory ($c_{scr} > 0$) and the cost of serving a customer via the alternative policy is higher than the regular service cost at all times (i.e., $p(u) \geqslant c_{se}$ for all $u \leqslant T$) as assumed in our standard assumptions then the conditions of Lemma 15 hold. Observe, under the convexity condition, we do not need to evaluate an upper bound $\underline{\mathrm{u}}_{\widetilde{P}_T}$ as discussed above. An optimal solution of optimization problem $(\widetilde{P}_T)$ is obtained simply by the first order condition

$$x^*_{\widetilde{P}_T} := \min\{x \in \mathbb{Z}_+ : c(x+1) - c(x) + \Delta_x \widetilde{C}(x,T) \geqslant 0\} \tag{4.35}$$

with $\Delta_x \widetilde{C}(x,T)$ given in relation (4.33) in Lemma 14. As before by computing iteratively these first order differences until the first order condition is satisfied one can also easily calculate the optimal objective value using

$$\widetilde{C}(x,T) = \sum_{k=0}^{x-1} \Delta_x \widetilde{C}(k,T) + \widetilde{C}(0,T).$$

In the next section we will discuss how to select the optimal static policy among the set of all static (x,τ) policies.

4.4 Analysis of Optimization Problem $(P_{\mathbb{D}})$

In this section we propose an algorithm to select among the class of all (x,τ)-policies, $x \in \mathbb{Z}_+$, $\tau \in \mathbb{D}$, $0 \leqslant \tau \leqslant T$ the optimal one having minimal expected discounted cost. This means we assume that at any deterministic time between 0 and T we can switch to the alternative policy and so our class of policies is given by all (x,τ)-policies with $x \in \mathbb{Z}_+$ and $\tau \in \mathbb{D}$, $0 \leqslant \tau \leqslant T$. Hence we need to solve optimization problem $(P_{\mathbb{D}})$. As shown in Section 4.1 the optimal solutions of optimization problem $(P_{\mathbb{D}})$ are the same as for optimization problem

$$\upsilon(\widetilde{P}_{\mathbb{D}}) = \inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{D},0\leqslant\tau\leqslant T}\{c(x) + \widetilde{C}(x,\tau)\}. \tag{$\widetilde{P}_{\mathbb{D}}$}$$

with $\widetilde{C}(x,\tau)$ given by (see relation 4.10)

$$\widetilde{C}(x,\tau) = \begin{cases} c_{scr}x + (1-q)\int_0^\tau e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]\mathbb{P}(N_0(u) < x)du \\ + \int_0^\tau e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du \\ +(h - \delta c_{scr})\int_0^\tau e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.36}$$

Also from relation (4.2) we know that

$$\upsilon(P_{\mathbb{D}}) = \upsilon(\widetilde{P}_{\mathbb{D}}) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{4.37}$$

In this section we will solve optimization problem $(\widetilde{P}_{\mathbb{D}})$ by means of a bi-level approach using the representation

$$\upsilon(\widetilde{P}_{\mathbb{D}}) = \inf_{0\leqslant\tau\leqslant T}\Phi(\tau) \tag{4.38}$$

with

$$\Phi(\tau) := \inf_{x\in\mathbb{Z}_+}\{c(x) + \widetilde{C}(x,\tau)\}. \tag{$\widetilde{P}_{\Phi(\tau)}$}$$

Note that optimization problem $(\widetilde{P}_{\Phi(\tau)})$ is a generalization of optimization problem $(\widetilde{P}_T)$ for any switching time $\tau \in \mathbb{D}$, $0 \leqslant \tau \leqslant T$. Before discussing how to solve for the general case the optimization problem $(P_{\mathbb{D}})$ we first identify under which parameter settings in the optimization problem $(P_{\mathbb{D}})$ it is optimal not to order. Again the next result is a special case of Lemma 1.

Lemma 16. *If $c_{ap}(0) \leqslant c_{se} + c_{scr}^-$ with c_{scr}^- listed in relation (3.5), then it is optimal not to order in optimization problem $(P_{\mathbb{D}})$ and the optimal objective value of $(P_{\mathbb{D}})$ equals*

$$\upsilon(P_{\mathbb{D}}) = \inf_{\tau\in\mathbb{D}}\left\{\int_0^\tau e^{-\delta u}\lambda(u)[q(c_{re} + c_{se} - c_a(u)) + (1-q)p(u)]du\right\}. \tag{4.39}$$

In almost all cases the sufficient conditions of Lemma 16 are not satisfied by our problem parameters and so we need to give an algorithm to solve optimization problem $(\widetilde{P}_{\mathbb{D}})$. In the general case we will use the approach as suggested by relation (4.38) and so part of the algorithm will consist of a procedure determining the optimal order size in case the closing time τ is known in advance. We can only execute this procedure solving problem $(\widetilde{P}_{\Phi(\tau)})$ a finite number of times. This means that we first need to construct from the set $[0,T]$ a finite set $\mathcal{D} = \{\tau_1, \tau_2, ...\tau_N\}$ satisfying $0 \leqslant \tau_1 < ...\tau_N \leqslant T$ such that with a known error bound we can replace the optimization problem in relation (4.38) by its discrete version $\inf_{\tau \in \mathcal{D}} \Phi(\tau)$. To select this finite set $\mathcal{D}$ we first introduce the set

$$D := \{0 \leqslant \tau \leqslant T : c_{ap}(\tau) \geqslant c_{se} - c_{scr}\}. \tag{4.40}$$

Since c_{ap} is non-increasing the set D is a (possibly empty) convex subset of $[0,T]$. In particular, if $c_{scr} \geqslant 0$ and we assume the natural condition $p(0) \geqslant c_{se}$ then it follows that 0 belongs to D. Hence it follows under these natural conditions that the set D is nonempty and we introduce

$$\tau_D := \sup\{0 \leqslant \tau \leqslant T : c_{ap}(\tau) \geqslant c_{se} - c_{scr}\} \leqslant T.$$

The next result yields some upperbound on the rate of growth of the function $\tau \mapsto \widetilde{C}(x,\tau)$. Note the so-called indicator function of the set $A \subset \mathbb{R}$ is given by

$$1_A(\tau) = \begin{cases} 1 & \text{if } \tau \in A \\ 0 & \text{if } \tau \notin A \end{cases} \tag{4.41}$$

Lemma 17. *For every $\tau, s \geqslant 0$ with $\tau + s \leqslant T$, and $x \in \mathbb{Z}_+$, we have*

$$\widetilde{C}(x,\tau+s)-\widetilde{C}(x,\tau) \geqslant \begin{cases} \widetilde{C}(0,\tau+s)-\widetilde{C}(0,\tau) \\ +e^{-\delta\tau}(c_{se}-c_{scr}-c_{ap}(\tau))(\Lambda_0(\tau+s)-\Lambda_0(\tau))1_D(\tau). \end{cases} \tag{4.42}$$

Proof. By relation (4.11) and $h-\delta c_{scr} \geqslant 0$ it follows

$$\begin{aligned} \widetilde{C}(x,\tau+s)-\widetilde{C}(x,\tau) &= \begin{cases} (1-q)\int_\tau^{\tau+s} e^{-\delta u}\lambda(u)([c_{se}-c_{scr}-c_{ap}(u)] \\ \mathbb{P}(N_0(u)<x)du + \widetilde{C}(0,\tau+s)-\widetilde{C}(0,\tau) \\ +(h-\delta c_{scr})\int_0^\tau e^{-\delta u}\mathbb{E}((x-N_0(u)^+)du \end{cases} \\ &\geqslant \begin{cases} (1-q)\int_\tau^{\tau+s} e^{-\delta u}\lambda(u)[c_{se}-c_{scr}-c_{ap}(u)] \\ \mathbb{P}(N_0(u)<x)du + \widetilde{C}(0,\tau+s)-\widetilde{C}(0,\tau). \end{cases} \end{aligned}$$

Hence for τ not belonging to D or $c_{ap}(\tau) < c_{se}-c_{scr}$ and using c_{ap} is decreasing we obtain

$$\widetilde{C}(x,\tau+s)-\widetilde{C}(x,\tau) \geqslant \widetilde{C}(0,\tau+s)-\widetilde{C}(0,\tau),$$

while for τ belonging to D or $c_{ap}(\tau) \geqslant c_{se}-c_{scr}$ it follows

$$\begin{aligned} \widetilde{C}(x,\tau+s)-\widetilde{C}(x,\tau) &\geqslant \begin{cases} (1-q)\int_\tau^{\tau+s} e^{-\delta u}\lambda(u)[c_{se}-c_{scr}-c_{ap}(u)]du \\ +\widetilde{C}(0,\tau+s)-\widetilde{C}(0,\tau) \end{cases} \\ &\geqslant \begin{cases} e^{-\delta\tau}[c_{se}-c_{scr}-c_{ap}(\tau)](\Lambda_0(\tau+s)-\Lambda_0(\tau)) \\ +\widetilde{C}(0,\tau+s)-\widetilde{C}(0,\tau). \end{cases} \end{aligned}$$

This shows the result. □

For $\tau, s \geqslant 0$ with $\tau + s \leqslant T$, the first difference in the right hand side of the inequality in (4.42) can be evaluated by simple integration as

$$\widetilde{C}(0, \tau + s) - \widetilde{C}(0, \tau) = \int_{\tau}^{\tau+s} e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du. \tag{4.43}$$

Using Lemma 17, we provide a growth condition for the function $\tau \mapsto \Phi(\tau)$ in terms of this difference.

Lemma 18. *For every $\tau, s \geqslant 0$ with $\tau + s \leqslant T$ we have*

$$\Phi(\tau + s) - \Phi(\tau) \geqslant \begin{cases} \widetilde{C}(0, \tau + s) - \widetilde{C}(0, \tau) \\ \\ +e^{-\delta\tau}(c_{se} - c_{scr} - c_{ap}(\tau))(\Lambda_0(\tau + s) - \Lambda_0(\tau))1_D(\tau). \end{cases} \tag{4.44}$$

Proof. Let $x^*_{\widetilde{P}_{\Phi(\tau+s)}} \in \mathbb{Z}_+$ denote the optimal order level of optimization problem $(\widetilde{P}_{\Phi(\tau+s)})$. It follows by the definition of $\Phi(\tau + s)$ that

$$\Phi(\tau + s) = c(x^*_{\widetilde{P}_{\Phi(\tau+s)}}) + \widetilde{C}(x^*_{\widetilde{P}_{\Phi(\tau+s)}}, \tau + s).$$

Since $x^*_{\widetilde{P}_{\Phi(\tau+s)}}$ is also a feasible solution for the optimization problem problem $(\Phi(\tau))$ it is obvious that

$$\Phi(\tau) \leqslant c(x^*_{\widetilde{P}_{\Phi(\tau+s)}}) + \widetilde{C}(x^*_{\widetilde{P}_{\Phi(\tau+s)}}, \tau).$$

Applying both the above equality and inequality and using Lemma 17 we obtain the inequality in relation (4.44). ◻

To identify a region on which the function Φ is increasing we introduce the (possibly empty) convex set

$$D_1 := \{0 \leqslant \tau \leqslant T : c_{se} + c_{re} - c_a(\tau) \geqslant 0\}$$

(due to c_a being non-increasing) and let

$$\tau_{D_1} := \begin{cases} \inf\{0 \leqslant \tau \leqslant T : c_{se} + c_{re} - c_a(\tau) \geqslant 0\} & \text{if } D_1 \neq \varnothing \\ T & \text{if } D_1 = \varnothing. \end{cases}$$

An immediate consequence of Lemma 18 is given by the following result.

Lemma 19. *If* $\overline{\tau} = \max\{\tau_D, \tau_{D_1}\}$ *then the function* Φ *is non-increasing on* $[\overline{\tau}, T]$.

Proof. For every $\tau, \tau + s \in [\overline{\tau}, T]$ and $s \geqslant 0$ it follows that τ does not belong to D. Hence by Lemma 18 and $c_{se} + c_{re} - c_a(u) \geqslant 0$ for every $\tau \leqslant u \leqslant \tau + s$ we obtain

$$\begin{aligned} \Phi(\tau + s) - \Phi(\tau) &\geqslant \widetilde{C}(0, \tau + s) - \widetilde{C}(0, s) \\ &= \int_\tau^{\tau+s} e^{-\delta u} \lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1 - q)(p(u)]du \\ &\geqslant 0 \end{aligned}$$

and this shows the result. $\square$

Note for $\tau \notin D$ we have

$$c_{se} > c_a(\tau) + p(\tau) + c_{scr} \geqslant c_a(\tau) + c_{scr}.$$

This means, if an item arrives at time τ and is non-repairable and we have a spare one in the inventory, it is better to scrap the item in the inventory immediately and use the alternative policy. Clearly, we also have $c_a(\tau) + p(\tau) \geqslant c_a(\tau)$ at all times. Hence, D', denoting the complement of D over $[0, T]$, is essentially the set of times where serving non-repairable items is more expensive under the repair-replacement policy. Also note that when $c_{se} + c_{re} \geqslant c_a(\tau)$, the cost of serving a repairable item at time τ under the repair-replacement policy is higher than (or equal to) that of using the alternative policy. Clearly, when this inequality holds for some τ, it also holds for all time points $u \geqslant \tau$. Then, Lemma 19 above gives sufficient conditions (on τ-values) under which serving a customer under the alternative policy is always better regardless of the repairability of the arriving item or the availability of a spare one in the inventory. Hence, we should not consider any value of $\tau \geqslant \bar{\tau}$ in the minimization problem $(\widetilde{P}_{\mathbb{D}})$ and so we may conclude

$$\upsilon(\widetilde{P}_{\mathbb{D}}) = \inf_{0 \leqslant \tau \leqslant \bar{\tau}} \Phi(\tau). \tag{4.45}$$

Before applying the result of Lemma 18 to construct a finite set $\mathcal{D} \subseteq [0, \bar{\tau}]$ having a predetermined error bound we first discuss solving optimization problem $(\widetilde{P}_{\Phi(\tau)})$ for some given $0 \leqslant \tau \leqslant T$. This procedure can only be applied a finite number of times and so it is crucial to construct this finite set $\mathcal{D} \subseteq [0, \bar{\tau}]$. For the general case as in the previous section we need to determine first an upperbound on the optimal order quantity for any fixed τ. By a similar proof as used in Lemma 12 replacing T by τ one can show the following result.

Lemma 20. *If*

$$u_{\tilde{P}_{\Phi(\tau)},0} := \min\{x \in \mathbb{Z}_+ : k(x, \tau) \geqslant g_1(\infty, \tau)\} \ < +\infty$$

and

$$u_{\tilde{P}_{\Phi(\tau)},n+1} := \min\{x \in \mathbb{Z}_+ : k(x, \tau) \geqslant g_1(u_{\tilde{P}_{\Phi(\tau)},n}, \tau)\}$$

with the functions k and g_1 defined in relation (4.22) *and* (4.23) *respectively then the sequence $u_{\tilde{P}_{\Phi(\tau)},n}$, $n \in \mathbb{Z}_+$ is non-increasing, and any optimal solution $x^*_{\tilde{P}_{\Phi(\tau)}}$ of optimization problem $(\widetilde{P}_{\Phi(\tau)})$ satisfies $x^*_{\tilde{P}_{\Phi(\tau)}} \leqslant u_{\tilde{P}_{\Phi(\tau)},n}$ for every $n \in \mathbb{Z}_+$.*

For any given $0 \leqslant \tau \leqslant T$ and functions c_a, c, p, λ and the constants c_{se}, c_{scr}, q, h, δ it is possible to compute the sequence $u_{\tilde{P}_{\Phi(\tau)},0}, u_{\tilde{P}_{\Phi(\tau)},1}, \ldots$ of non-increasing upper bounds. The procedure will stop if at a certain iteration step the new computed upper bound is the same as the previous upper bound, and this takes at most $u_{\Phi(\tau),0}$ iteration steps. After having stopped at some upper bound, call it $\underline{\mathrm{u}}_{\tilde{P}_{\Phi(\tau)}}$, we can carry out a complete enumeration and evaluate the function values

$$c(x) + \widetilde{C}(x,\tau), \qquad x = 0, ..., \underline{\mathrm{u}}_{\tilde{P}_{\Phi(\tau)}}$$

to determine an optimal solution of this optimization problem. To compute the value $\widetilde{C}(x,\tau)$ at a given point x we introduce the first order difference operator for $\tau \leqslant T, \tau \in \mathbb{D}$ given by (see Lemma 8)

$$\begin{aligned} \Delta_x \widetilde{C}(x,\tau) &= \widetilde{C}(x+1,\tau) - \widetilde{C}(x,\tau) \\ &= \begin{cases} c_{scr} + (h - \delta c_{scr}) \int_0^\tau e^{-\delta u} \mathbb{P}(N_0(u) \leqslant x) du \\ \\ +(1-q) \int_0^\tau e^{-\delta u} \lambda(u)[c_{se} - c_{scr} - c_{ap}(u)] \mathbb{P}(N_0(u) = x) du. \end{cases} \end{aligned} \tag{4.46}$$

Once we have computed these differences iteratively we may also compute the objective value observing

$$\widetilde{C}(x,\tau) = \sum_{k=0}^{x-1} \Delta_x \widetilde{C}(k,\tau) + \widetilde{C}(0,\tau). \tag{4.47}$$

For some values of τ it is possible to avoid calculating these upperbounds and do a complete enumeration. Under some conditions one can show that the function $x \mapsto C(x,\tau)$ is discrete convex for a given τ. Observe that if $c_{scr} \geqslant 0$ and $p(u) \geqslant c_{se}$ it follows that $c_{ap}(T) \geqslant c_{se}$ and the condition in the next lemma is always satisfied for every $0 \leqslant \tau \leqslant T$. Also in this case it follows that $D = [0,T]$. The following result is a special case of Lemma 4.

Lemma 21. *If the procurement function c is discrete convex on $\mathbb{Z}_+$ and $c_{ap}(\tau) \geqslant c_{se} - c_{scr}$ or equivalently $\tau \in D$, then both the functions $x \mapsto c(x) + C(x,\tau)$ and*

$x \mapsto c(x) + \widetilde{C}(x,\tau)$ *are discrete convex on* $\mathbb{Z}_+$.

Proof. Apply the observation after Lemma 4. $\square$

Under the conditions of Lemma 21, we obtain that an optimal solution of optimization problem $(\widetilde{P}_{\Phi(\tau)})$ is given by

$$x^*_{\widetilde{P}_{\Phi(\tau)}} = \min\{x \in \mathbb{Z}_+ : c(x+1) - c(x) + \Delta_x \widetilde{C}(x,\tau) \geqslant 0\} \tag{4.48}$$

with $\Delta_x \widetilde{C}(x,\tau) := \widetilde{C}(x+1,\tau) - \widetilde{C}(x,\tau)$, for $x \in \mathbb{Z}_+$. Using this difference operator we can also easily evaluate the optimal objective value $\Phi(\tau)$ and this is given by

$$\Phi(\tau) = c(x^*_{\widetilde{P}_{\Phi(\tau)}}) + \widetilde{C}(x^*_{\widetilde{P}_{\Phi(\tau)}},\tau) = c(x^*_{\widetilde{P}_{\Phi(\tau)}}) + \widetilde{C}(0,\tau) + \sum\nolimits_{k=0}^{x^*_{\widetilde{P}_{\Phi(\tau)}}-1} \Delta\widetilde{C}(k,\tau). \tag{4.49}$$

We finally conclude this section with an algorithm describing how to solve problem $(\widetilde{P}_{\mathbb{D}})$. The algorithm brings a clever discretization of the interval $[0, T \wedge \bar{\tau}]$ to search for the best switching time. Recall that the function $\tau \mapsto \Phi(\tau)$ by Lemma 18 satisfies the growth condition (4.44). Using this inequality, we construct the finite collection of discretization points $\mathcal{D}_0 = \{\tau_1, \ldots, \tau_N\}$ where $\tau_1 = 0$, and for $i \geqslant 1$, we set

$$\tau_{i+1} = \min\Big\{ s > \tau_i \;:\; \widetilde{C}(0,s) - \widetilde{C}(0,\tau_i) + e^{-\delta\tau_i}(c_{se} - c_{scr} - c_{ap}(\tau_i)) \tag{4.50}$$

$$\big[\Lambda_0(s) - \Lambda_0(\tau_i)\big]1_D(\tau_i) \leqslant -\varepsilon \Big\}, \tag{4.51}$$

where N is the iteration number for which the resulting point in (4.50) exceeds $T \wedge \bar{\tau}$, and we set $\tau_N = T \wedge \bar{\tau}$ (see Lemma 19 for the definition of $\bar{\tau}$). It follows by (4.44) that

$$\Phi(s) - \Phi(\tau_i) \geqslant -\varepsilon \qquad \text{for every } s \in [\tau_i, \tau_{i+1}].$$

This further implies

$$\min_{\tau_i \in \mathcal{D}} \Phi(\tau_i) \geqslant \inf_{\tau \in [0,T]} \Phi(\tau) \equiv \upsilon(\widetilde{P}_{\mathbb{D}}) \geqslant \min_{\tau_i \in \mathcal{D}} \Phi(\tau_i) - \varepsilon.$$

Hence, for a given numerical tolerance level $\varepsilon > 0$, if we search the best switching time τ in $\mathcal{D}$, the loss in the objective value is no more than ε. This observation gives us the following algorithm.

Algorithm 1. ***Numerical Algorithm to Solve Optimization Problem*** $(\widetilde{P}_{\mathbb{D}})$

1. *Select* $\epsilon > 0$ *and construct on the interval* $[0, T \wedge \bar{\tau}]$ *the finite discretization* $\mathcal{D} = \{\tau_1, \ldots, \tau_N\}$ *described in* (4.50) *with* $\tau_1 = 0$ *and* $\tau_N = T \wedge \bar{\tau}$.

2. *For every* $\tau_i \in \mathcal{D}$, *if the convexity conditions of Lemma 21 hold, compute* $x^*_{\widetilde{P}_{\Phi(\tau_i)}}$ *using* (4.48) *and evaluate* $\Phi(\tau_i)$ *via* (4.49). *Otherwise, find the smallest upper bound* $\underline{u}_{\widetilde{P}_{\Phi(\tau_i)}}$ *following Lemma 20 and compute*

$$x^*_{\widetilde{P}_{\Phi(\tau_i)}} = \arg\min_{x \leqslant \underline{u}_{\widetilde{P}_{\Phi(\tau_i)}}} \{c(x) + \widetilde{C}(x, \tau_i)\}$$

and

$$\Phi(\tau_i) = c(x^*_{\widetilde{P}_{\Phi(\tau_i)}}) + \widetilde{C}(x^*_{\widetilde{P}_{\Phi(\tau_i)}}, \tau_i).$$

3. *Find the best* τ_i^* *attaining* $\min_{\tau_i \in \mathcal{D}_0} \Phi(\tau_i)$. *Report* $(x^*_{\widetilde{P}_{\Phi(\tau_i^*)}}, \tau_i^*)$ *as the* ε *-optimal policy.*

In the next subsection we will start analyzing the class of pseudo static policies.

4.5 Analysis of Optimization Problem $(P_{T\wedge\sigma})$

In this section we propose an algorithm to select among the class of all pseudo static policies of the form $T \wedge \sigma_x$ the optimal one having minimal expected discounted cost. This means we assume that we will apply the repair-replacement policy up to the time that the inventory level drops to zero. In this problem we only need to determine the optimal order quantity and so we need to solve optimization problem $(P_{T\wedge\sigma})$ given by

$$\upsilon(P_{T\wedge\sigma}) = \inf_{x\in\mathbb{Z}_+}\{c(x) + C(x, T\wedge\sigma_x)\}. \tag{$P_{T\wedge\sigma}$}$$

As shown in Section 4.1 the optimal solution of optimization problem $(P_{T\wedge\sigma})$ is the same as for optimization problem

$$\upsilon(\widetilde{P}_{T\wedge\sigma}) = \inf_{x\in\mathbb{Z}_+}\{c(x) + \widetilde{C}(x, T\wedge\sigma)\}. \tag{$\widetilde{P}_{T\wedge\sigma}$}$$

with $\widetilde{C}(x, T\wedge\sigma_x)$ given by (see relation (4.16) for $\tau = T$)

$$\widetilde{C}(x, T\wedge\sigma_x) = \begin{cases} c_{scr}x + \int_0^T e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)]\mathbb{P}(N_0(u) < x)du \\ \\ +(h - \delta c_{scr})\int_0^T e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.52}$$

Also from relation (4.3) we know that

$$\upsilon(P_{T\wedge\sigma}) = \upsilon(\widetilde{P}_{T\wedge\sigma}) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{4.53}$$

Due to the previous remarks we will analyze in this section the optimization problem $(\widetilde{P}_{T\wedge\sigma})$. The analysis is similar to the problem $(\widetilde{P}_T)$ discussed in subsection 4.3 since both problems are essentially one dimensional problems with the same decision variable $x \in \mathbb{Z}_+$. Since not ordering and immediately applying the alternative policy is feasible within the class of considered $(T\wedge\sigma_x)$ policies, $x \in \mathbb{Z}_+$ we first list some set of sufficient conditions on the cost function c_a and the parameters c_{se}, c_{re} and c_{scr} under which this is optimal. The next result is a special case of Lemma 2.

Lemma 22. *If $c_a(0) \leqslant c_{se} + qc_{re} - (1-q)c_{scr}^-$, then it is optimal not to order in optimization problem ($P_{T\wedge\sigma}$) and immediately apply the alternative policy. The optimal objective value is equal to $\int_0^T e^{-\delta u}\lambda(u)c_a(u)du$.*

As for the optimization problem ($\widetilde{P}_T$) we first construct in the general case with no restrictions on the parameters an upper bound for the optimal order quantity in problem $\widetilde{P}_{T\wedge\sigma}$. The construction is similar to that described in Lemmas 12 and 29 above for the optimization problem ($\widetilde{P}_T$). We first introduce for $x \in \mathbb{Z}_+$, $\tau \in \mathbb{D}$, $\tau \leqslant T$ the function

$$g_2(x,\tau) := \mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)(c_a(u) - c_{se} - qc_{re} + (1-q)c_{scr})^+ du\right). \tag{4.54}$$

Clearly this function is non-decreasing in x. By the monotone convergence theorem we obtain

$$g_2(\infty,\tau) = \int_0^{\tau} e^{-\delta u}\lambda(u)(c_a(u) - c_{se} - qc_{re} + (1-q)c_{scr})^+ du < +\infty.$$

Using a similar proof as in Lemma 29 one can show the following result.

Lemma 23. *There exists an optimal solution of problem ($\widetilde{P}_{T\wedge\sigma}$) and any optimal solution $x^*_{\widetilde{P}_{T\wedge\sigma}}$ of optimization problem ($\widetilde{P}_{T\wedge\sigma}$) satisfies $x^*_{\widetilde{P}_{T\wedge\sigma}} \leqslant u_{\widetilde{P}_{T\wedge\sigma}}$ with*

$$u_{\widetilde{P}_{T\wedge\sigma}} := \min\{x \in \mathbb{Z}_+ : k(x,T) > g_2(\infty,T)\} \ < +\infty. \tag{4.55}$$

The next result follows a similar line of proof as the one followed in Lemma 12.

Lemma 24. *If $u_{\widetilde{P}_{T\wedge\sigma},0} = u_{\widetilde{P}_{T\wedge\sigma}}$ and for every $n \in \mathbb{Z}_+$*

$$u_{\widetilde{P}_{T\wedge\sigma},n+1} = \min\{x \in \mathbb{Z}_+ : k(x,T) \geqslant g_2(u_{\widetilde{P}_{T\wedge\sigma},n},T)\} \tag{4.56}$$

where the function k is defined in (4.22), then the sequence $u_{\tilde{P}_{T\wedge\sigma},n}$, $n \in \mathbb{Z}_+$, is non-increasing and any optimal solution $x^*_{\tilde{P}_{T\wedge\sigma}}$ of optimization problem $(\widetilde{P}_{T\wedge\sigma})$ satisfies $x^*_{\tilde{P}_{T\wedge\sigma}} \leqslant u_{\tilde{P}_{T\wedge\sigma},n}$ for every $n \in \mathbb{Z}_+$.

For any selection of the functions c_a, c_0, p, λ and the constants c_{se}, c_{scr}, q, h, δ, it is easy to evaluate the sequence of non-increasing upper bounds $u_{\tilde{P}_{T\wedge\sigma},0}, u_{\tilde{P}_{T\wedge\sigma},1}, \ldots$. The procedure stops if in a certain iteration step the new computed upper bound is the same as the previous one, and this procedure takes at most $u_{\tilde{P}_{T\wedge\sigma},0}$ iterations. After having stopped at some upper bound $\underline{\mathrm{u}}_{\tilde{P}_{T\wedge\sigma}}$ we evaluate the function values

$$c(x) + \widetilde{C}(x, T \wedge \sigma_x), \qquad x = 0, ..., \underline{\mathrm{u}}_{P_{\tilde{P}_{T\wedge\sigma}}}$$

to identify an optimal procurement quantity. Here, the cost term $\widetilde{C}(x, T \wedge \sigma_x)$ can be obtained via the difference operator (see Lemma 9 for $\tau = T$)

$$\begin{aligned}\Delta_x \widetilde{C}(x, T \wedge \sigma_x) &:= \widetilde{C}(x+1, T \wedge \sigma_{x+1}) - \widetilde{C}(x, T \wedge \sigma_x) \\ &= \begin{cases} c_{scr} + (h - \delta c_{scr}) \displaystyle\int_0^T e^{-\delta u} \mathbb{P}(N_0(u) \leqslant x) du \\ + \displaystyle\int_0^T e^{-\delta u} \lambda(u) [c_{se} + q c_{re} - (1-q) c_{scr} - c_a(u)] \mathbb{P}(N_0(u) = x) du. \end{cases}\end{aligned} \tag{4.57}$$

Under some additional conditions on the function c_a and the parameters c_{se}, c_{re}, c_{scr} and q, one can show that the function $x \mapsto c(x) + \widetilde{C}(x, T \wedge \sigma_x)$ is discrete convex. This simplifies the computation of an optimal procurement amount. The next result is a special case of Lemma 5.

Lemma 25. *If the procurement cost function c is discrete convex on $\mathbb{Z}_+$ and $c_a(T) \geqslant c_{se} + q c_{re} - (1-q) c_{scr}$, then the function $x \mapsto c(x) + \widetilde{C}(x, T \wedge \sigma_x)$ is discrete convex on $\mathbb{Z}_+$.*

Under the convexity structure of Lemma 25, an optimal procurement quantity is given by

$$x^*_{\widetilde{P}_{T\wedge\sigma}} := \min\{x \in \mathbb{Z}_+ : c(x+1) - c(x) + \Delta_x \widetilde{C}(x, T\wedge\sigma_x) \geqslant 0\} \tag{4.58}$$

with $\Delta\widetilde{C}(x, T\wedge\sigma_x)$ listed in relation (4.57).

In the next subsection we will analyze the last optimization problem discussed in Section 4.1.

4.6 Analysis of Optimization Problem $(P_{\mathbb{D}\wedge\sigma})$.

In this final subsection, we consider the problem of minimizing the cost function over the initial procurement amount $x \in \mathbb{Z}_+$ and policy switching time of the form $\tau\wedge\sigma_x$ for $\tau \in \mathbb{D}$, $0 \leqslant \tau \leqslant T$. That is, we fix a deterministic time τ, and the alternative policy is adopted at time τ or at the time the inventory is depleted, whichever occurs first. As shown in Section 4.1 the optimal solutions of optimization problem $(P_{\mathbb{D}\wedge\sigma})$ are the same as for optimization problem

$$\upsilon(\widetilde{P}_{\mathbb{D}\wedge\sigma}) = \inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{F},0\leqslant\tau\leqslant T}\{c(x) + \widetilde{C}(x,\tau\wedge\sigma)\}. \tag{$\widetilde{P}_{\mathbb{D}\wedge\sigma}$}$$

with $\widetilde{C}(x,\tau\wedge\sigma)$ given by (see relation 4.16)

$$\widetilde{C}(x,\tau\wedge\sigma_x) = \begin{cases} c_{scr}x + \int_0^\tau e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)]\mathbb{P}(N_0(u) < x)du \\ \\ +(h - \delta c_{scr})\int_0^\tau e^{-\delta u}\mathbb{E}((x - N_0(u))^+)du. \end{cases} \tag{4.59}$$

Also from relation (4.3) we know that

$$\upsilon(P_{\mathbb{D}\wedge\sigma}) = \upsilon(\widetilde{P}_{\mathbb{D}\wedge\sigma}) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{4.60}$$

Due to the previous remarks we will analyze in this section optimization problem $(\widetilde{P}_{\mathbb{D}\wedge\sigma})$ As in Section 4.4, this joint minimization problem can be solved in two stages; we first find the best procurement quantity for a given τ, and then search for the best τ. In the first stage, we follow the arguments of Section 4.5. The proofs for the first stage follow mostly by simple modification of those in Section 4.5 by replacing T with τ. For the second stage, we follow the footprints of the analysis of Section 4.4. To start our analysis we define

$$\Psi(\tau) := \inf_{x\in\mathbb{Z}_+}\{c(x) + \widetilde{C}(x, \tau \wedge \sigma_x)\}. \tag{$\widetilde{P}_{\Psi(\tau)}$}$$

It now follows that

$$\upsilon(\widetilde{P}_{\mathbb{D}\wedge\sigma}) = \inf_{0\leqslant\tau\leqslant T} \Psi(\tau). \tag{4.61}$$

Before discussing the solution procedure for the above optimization problem we first give some sufficient conditions on the cost functions and the parameters under which it is optimal not to order and start at time 0 with the alternative policy. The next result is a special case of Lemma 2.

Lemma 26. *If $c_a(0) \leqslant c_{se} + qc_{re} + (1-q)c^-_{scr}$ with c^-_{scr} given by relation (3.5), then it is optimal not to order in optimization problem $(P_{\mathbb{D}\wedge\sigma})$ and to start immediately at time 0 with the alternative policy. In this case the optimal objective value of problem $(P_{\mathbb{D}\wedge\sigma})$ is given by $\int_0^T e^{-\delta u}\lambda(u)c_a(u)du$.*

In almost all cases the sufficient conditions of Lemma 26 are not satisfied by our problem parameters and so we need to give an algorithm to solve optimization problem $(\widetilde{P}_{\mathbb{D}\wedge\sigma})$. In the general case we will use the approach as suggested by relation (4.61) and so part of the algorithm will consist of a procedure determining the optimal order size in case the switching time τ is known in advance. We can only execute this procedure by solving problem $(\widetilde{P}_{\Psi(\tau)})$ a finite number of times. This means that we first need to construct from the set $[0, T]$ a finite set $\mathcal{D} = \{\tau_1, \tau_2, ...\tau_N\}$

satisfying $0 \leqslant \tau_1 < ...\tau_N \leqslant T$ such that with a known error bound we can replace the optimization problem in relation (4.61) by its discrete version $\inf_{\tau \in \mathcal{D}} \Psi(\tau)$. To select this finite set $\mathcal{D}$ introduce the set

$$S := \{0 \leqslant \tau \leqslant T : c_a(\tau) \geqslant c_{se} + qc_{re} - (1-q)c_{scr}\}. \tag{4.62}$$

The next result yields some upperbound on the rate of growth of the function $\tau \mapsto \widetilde{C}(x, \tau \wedge \sigma_x)$ for a fixed $x \in \mathbb{Z}_+$.

Lemma 27. *For every $\tau, s \geqslant 0$ with $\tau + s \leqslant T$ and $x \in \mathbb{Z}_+$, we have*

$$\begin{aligned} & \widetilde{C}(x, (\tau+s) \wedge \sigma_x) - \widetilde{C}(x, \tau \wedge \sigma_x) \\ \geqslant \; & e^{-\delta\tau}(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau))(\Lambda(\tau+s) - \Lambda(\tau))1_S(\tau) \end{aligned} \tag{4.63}$$

with 1_S the indicator function of the set S defined in relation (4.41).

Proof. By relation (4.15) and $h - \delta c_{scr} \geqslant 0$ we obtain

$$\begin{aligned} & \widetilde{C}(x, (\tau+s) \wedge \sigma_x) - \widetilde{C}(x, \tau \wedge \sigma_x) \\ \geqslant \; & \mathbb{E}\left(\int_{\tau \wedge \sigma_x}^{(\tau+s)\wedge\sigma_x} e^{-\delta u}\lambda(u)(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u))du\right) \\ = \; & \mathbb{E}\left(\int_{\tau}^{(\tau+s)\wedge\sigma_x} e^{-\delta u}\lambda(u)(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)1_{\{\sigma_x > \tau\}}du\right). \end{aligned} \tag{4.64}$$

Since the function c_a is non-increasing it follows for τ not belonging to the set S that $c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u) \geqslant 0$ for every $u \geqslant \tau$. This implies by relation (4.64)

$$\widetilde{C}(x, (\tau+s) \wedge \sigma_x) - \widetilde{C}_\delta(x, \tau \wedge \sigma_x) \geqslant 0. \tag{4.65}$$

Using again c_a is non-increasing we obtain for every τ belonging to the set S and

$u \geqslant \tau$ that

$$e^{-\delta u}(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(u)) \geqslant e^{-\delta\tau}(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau))$$

and

$$e^{-\delta\tau}(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau)) \leqslant 0.$$

This shows by relation (4.64) that

$$\begin{aligned}
& \mathbb{E}\left(\int_\tau^{(\tau+s)\wedge\sigma_x} e^{-\delta\tau}\lambda(u)(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau))1_{\{\sigma_x>\tau\}}du\right) \\
\geqslant\ & e^{-\delta\tau}((c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau))\mathbb{E}\left(\int_\tau^{(\tau+s)\wedge\sigma_x} \lambda(u)1_{\{\sigma_x>\tau\}}du\right) \\
\geqslant\ & e^{-\delta\tau}((c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau)(\Lambda(\tau+s) - \Lambda(\tau)).
\end{aligned}$$

This shows the result. □

Applying Lemma 27 the following result follows immediately using a similar proof as in Lemma 18.

Lemma 28. *For every $\tau, s \geqslant 0$ with $\tau + s \leqslant T$ we have*

$$\Psi(\tau+s) - \Psi(\tau) \geqslant e^{-\delta\tau}(c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau))(\Lambda(\tau+s) - \Lambda(\tau))1_S(\tau). \quad (4.66)$$

An immediate consequence of Lemma 28 is given by the following important corollary.

Corollary 2. *The function $\tau \mapsto \Psi(\tau)$ is non-decreasing on the complement of the set S on $[0, T]$.*

An important consequence of the above corollary is given by the observation that an optimal solution of optimization problem $(\widetilde{P}_{\mathbb{D}\wedge\sigma})$ belongs to the set S and so we obtain

$$\upsilon(\widetilde{P}_{\mathbb{D}\wedge\sigma}) = \inf_{\tau\in S}\Psi(\tau). \tag{4.67}$$

Before applying the result of Lemma 28 to construct a finite set $\mathcal{D} \subset S$ having a predetermined error bound we first discuss solving optimization problem $(\widetilde{P}_{\Psi(\tau)})$ for some given τ belonging to the set S. This procedure can only be applied a finite number of times and so it is crucial to construct this finite set $\mathcal{D} \subset S$. In case the function c is not discrete convex one can show in a similar way as done in Lemma 29 with some obvious modifications the following upper bounding result.

Lemma 29. *There exists an optimal solution of problem* $(\widetilde{P}_{\Psi(\tau)})$ *and any optimal solution* $x^*_{\widetilde{P}_{\Psi(\tau)}}$ *of optimization problem* $(\widetilde{P}_{\Psi(\tau)})$ *satisfies* $x^*_{\widetilde{P}_{\Psi(\tau)}} \leqslant u_{\widetilde{P}_{\Psi(\tau)}}$ *with*

$$u_{\widetilde{P}_{\Psi(\tau)}} := \min\{x \in \mathbb{Z}_+ : k(x,\tau) > g_2(\infty,\tau)\} \ < +\infty \tag{4.68}$$

with the function g_2 *listed in relation* (4.54) *and the function* k *in relation* (4.22).

As for the other problems one can improve the above upper bound. The next result can also be verified in similar way as done in Lemma 12 with some obvious modifications.

Lemma 30. *If* $u_{\widetilde{P}_{\Psi(\tau)},0} := u_{\widetilde{P}_{\Psi(\tau)}}$ *and*

$$u_{\widetilde{P}_{\Psi(\tau)},n+1} := \min\{x \in \mathbb{Z}_+ : k(x,\tau) \geqslant g_2(u_{\widetilde{P}_{\Psi(\tau)},n},\tau)\}, \tag{4.69}$$

then the sequence $u_{\widetilde{P}_{\Psi(\tau)},n}$*,* $n \in \mathbb{Z}_+$ *is non-increasing, and any optimal solution* $x^*_{\widetilde{P}_{\Psi(\tau)}}$ *of optimization problem* $(\widetilde{P}_{\Psi(\tau)})$ *satisfies* $x^*_{\widetilde{P}_{\Psi(\tau)}} \leqslant u_{\widetilde{P}_{\Psi(\tau)},n}$ *for every* $n \in \mathbb{Z}_+$.

After at most $u_{\widetilde{P}_{\Psi(\tau)}}$ many iterations, the upper bound at an iteration will be the same as the previous one. Hence, the construction will stop at some upper bound $\underline{\mathrm{u}}_{\widetilde{P}_{\Psi(\tau)}}$, and we can search for the minimal procurement amount in $(\widetilde{P}_{\Psi(\tau)})$ over the set $\{0,1,\ldots,\underline{\mathrm{u}}_{\widetilde{P}_{\Psi(\tau)}}\}$. We leave the details to the reader.

If the procurement cost function c is discrete convex on $\mathbb{Z}_+$ we do not need to use the upper bounding procedure. The next convexity result is a special case of Lemma 5.

Lemma 31. *If the procurement function c is discrete convex on $\mathbb{Z}_+$ and $c_a(\tau) \geqslant c_{se} + qc_{re} - (1-q)c_{scr}$ or equivalently $\tau \in S$, then both the function $x \mapsto c(x) + C(x, \tau \wedge \sigma_x)$ and $x \mapsto c(x) + \widetilde{C}(x, \tau \wedge \sigma_x)$ are discrete convex on $\mathbb{Z}_+$.*

Proof. Apply relation (3.35). □

Lemma (31) eases the search for the minimal procurement amount for the problem $(\widetilde{P}_{\Psi(\tau)})$ for $\tau \in S$ under the discrete convexity of the procurement cost function c. The first order condition

$$x^*_{\widetilde{P}_{\Psi(\tau)}} := \min\{x \in \mathbb{Z}_+ : c(x+1) - c(x) + \Delta_x \widetilde{C}(x, \tau \wedge \sigma_x) \geqslant 0\} \tag{4.70}$$

in terms of the difference operator in (4.57) is simply sufficient, and no upper bound is needed. It is also easy to evaluate the objective value at $x^*_{\widetilde{P}_{\Psi(\tau)}}$ by observing

$$\widetilde{C}\left(x^*_{\widetilde{P}_{\Psi(\tau)}}, \tau \wedge \sigma_{x^*_{\widetilde{P}_{\Psi(\tau)}}}\right) = \sum\nolimits_{k=0}^{x^*_{\widetilde{P}_{\Psi(\tau)}}-1} \Delta_x \widetilde{C}(k, \tau \wedge \sigma_k). \tag{4.71}$$

Under the assumption of discrete convexity of the procurement cost function c stated in Lemma 31, one can establish sufficient conditions for which optimal order quantity $x^*_{\Psi(\tau)}$ is equal to zero for $\tau \in S$. Clearly, $x^*_{\Psi(\tau)} = 0$ if and only if $c_0(1) + \Delta\widetilde{C}(0, \tau \wedge \sigma_0) \geqslant 0$ by (4.70). Since $\mathbb{P}(N_0(u) = 0) = e^{-\Lambda_0(u)}$, $u \geqslant 0$, we have

$$\begin{aligned}\Delta\widetilde{C}(0, \tau \wedge \sigma_0) = c_{scr} + \int_0^\tau e^{-[\delta u + \Lambda_0(u)]}\Big[&\lambda(u)(c_{se} + qc_{re} - (1-q)c_{csr} - c_a(u)) \\ &h - \delta c_{scr}\Big] du.\end{aligned} \tag{4.72}$$

The following is now immediate from the integral term in (4.72). No proof is needed.

Lemma 32. *If the function* $\tau \mapsto \lambda(\tau)(c_{se} + qc_{re} - (1-q)c_{csr} - c_a(\tau))$ *is non-decreasing on* S*, then the function* $\tau \mapsto \Delta\widetilde{C}(0, \tau \wedge \sigma_0)$ *is unimodal, if not monotone, on* S*. Let us define*

$$\breve{u} = \inf\{\tau \in [0,T]\,;\, \lambda(\tau)(c_{se} + qc_{re} - (1-q)c_{csr} - c_a(\tau)) \geqslant 0\} \tag{4.73}$$

with the convention $\inf \varnothing = 0$*. Then,* $\tau \mapsto \Delta\widetilde{C}(0, \tau \wedge \sigma_0)$ *is non-increasing on* $[0, \breve{u}]$*, and it is non-decreasing on* $[\breve{u}, T]$*.*

Since on S the function $\tau \mapsto c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau)$ is non-positive and non-decreasing, a sufficient condition for the monotonicity assumption in Lemma 32 is to assume that the arrival rate function is decreasing. This is indeed a realistic assumption since the number of products/items (with a warranty contract) owned by customers is clearly non-increasing over time.

Corollary 3. *Recall that to avoid pathological cases we assume that* $c_0(x) + c_{scr}x$ *is increasing, hence we have* $c(1) + c_{scr} > 0$*. Then, if*

$$\Delta\widetilde{C}(0, \tau \wedge \sigma_0) - c_{scr} \geqslant 0 \tag{4.74}$$

we have $x^*_{\Psi(\tau)} = 0$ *and* $\Psi(\tau) = 0$*. Under the stronger condition* $\Delta\widetilde{C}(0, \breve{u} \wedge \sigma_0) - c_{scr} \geqslant 0$*, the inequality* (4.74) *holds for every* $\tau \in S$*, and* $x^*_{\widetilde{P}_{\Psi(\tau)}} = 0$*, and* $\Psi(\tau) = 0$ *for every* $\tau \in S$ *again.*

We now conclude our analysis with the following algorithm to solve optimization problem $(\widetilde{P}_{\mathbb{D}\wedge\sigma})$. As in the optimization problem $(\widetilde{P}_{\mathbb{D}})$, we discretize the space S to search for a value of τ within a given computational tolerance level $\varepsilon > 0$. For that we define the set $\mathcal{D} := \{\tau_1, \ldots, \tau_{\hat{N}}\} \subseteq S$ where $\tau_1 = 0$, and for $i \geqslant 1$,

$$\tau_{i+1} = \min\left\{ s > \tau_i \,:\, e^{-\delta\tau_i}[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau_i)](\Lambda(s) - \Lambda(\tau_i)) \leqslant -\varepsilon \right\}. \tag{4.75}$$

Here, $\widehat{N}$ is the iteration number for which the relation (4.75) exceeds $\sup S$, and at which we set $\tau_{\widehat{N}} = \sup S$. By Lemma 28, we have for every $s \in [\tau_i, \tau_{i+1}]$

$$\Psi(s) - \Psi(\tau_i) \geqslant e^{-\delta\tau}[c_{se} + qc_{re} - (1-q)c_{scr} - c_a(\tau)](\Lambda(\tau + s) - \Lambda(\tau)),$$

which further implies that

$$\min_{\tau_i \in \mathcal{D}} \Psi(\tau_i) \geqslant \upsilon(P_{\widetilde{P}_{\Psi(\tau)}}) \geqslant \min_{\tau_i \in \mathcal{D}} \Phi(\tau_i) - \varepsilon.$$

Hence, the error associated with searching the best τ in the set $\mathcal{D}$ is no more that the given error level $\varepsilon > 0$.

Algorithm 4. *Numerical Algorithm to Solve Optimization Problem ($\widetilde{P}_{\mathbb{D}\wedge\sigma}$):*

1. *Select $\epsilon > 0$ and construct on the compact interval S the finite discretization $\mathcal{D} = \{\tau_1, \ldots, \tau_{\widehat{N}}\}$ with $\tau_1 = 0$ and τ_i's are as defined using (4.75). The construction (4.75) continues until $\sup S$ is exceeded, and we set $\tau_{\widehat{N}} = \sup S$.*

2. *For every $\tau_i \in \mathcal{D}$, if the function c is convex (see Lemma 31), find $x^*_{\widetilde{P}_{\Psi(\tau)}}$ using (4.70) and evaluate $\Psi(\tau_i)$ via (4.71). Otherwise find the smallest upper bound described in $\underline{u}_{\widetilde{P}_{\Psi(\tau)}}$ and compute*

$$x^*_{\widetilde{P}_{\Psi(\tau)}} = \arg\min_{x \leqslant \underline{u}_{\widetilde{P}_{\Psi(\tau_i)}}} \{c(x) + \widetilde{C}(x, \tau_i \wedge \sigma_x)\}$$

with

$$\Psi(\tau_i) = c(x^*_{\tilde{P}_{\Psi(\tau)}}) + \widetilde{C}\Big(x^*_{\tilde{P}_{\Psi(\tau_i)}}, \tau_i \wedge \sigma_{x^*_{\tilde{P}_{\Psi(\tau_i)}}}\Big).$$

3. *Find the value of* τ_i^* *attaining* $\min_{\tau_i \in \mathcal{D}} \Psi(\tau_i)$*, and output* $(x^*_{\tilde{P}_{\Psi(\tau_i)}}, \tau_i^*)$ *as an* ϵ-*optimal solution.*

Using now the lower bound in Chapter 3 one can also set a predetermined relative error.

4.7 Numerical Examples

In this section, we give examples for the problems analyzed in the Sections 4.3 - 4.6. For that, we adopt the experimental setup discussed in Pourakbar et al. [2012], which originated from a case study of a well-known supplier of consumer electronics products in the European market. The product under consideration is the cathode ray tube (CRT) which is an essential component of old generation TV sets or monitors in the 1990s. Due to the introduction of liquid crystal display (LCD), plasma and organic light emitting diode (OLED) screens, CRTs have become obsolete and their production has been terminated. So, for the company the end-of-life problem of this service part is an essential challenge.

The parameters and their values for the base case scenario are summarized in Table 4.1.

The planning horizon (or the time all warranties expire) is taken as $T = 66$ months. The procurement cost function is linear and has the form $c(x) = c_p x$ for some constant c_p. The cost of the alternative policy/product decays exponentially and is given by $c_a(u) = c_a e^{-\gamma u}$ for some constants c_a and γ. The penalty cost of using the alternative policy stays steady and is given by a constant p. It simply represents

Table 4.1: Parameter setting for the base case scenario

Notation	Definition	Cost
c_p	Procurement cost per item	225
h	Holding cost per item per time	3.25
c_{se}	Service cost per item	30
c_{re}	Repair cost per repairable item	20
p	Penalty cost per item	100
c_a	Alternative policy cost per item	645
c_{scr}	Scrapping cost per item	30
γ	Price erosion factor per month	0.02
δ	Discounting factor per month	0.005
q	Repair probability of an item	0.5

the additional costs associated with an emergency order for the alternative product. As in Pourakbar et al. [2012], we assume that the customer arrival process N (see (3.1)) has the arrival intensity $\lambda(u) = au^2e^{-bu}$, $u \geqslant 0$. This implies

$$\Lambda(u) = \int_0^u \lambda(s)ds = \frac{2a}{b^3}\left[1 - e^{-bu}\left(1 + bu + \frac{b^2u^2}{2}\right)\right]. \tag{4.76}$$

Below, we set $b = 1$, and for a we consider two different values 100 and 1000. For $a = 100$ the expected total demand over $[0, 66]$ is $\Lambda(66) \approx 200$, and for $a = 1000$ it is approximately 2000 (the expected demand is proportional to a). Tables 4.2 and 4.3 report our results for the various choices of the problem parameters. Table 4.2 summarizes our results for the case $a = 100$, and Table 4.3 is for $a = 1000$. In constructing these tables, we select the base case scenario whose parameter values are given in Table 4.1. In each row of Tables 4.2 and 4.3, consisting of three lines, we increment and decrement the value of one of the parameters by several folds (compared to the base case) while fixing the others. In Tables 4.2 and 4.3 the lines with $\delta = 0.005$ correspond to our base case scenario. Column headers in Tables 4.2 and 4.3 are the labels of the problems discussed in Sections 4.3 - 4.6. For a given parameter set in these tables, the first number indicates the optimal initial order quantity and the second is the minimal expected total cost. In columns/problems $(P_{\mathbb{D}})$ and $(P_{\mathbb{D}\wedge\sigma})$ the optimal τ values are given as the last entries. Recall that we apply the Algorithms 1 and 4 to solve the problems $(\widetilde{P}_{\mathbb{D}})$ and $(\widetilde{P}_{\mathbb{D}\wedge\sigma})$ respectively. In $(P_{\mathbb{D}})$ we know that the value L given in (3.41) yield a lower bound for the minimal expected cost when $p(u) \geqslant c_{se}$ for all $u \in [0,T]$ (as in the examples that we consider in this section). Moreover, it is straightforward to verify that L is proportional to a. Therefore in Algorithm 1 we set $\epsilon \leqslant (L/a)/100$ to fill in the entries of $(P_{\mathbb{D}})$ in Tables 4.2 and 4.3. The number L does not act as a lower bound for the minimal cost in $(\widetilde{P}_{\mathbb{D}\wedge\sigma})$ since the pair $(0,\tau)$ is not a feasible solution in that problem. Hence, in Algorithm 4 we simply set $\varepsilon \leqslant c_p/1000$ to obtain the results in Tables 4.2 and 4.3. Also, in most of the considered cases, the convexity conditions of Lemmas 15, 21, 25, and 31 hold. Therefore the optimal order quantities are easily identified using first order conditions. In the remaining cases, the order quantity is searched over the set of values restricted with an upperbound; see Lemmas 12, 24, 20, and 30.

Table 4.2: Optimization problem (a,b)=(100,1) (x^*, Cost, τ^*)

	P_T	$P_{T\wedge\sigma}$	$P_{\mathbb{D}}$	$P_{\mathbb{D}\wedge\sigma}$
$\delta = 0.001$	(99, 34754.0)	(104, 36297.5)	(102, 34155.0, 12.70)	(106, 35149.2, 11.90)
$\delta = 0.005$	(99,34561.0)	(104, 35918.9)	(101, 33984.7, 12.85)	(106, 34984.3, 11.85)
$\delta = 0.025$	(99, 33640.2)	(104, 34516.8)	(100, 321585.3, 13.00)	(105, 34172.6, 11.60)
$\gamma = 0.05$	(98, 34090.2)	(102, 35349.9)	(100, 33611.4, 12.40)	(104, 34566.2, 11.45)
$\gamma = 0.09$	(95, 33476.9)	(100, 34614.7)	(97, 33115.4, 11.70)	(102, 34010.5, 10.90)
$\gamma = 0.13$	(93, 32886.8)	(97, 33826.5)	(95, 32621.0, 10.95)	(99, 33401.2, 10.40)
$q = 0.2$	(159, 48283.0)	(159, 48437.0)	(162, 47702.3, 11.90)	(162, 47702.3, 11.90)
$q = 0.4$	(119, 39157.7)	(122, 40110.0)	(121, 38525.9, 12.65)	(125, 39262.4, 11.85)
$q = 0.8$	(40, 20511.4)	(46, 22728.6)	(41, 20147.7, 12.85)	(48, 21768.4, 11.80)
$c_p = 100$	(103, 21923.4)	(107, 22774.1)	(107, 21004.9, 12.35)	(110, 21484.1, 12.00)
$c_p = 350$	(95, 46724.5)	(101, 48695.1)	(97,46367.5, 12.95)	(103, 48049.2, 11.35)
$c_p = 450$	(92, 56088.8)	(99, 58700.9)	(93, 55852.2, 12.95)	(100, 58193.7, 11.05)
$c_a = 345$	(93, 32989.9)	(97, 34034.6)	(95, 32743.6, 12.05)	(99, 33613.2 , 10.45)
$c_a = 945$	(102, 35485.3)	(107, 36908.8)	(105, 34671.0, 13.00)	(109, 35685.5, 12.30)
$c_a = 1245$	(104, 36135.4)	(109, 37579.1)	(107, 35139.6, 13.15)	(111, 36155.7, 12.65)
$c_{se} = 0$	(100, 28778.0)	(104, 30126.3)	(102, 28168.4, 13.00)	(106, 29161.2, 11.95)
$c_{se} = 60$	(99, 40339.1)	(103, 41655.9)	(101, 39794.3, 12.20)	(106, 40806.9, 11.80)
$c_{se} = 90$	(98, 46107.5)	(103, 47429.5)	(100, 45603.7, 11.70)	(105, 46582.3, 11.70)
$h = 5$	(98, 35347.5)	(102, 36938.0)	(101, 34560.2, 12.35)	(106, 35656.7, 11.70)
$h = 10$	(95, 37299.1)	(99, 39543.1)	(99, 36148.4, 12.00)	(104, 37425.5, 11.00)
$h = 15$	(93 ,38981.9)	(97, 41782.2)	(98, 37670.1, 12.00)	(103, 39156.2, 10.55)
$c_{re} = 0$	(99, 32590.7)	(104, 33988.0)	(101, 32014.9, 12.85)	(106, 33043.1, 11.90)
$c_{re} = 40$	(99, 36531.3)	(104, 37849.8)	(101, 35954.7, 12.65)	(106, 36925.1, 11.85)
$c_{re} = 60$	(99, 38501.6)	(103, 39731.3)	(101, 37924.9, 12.45)	(106, 38866.2, 11.80)
$c_{scr} = 0$	(100, 34483.2)	(104, 35766.8)	(102, 33844.6, 12.65)	(107, 34772.3, 11.80)
$c_{scr} = 40$	(99, 34586.9)	(104, 35969.6)	(101, 34027.3, 12.85)	(106, 35064.8, 11.85)
$c_{scr} = 60$	(99, 34638.8)	(103, 36019.8)	(101, 34112.4, 12.90)	(105, 35186.3, 11.90)
$p = 0$	(97, 34084.4)	(104, 35918.9)	(100, 33622.2, 13.00)	(106, 34984.3, 11.85)
$p = 300$	(102, 35296.2)	(104, 35918.9)	(104, 34522.2, 12.45)	(106, 34984.3, 11.85)
$p = 500$	(104, 35852.2)	(104, 35918.9)	(106, 34917.0, 12.25)	(106, 34984.3, 11.85)

The results in Tables 4.2 and 4.3 numerically illustrate the sensitivity of the solutions with respect to the problem parameters. For example, as expected, we observe that the total costs are decreasing when the discount factor δ increases. We see that the order quantities are non-increasing in δ. Roughly speaking, we order less at time zero to protect ourselves against future costs when these costs are discounted at a heavier rate. On the other hand, when the price erosion factor γ increases, we know that the price of the alternative product decreases faster. As a result, we observe that we switch to the alternative policy earlier in problems ($P_{\mathbb{D}}$) and ($P_{\mathbb{D}\wedge\sigma}$) as expected, and this causes us to order less initially. In problem ($P_{T\wedge\sigma}$), we also order less because the lower the initial inventory is, the sooner it will be depleted and the alternative policy will be adopted. In problem (P_T), we order less again since an unscheduled use of the alternative product in the future has a lower cost impact. Compared to the erosion factor γ, we see that the total expected cost is more sensitive to the repair probability q. In particular, when q increases, on the average more products will be repairable and we therefore need to order less to satisfy the customer requests during the repair replacement phase. This in turn decreases the total cost. Note that lower initial inventory does not necessarily imply an earlier switching time since it may be cost efficient to take advantage of the repairability of the items. The sensitivity in c_p is also as expected. The higher c_p is, the higher the total expected costs and the lower the initial order quantities are. We observe the opposite effect for c_a on the order quantity, since the alternative product becomes more expensive as c_a increases, we switch to the alternative policy later and therefore we order more initially. Obviously the total cost is increasing in c_a.

Recall that in the repair replacement policy, the service cost c_{se} is incurred if an arriving item is repairable or there is a spare item in inventory if it is not. Clearly, when we increase the value of c_{se} we observe that the total cost increases and the initial order quantity decreases in each problem. In our base case scenario we have $q = 0.5$, which means on the average half of the arriving items are repairable. As a result, c_{se} has a significant impact on the total costs in our examples. Compared to c_{se}, the repair cost c_{re} has a similar effect on the total cost and on the order quantity x. However, its effect on the total cost is relatively less since it applies to repairable items only. The sensitivity of our solutions to the holding cost is as expected. As h increases, the total cost increases, and so we order less. Compared to other parameters, we observe that the solutions reported in Table 4.2 are less sensitive to h. The impact of the cost of scrapping c_{scr} is also as expected, and the solutions are not highly sensitive to it. We believe that this is mainly because of the fact in the

Table 4.3: Optimization problem (a,b)=(1000,1) (x^*, Cost, τ^*)

	P_T	$P_{T\wedge\sigma}$	$P_{\mathbb{D}}$	$P_{\mathbb{D}\wedge\sigma}$
$\delta = 0.001$	(996, 324613.4)	(1011, 329315.5)	(1004, 322866.3, 14.10)	(1018, 325978.5, 13.60)
$\delta = 0.005$	(996, 323301.7)	(1011, 327431.0)	(1004, 313645.4, 14.10)	(1018, 324704.8, 13.65)
$\delta = 0.025$	(996, 317022.5)	(1011, 319598.5)	(999, 315458.3, 14.70)	(1015, 318611.7, 13.95)
$\gamma = 0.05$	(990, 321450.7)	(1004, 325292.6)	(996, 320113.9, 13.80)	(1011, 323108.9, 13.30)
$\gamma = 0.09$	(980, 319020.1)	(994, 322306.2)	(986, 318042.2, 13.70)	(1001, 320818.5, 12.80)
$\gamma = 0.13$	(969, 316624.5)	(982, 319115.3)	(973, 316030.3, 12.00)	(988, 318247.2, 12.40)
$q = 0.2$	(1595, 454789.6)	(1597, 455216.1)	(1603, 452638.8, 15.00)	(1606, 453064.3, 13.65)
$q = 0.4$	(1196, 367200.2)	(1207, 370157.9)	(1203, 365326.6, 14.70)	(1214, 367605.0, 13.65)
$q = 0.8$	(398, 190799.6)	(420, 197617.9)	(402, 189719.1, 14.90)	(424, 194615.8, 13.75)
$c_p = 100$	(1009, 197963.8)	(1020, 200491.8)	(1019, 195234.7, 14.70)	(1031, 196726.2, 14.00)
$c_p = 350$	(984, 447079.6)	(1002, 453245.3)	(988, 446028.7, 14.55)	(1008, 451284.1, 13.80)
$c_p = 450$	(972, 544870.0)	(996, 553163.0)	(974, 544162.8, 14.95)	(1000, 551662.2, 13.65)
$c_a = 345$	(975, 318267.4)	(990, 321530.5)	(980, 317551.2, 14.00)	(997, 320293.5, 13.25)
$c_a = 945$	(1007, 326208.9)	(1021, 330529.0)	(1013, 323781.2, 15.00)	(1027, 326910.8, 13.95)
$c_a = 1245$	(1013, 328240.5)	(1027, 332589.9)	(1020, 325284.2, 15.50)	(1033, 328368.6, 14.15)
$c_{se} = 0$	(998, 264603.2)	(1012, 268767.0)	(1004, 262832.7, 16.00)	(1019, 265918.0, 13.70)
$c_{se} = 60$	(995, 381972.9)	(1009, 386051.1)	(1001, 380357.2, 14.15)	(1016, 383457.5, 13.60)
$c_{se} = 90$	(993, 440617.1)	(1008, 444650.0)	(1000, 439104.9, 14.05)	(1015, 442193.8, 13.55)
$h = 5$	(992, 329254.3)	(1007, 334226.5)	(1001, 326953.7, 14.50)	(1016, 330334.8, 13.15)
$h = 10$	(982, 345287.0)	(997, 352376.1)	(995 , 342061.9, 14.50)	(1012, 346198.2, 12.40)
$h = 15$	(974, 36040.1)	(990, 369325.5)	(989, 356861.0, 14.35)	(1008, 361788.7, 12.00)
$c_{re} = 0$	(997, 303598.8)	(1011, 307875.4)	(1003, 301883.3, 14.20)	(1018, 305108.3, 13.65)
$c_{re} = 40$	(997, 343004.7)	(1010, 346970.5)	(1003, 341286.2, 14.10)	(1017, 344385.0, 13.60)
$c_{re} = 60$	(997, 362707.7)	(1010, 366519.1)	(1003, 360987.6, 14.10)	(1017, 363879.5, 13.55)
$c_{scr} = 0$	(997, 323053.3)	(1012 ,327014.8)	(1005, 321170.4, 14.05)	(1020, 324022.7, 13.50)
$c_{scr} = 40$	(996, 323381.5)	(1011, 327572.8)	(1002, 321717.3, 14.20)	(1017, 324920.0, 13.70)
$c_{scr} = 60$	(996, 323540.8)	(1010, 327833.5)	(1001, 321971.4, 14.30)	(1016, 325348.9, 13.80)
$p = 0$	(990, 321730.3)	(1011, 327431.0)	(997, 320374.9, 14.45)	(1018, 324704.8, 13.65)
$p = 300$	(1005, 325677.8)	(1011, 327431.0)	(1012, 323354.2, 14.00)	(1018, 324704.8, 13.65)
$p = 500$	(1011, 327444.8)	(1011, 327431.0)	(1017, 324627.2, 13.60)	(1018, 324704.8, 13.65)

optimal solution the initial order is set so that a considerable portion of this inventory will be used by the time the alternative policy is adopted, or if not, by the end of the horizon T.

Chapter 5

On Dynamic Policies

In this chapter, we study the dynamic policies for end-of-life inventory problem. In this framework, the final order quantity is a static variable and its value is determined initially at the beginning of the final phase as before. The switching time, on the other hand, is a stopping time of the arrival process. In this chapter, we re-work on some of the cost expressions to make the analysis easier. Then, a lower bound on the objective function is given, and an easily computable upper bound on the optimal order quantity is driven. After that we discuss how to approximate the continuous stopping problem with a discrete one, and try to solve this discrete optimal stopping problem by using the standard arguments of discrete time dynamic programming. We consider the case where the intensity function of the non-homogenous Poisson process is piecewise constant. Finally, we apply our method on some numerical examples and conduct some sensitivity analysis over the problem parameters, and also compare our optimal dynamic policy with the static policies of Teunter and Fortuin [1998] and with one of the static policies studies in Chapter 4.

5.1 Introduction

To solve any optimal stopping time problem, one can consider two classes of stopping times, and they yield two different formulations requiring different techniques. In the first one, the switching decision can be made at any point in time. This gives rise to a *continuous time optimal stopping problem* which is relatively more difficult to analyze; see Gugerui [1986] and Davis [1993] for the theory of optimal stopping for piecewise deterministic Markov process. In the second formulation, on the other hand, the stopping times are assumed to take only values on some pre-determined discrete set $\Theta = \{0, t_1, t_2, \ldots, T\}$, where T denotes the time when all service obligations expire. This second formulation is called a *discrete time optimal stopping problem* which is easier to analyze as it requires standard tools of discrete time Markov dynamic programming; see for example Bertsekas [1995]. Since in our problem it is possible to give a computable error bound beforehand replacing the continuous time optimal stopping problem by a discrete time optimal stopping problem and compute the size of the mesh of the set Θ, we will use the second approach to solve the most general formulation given by optimization problem $(P_{\mathbb{F}})$ of the end-of life inventory problem.

To start with the analysis of this problem we recall using relation (3.20) in Chapter 3 that the expected discounted operation cost of an arbitrary (x,τ)-policy, $x \in \mathbb{Z}_+$, $\tau \in \mathbb{F}$, $0 \leqslant \tau \leqslant T$ is given by

$$C(x,\tau) = \begin{cases} c_{scr}x + (1-q)\mathbb{E}\left(\int_0^{\tau\wedge\sigma_x} e^{-\delta u}\lambda(u)[c_{se} - c_{scr} - c_{ap}(u)]du\right) \\ +\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}\lambda(u)[q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u)]du\right) \\ +(h - \delta c_{scr})\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}((x - N_0(u))^+)du\right) + \int_0^T e^{-\delta u}c_a(u)\lambda(u)du, \end{cases} \tag{5.1}$$

and we need to solve the optimization problem $(P_{\mathbb{F}})$ given by

$$\upsilon(P_{\mathbb{F}}) = \inf_{x\in\mathbb{Z}_+, \tau\in\mathbb{F}, \tau\leqslant T}\{c(x) + C(x,\tau)\}. \tag{5.2}$$

In relation (3.22) in Chapter 3 we introduced the function

$$\widetilde{C}(x,\tau) := C(x,\tau) - \int_0^T e^{-\delta u}\lambda(u)c_a(u)du \tag{5.3}$$

and so the set of optimal solutions of optimization problem $(P_{\mathbb{F}})$ is the same as the set of optimal solutions of optimization problem

$$\upsilon(\widetilde{P}_{\mathbb{F}}) = \inf_{x\in\mathbb{Z}_+,\tau\in\mathbb{F}}\{c(x) + \widetilde{C}(x,\tau)\}. \tag{$\widetilde{P}_{\mathbb{F}}$}$$

Also it follows that

$$\upsilon(P_{\mathbb{F}}) = \upsilon(\widetilde{P}_{\mathbb{F}}) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du. \tag{5.4}$$

In this chapter (as in the previous chapter for static policies) we solve the equivalent optimization problem $(\widetilde{P}_{\mathbb{F}})$. To do so, we use a bi-level approach and introduce for every $x \in \mathbb{Z}_+$ the optimization problem

$$\widetilde{\varphi}(x) := \inf_{\tau\in\mathbb{F},\tau\leqslant T}\{c(x) + \widetilde{C}(x,\tau)\} \tag{$\widetilde{P}_{\mathbb{F}}(x)$}$$

and observe that solving optimization problem $(\widetilde{P}_{\mathbb{F}})$ is the same as solving problem

$$\inf_{x\in\mathbb{Z}_+}\{\widetilde{\varphi}(x)\} \tag{5.5}$$

For every x the minimization problem $(\widetilde{P}_{\mathbb{F}}(x))$ can be considered as continuous time optimal stopping problem with initial inventory level x in which all the costs are accumulated until the policy switching time τ; beyond τ no additional cost is incurred.

5.2 An Upperbound on the Optimal Order Quantity

In this section we derive a computable upper bound on the optimal order quantity for the general version of the end-of-life problem. Note in this section we always assume our basic assumptions given at the end of Section 3.1. hold. To start with deriving such a bound introduce the function $g_* : \mathbb{Z}_+ \to \mathbb{R}$ given by

$$g_*(x) := \inf_{\tau \in \mathbb{F}} g(x, \tau) \tag{5.6}$$

with the function g listed in relation (3.36). Recall that

$$g(x,\tau) = \begin{cases} c(x) + c_{scr}x + c_{scr}^{-}(1-q)\mathbb{E}(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du) \\ \\ +\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - c_a(u)]du\right) + \int_0^T e^{-\delta u}\lambda(u)c_a(u)du \end{cases} \tag{5.7}$$

with c_{scr}^{-} listed in relation (3.5). Since by Lemma 6 the function $x \to g(x,\tau)$ is non-decreasing for every $\tau \in \mathbb{F}$ it follows that the function g_* is non-decreasing. Also by the same lemma we obtain for every $\tau \in \mathbb{F}$ that

$$c(x) + C(x,\tau) \geqslant g(x,\tau). \tag{5.8}$$

This shows

$$\inf_{\tau\in\mathbb{F}}\{c(x) + C(x,\tau)\} \geqslant \inf_{\tau\in\mathbb{F}} g(x,\tau) = g_*(x) \tag{5.9}$$

To obtain an elementary expression for g_* we observe by the definition of $g(x,\tau)$ that

$$g_*(x) = \begin{cases} c(x) + c_{scr}x + c_{scr}^{-}\mathbb{E}\left(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right) \\ \\ +\int_0^T e^{-\delta u}\lambda(u)c_a(u)du + \inf_{\tau\in\mathbb{F}}\left\{\mathbb{E}\left(\int_0^{\tau} e^{-\delta u}\lambda(u)[c_{se} + qc_{re} - c_a(u)]du\right)\right\}. \end{cases}$$

As in the last part in the proof of Lemma 6 one can show that

$$\inf_{\tau\in\mathbb{F}}\left\{\mathbb{E}\left(\int_0^\tau e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du\right)\right\} = \inf_{\tau\in\mathbb{D}}\left\{\int_0^\tau e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du\right\}$$

$$= \int_0^\varsigma e^{-\delta u}\lambda(u)[c_{se}+qc_{re}-c_a(u)]du$$

with ς listed in relation (3.38). Hence we obtain that

$$g_*(x) = \begin{cases} c(x) + c_{scr}x + c_{scr}^{-}\mathbb{E}\left(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right) \\ +(c_{se}+qc_{re})\int_0^\varsigma e^{-\delta u}\lambda(u)du + \int_\varsigma^T e^{-\delta u}\lambda(u)c_a(u)du. \end{cases} \tag{5.10}$$

Introducing now the function $g:\mathbb{Z}_+\to\mathbb{R}$ given by

$$g(x) = \begin{cases} c(x) + c_{scr}x + c_{scr}^{-}\mathbb{E}\left(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right) \\ +\int_0^T e^{-\delta u}\lambda(u)\min\{c_{se}+qc_{re}, c_a(u)\}du \end{cases} \tag{5.11}$$

it follows easily from relation (5.10) that

$$g_*(x) \geqslant g(x) \tag{5.12}$$

for every $x\in\mathbb{Z}_+$. Also since we know that $x\mapsto c(x)+c_{scr}x+c_{scr}^{-}\mathbb{E}\left(\int_0^{T\wedge\sigma_x} e^{-\delta u}\lambda(u)du\right)$ is nondecreasing (see proof of Lemma 6) we obtain that the function g is nondecreasing and again by our standard assumptions $g(\infty)=\infty$. One can now show the following result.

Lemma 33. *Let $(\overline{x},\overline{\tau})$ be an arbitrarily selected policy with $\overline{\tau}\in\mathbb{D}$, and for such a policy define*

$$x_U(\overline{x},\overline{\tau}) := \min\{x\in\mathbb{Z}_+ : g(x) > c(\overline{x}) + C(\overline{x},\overline{\tau})\}. \tag{5.13}$$

Then any optimal order quantity of optimization problem $(P_{\mathbb{F}})$ is bounded above by $x_U(\overline{x},\overline{\tau})$.

Proof. By relations (5.8) and (5.12) it follows for a policy (x, τ) with $x > x_U(\overline{x}, \overline{\tau})$ that $c(x) + C(x, \tau) \geqslant g(x) > c(\overline{x}) + C(\overline{x}, \overline{\tau}) \geqslant v(P_{\mathbb{F}})$. This shows the result. $\square$

Clearly the same result also applies to problem $(\widetilde{P}_{\mathbb{F}})$ since both problems have the same set of optimal solutions.

An obvious choice for the policy $(\overline{x}, \overline{\tau})$ in Lemma 33 is $\overline{x} = 0$ and $\overline{\tau} = 0$. For this policy we obtain

$$c(\overline{x}) + C(\overline{x}, \overline{\tau}) = C(0,0) = \int_0^T e^{-\delta u} \lambda(u) c_a(u) du. \tag{5.14}$$

If we select another policy $(\overline{x}, \overline{\tau})$, $\overline{\tau} \in \mathcal{F}_0$, with a lower expected cost than $C(0,0)$, it follows by Lemma 33 that the value $x_U(\overline{x}, \overline{\tau})$ for that selected policy will be smaller. This improves the computational efficiency of our proposed algorithm. However it may require additional computational time to obtain such an alternative policy. In the remainder, we assume that the policy $(\overline{x}, \overline{\tau})$ is fixed and leave the exact choice open. Applying Lemma 33 it is equivalent to solve the optimal stopping problem

$$\widetilde{\varphi}(x) := \inf_{\tau \in \mathbb{F}, \tau \leqslant T} \{c(x) + \widetilde{C}(x, \tau)\} \tag{$\widetilde{P}_{\mathbb{F}}(x)$}$$

for each $x = 0, 1, ..., x_U(\overline{x}, \overline{\tau})$. In the next section we will propose a discretization scheme to approximately solve this continuous time optimal stopping problem by a discrete time optimal stopping problem.

5.3 An Approximation Argument

Let Θ be a discrete set of time points $\{t_0, ..., t_N\} \subset [0, T]$ with $t_0 = 0$, $t_N = T$, and mesh

$$\Delta := \max_{0 \leqslant j \leqslant N-1} \mid t_{j+1} - t_j \mid . \tag{5.15}$$

The stopping times taking values in Θ are denoted by $\tau \in \Theta$. Also define

$$\widetilde{\varphi}_{\Theta}(x) := \inf_{\tau \in \Theta}\{c(x) + \widetilde{C}(x,\tau)\}. \qquad (\widetilde{P}_{\Theta}(x))$$

The function $\widetilde{\varphi}_{\Theta}(x)$ can be regarded as an approximation for $\widetilde{\varphi}(x)$ in problem $\widetilde{P}_{\mathbb{F}}(x)$, and it is immediate that $\widetilde{\varphi}_{\Theta}(x) \geqslant \widetilde{\varphi}(x)$. The quality of the approximation is determined by the selection of the set Θ. To measure the approximation error, let us introduce the supnorm of a function f on $[0,T]$ as

$$\| f \|_{\infty} := \sup_{0 \leqslant u \leqslant T} | f(u) | . \qquad (5.16)$$

Using the modified cost expression for $\widetilde{C}(x,\tau)$ in relation (5.1) and (5.3)and introducing for notational convenience the functions $f_i : [0,T] \to \mathbb{R}, i = 1,2$ given by

$$f_1(u) := (1-q)(c_{se} - c_{scr} - c_{ap}(u)), u \geqslant 0 \qquad (5.17)$$

and

$$f_2(u) := q(c_{se} + c_{re} - c_a(u)) + (1-q)p(u), u \geqslant 0, \qquad (5.18)$$

one can show the following result.

Lemma 34. *If $f_0 : \mathbb{Z}_+ \mapsto \mathbb{R}$ is given by*

$$f_0(x) := (h - \delta c_{scr})x + \| \lambda f_1 \|_{\infty} + \| \lambda f_2 \|_{\infty} \qquad (5.19)$$

with λ denoting the arrival intensity function of the Poisson arrival process and f_i, $i = 1,2$, listed in relation (5.17) and (5.18) then for the set $\Theta = \{t_0, ..., t_N\}$ with mesh Δ in (5.15) *it follows*

$$0 \leqslant \widetilde{\varphi}_{\Theta}(x) - \widetilde{\varphi}(x) \leqslant f_0(x)\Delta \qquad (5.20)$$

for every $x \in \mathbb{Z}_+$ and $\delta \geqslant 0$.

Proof. If the discrete set Θ is given by $\Theta = \{t_0, ..., t_N\}$ with $t_0 = 0$ and $t_N = T$ and mesh Δ it is easy to see that for every stopping time $\tau \in \mathcal{F}, \tau \leqslant T$ there exists a stopping time $\tau_\theta \in \theta$ satisfying $0 \leqslant \tau_\theta - \tau \leqslant \Delta$ with probability 1. As an example take $\tau_\theta = d_\theta(\tau)$ with the function $d_\theta : [0, T] \to \mathbb{R}$ given by

$$d_\theta(r) = \sum\nolimits_{k=0}^{N-2} t_{k+1}\, 1_{[t_k, t_{k+1})}(r) + T\, 1_{[t_{N-1}, T]}(r).$$

Using these two stopping times it is sufficient to show for every $x \in \mathbb{Z}_+$ and $\delta \geqslant 0$ that

$$|\ \widetilde{C}(x, \tau_\theta) - \widetilde{C}(x, \tau)\ | \leqslant f_0(x)\Delta \tag{5.21}$$

with f_0 defined in relation (5.19). To verify this inequality it follows by relations (5.1) and (5.3) for every $\delta \geqslant 0$ that

$$|\ \widetilde{C}(x, \tau_\theta) - \widetilde{C}(x, \tau)\ | \leqslant \begin{cases} \left|\mathbb{E}\left(\int_{\tau \wedge \sigma_x}^{\tau_\theta \wedge \sigma_x} e^{-\delta u} \lambda(u) f_1(u) du\right)\right| + \left|\mathbb{E}\left(\int_{\tau}^{\tau_\theta} e^{-\delta u} \lambda(u) f_2(u) du\right)\right| \\ \\ +(h - \delta c_{scr}) \left|\mathbb{E}\left(\int_{\tau}^{\tau_\theta} e^{-\delta u} (x - N_0(u))^+ du\right)\right| \end{cases} \tag{5.22}$$

Since $0 \leqslant (x - N_0(u))^+ \leqslant x$ and $0 \leqslant \tau_\theta - \tau \leqslant \Delta$ with probability 1 we obtain for every $\delta \geqslant 0$

$$\left|\mathbb{E}\left(\int_{\tau}^{\tau_\theta} e^{-\delta u} (x - N_0(u))^+ du\right)\right| \leqslant x\mathbb{E}(\tau_\theta - \tau) \leqslant x\Delta$$

Similarly it can be verified for every $\delta \geqslant 0$

$$\mathbb{E}\left(\int_{\tau}^{\tau_\theta} e^{-\delta u} \lambda(u) f_2(u) du\right) \leqslant \|\lambda f_2\|_\infty \Delta \quad \text{and}$$

$$\mathbb{E}\left(\int_{\tau \wedge \sigma_x}^{\tau_\theta \wedge \sigma_x} e^{-\delta u} \lambda(u) f_1(u) du\right) \leqslant \|\lambda f_1\|_\infty \Delta$$

Using these upper bounds in relation (5.22) yields relation (5.21). □

Corollary 5. *With our usual notation let us define*

$$\upsilon(\widetilde{P}_\Theta) := \inf_{x \leqslant x_U(\overline{x},\overline{\tau}),\, \tau \in \Theta} \{c(x) + \widetilde{C}(x,\tau)\} = \inf_{x \leqslant x_U(\overline{x},\overline{\tau})} \widetilde{\varphi}_\Theta(x) \qquad (\widetilde{P}_\Theta)$$

approximating the value $\upsilon(\widetilde{P}_{\mathbb{F}})$ of optimization problem $(\widetilde{P}_{\mathbb{F}})$. It follows from Lemmas 33 and 34 that

$$0 \leqslant \upsilon(\widetilde{P}_\Theta) - \upsilon(\widetilde{P}) \leqslant f_0(x_U(\overline{x},\overline{\tau}))\Delta. \qquad (5.23)$$

Hence, for a given absolute tolerance level $\epsilon > 0$, selecting a mesh $\Delta \leqslant \epsilon / f_0(x_U(\overline{x},\overline{\tau}))$ will guarantee that $\upsilon(\widetilde{P}_\Theta)$ is no more than ϵ away from the compensated minimal cost $\upsilon(\widetilde{P})$.

Corollary 6. *Clearly, we have $\upsilon(\widetilde{P}_\Theta) - \upsilon(\widetilde{P}_{\mathbb{F}}) = \upsilon(P_\Theta) - \upsilon(P_{\mathbb{F}})$ since $C(x,\tau)$ and $\widetilde{C}(x,\tau)$ differ by the same constant (for any (x,τ)). Hence, the upper and lower bounds in (5.23) also holds for the difference $\upsilon(P_\Theta) - \upsilon(P)$. Re-arranging these inequalities and using the inequality $\upsilon(P) \geqslant g(0)$ (see Lemma 7 and relation (5.11) give*

$$1 \leqslant \frac{\upsilon(P_\Theta)}{\upsilon(P)} \leqslant 1 + \frac{f_0(x_U(\overline{x},\overline{\tau}))\Delta}{\upsilon(P)} \leqslant 1 + \frac{f_0(x_U(\overline{x},\overline{\tau}))\Delta}{g(0)}. \qquad (5.24)$$

These inequalities indicate that to achieve a relative *error of $100 \cdot \epsilon$ % (relative to the magnitude of the objective function) we may simply set $\Delta \leqslant \frac{g(0)}{f_0(x_U(\overline{x},\overline{\tau}))}\epsilon$.*

5.4 The Discrete Optimal Stopping Problem

Let us now assume that the set Θ is fixed. For a given Θ, the problem $(\widetilde{P}_\Theta)$ is Markovian in both the current inventory level and also the time (or the time index) and hence we can solve this problem by using the standard and powerful arguments of discrete time dynamic programming. For that, we let $V_n(x)$ denote the minimum expected incremental discounted cost that the supplier incurs given that the current time is t_n and the inventory level at the current time is x. Also we let $B_n(x)$ denote the expected discounted one period costs from t_n to t_{n+1} consisting of the inventory cost and compensated service costs during that period if the inventory level at the current time is x and the current time is not a switching time. Clearly, for $t_N = T$, we have the boundary condition

$$V_N(x) = c_{scr}x, \tag{5.25}$$

and for $n = 0, ..., N-1$ and $x = 0, 1, \ldots, x_U(\overline{x}, \overline{\tau})$, in terms of the difference term $\Delta t_n = t_{n+1} - t_n$, we have the following Bellman optimality equation

$$V_n(x) = \min\left\{c_{scr}x\ ,\ B_n(x) + \mathbb{E}\left[e^{-\delta\Delta t_n}V_{n+1}((x - N_0^{t_n}(t_{n+1}))^+)\right]\right\} \tag{5.26}$$

where

$$N_0^{(t_n)} := \{N_0^{(t_n)}(s) : s \geqslant 0\} = \{N_0(s+t_n) - N_0(t_n) : s \geqslant 0\} \tag{5.27}$$

is a non-homogeneous Poisson process with arrival intensity $\lambda_0(t_n + u)$, for $u \geqslant 0$. In (5.26), $c_{scr}x$ gives the immediate cost of stopping. Then, we simply compare the cost of immediate stopping with the value of continuing, and we select the minimum to identify the best action at time t_n. The value of continuing consists of multiple components: i) inventory cost from t_n until t_{n+1}, ii) compensated service cost from t_n until t_{n+1}, and iii) minimal cost from time t_{n+1} onwards. The first two is given by the term $B_n(x)$ and the third is given by the expectation term in (5.26). All costs should be discounted to time t_n for a proper comparison. We iterate the Bellman equation (5.26) recursively backwards for $n = 0, ..., N-1$ and $x = 0, 1, \ldots, x_U(\overline{x}, \overline{\tau})$, and at $n = 0$, we set $\widetilde{\varphi}_\theta(x) = V_0(x)$. The value of x solving the problem

$$\upsilon(\widehat{P}_\Theta) = \min_{x \leqslant x_U(\bar{x},\bar{\tau})} c(x) + \widehat{\varphi}_\theta(x) = \min_{x \leqslant x_U(\bar{x},\bar{\tau})} c(x) + V_0(x) \tag{5.28}$$

gives the optimal initial order quantity. In general, the function $x \mapsto \widehat{\varphi}_\theta(x)$ does not give useful properties (like convexity) for the sufficiency of first order conditions. Therefore a numerical search method should be employed to find the best order quantity (depending on the problem size, even a complete enumeration can be used). In implementation, one starts with this order quantity and continue serving customers with repair-replacement policy until the first time the value V_n is equal to the cost of scrapping. More precisely, let x^* be the optimal order quantity and let $X_t = (x^* - N_0(t))^+$ be the level of the corresponding inventory process at time t. Then the optimal stopping time can be represented as

$$\tau^* = \min\{t_n \in \Theta \,:\, V_n(X_{t_n}) = c_{scr}X_{t_n}\}. \tag{5.29}$$

Below, in the next subsection, we explain how $B_n(x)$ (and its inventory and service components) can be computed.

5.4.1 Computing the Expected One Period Discounted Cost $B_n(x)$

At time t_n, with x-many items in the inventory, the expected discounted inventory holding cost (until t_{n+1}) is given by

$$H_n(x) := h\mathbb{E}\left(\int_{t_n}^{t_{n+1}} e^{-\delta(u-t_n)}(x - N_0^{(t_n)}(u))^+ du\right) = h\int_0^{\Delta t_n} e^{-\delta s}\mathbb{E}\left((x - N_0^{(t_n)}(s))^+\right) ds.$$

This can be obtained by evaluating the increments $\Delta H_n(x) := H_n(x+1) - H_n(x)$ for $x = 0, ..., x_U(\overline{x}, \overline{\tau})$ having the simpler form

$$\Delta H_n(x) = h\int_0^{\Delta t_n} e^{-\delta s}\mathbb{P}(N_0^{(t_n)}(s) \leqslant x)ds. \tag{5.30}$$

On the other hand, using Doob's stopping theorem, the expected total discounted compensated service costs from time t_n until time t_{n+1} can be written as

$$S_n(x) \quad = \quad \begin{cases} \mathbb{E}\left(\int_{t_n}^{t_{n+1}} e^{-\delta(u-t_n)}\lambda(u)q(c_{se}+c_{re})du\right) + \\ \mathbb{E}\left(\int_{t_n}^{t_n \wedge \sigma_x^{(t_n)}} e^{-\delta(u-t_n)}\lambda(u)(1-q)c_{se}du\right) + \\ \mathbb{E}\left(\int_{t_n \wedge \sigma_x^{(t_n)}}^{t_{n+1}} e^{-\delta(u-t_n)}\lambda(u)(1-q)c_{ap}(u)du\right) - \\ \mathbb{E}\left(\int_{t_n}^{t_{n+1}} e^{-\delta(u-t_n)}\lambda(u)c_a(u)du\right) \end{cases} \tag{5.31}$$

with $\sigma_x^{(t_n)}$ denoting the arrival time of the x'th non-repairable item after time t_n. By standard re-arrangements of the expressions in (5.31) it follows that

$$S_n(x) = \begin{cases} \int_0^{\Delta t_n} e^{-\delta s}\lambda(s+t_n)[q(c_{se}+c_{re}-c_a(s+t_n)) + (1-q)p(s+t_n)]ds \\ + \int_0^{\Delta t_n} e^{-\delta s}\lambda(s+t_n)(1-q)[c_{se}-c_{ap}(s+t_n)]\mathbb{P}(N_0^{(t_n)}(s) < x)ds. \end{cases} \tag{5.32}$$

This shows for every $x = 0, ..., x_U(\overline{x}, \overline{\tau})$ and $\Delta S_n(x) := S_n(x+1) - S_n(x)$, for $x = 0, ..., x_U(\overline{x}, \overline{\tau})$, that

$$\Delta S_n(x) = \int_0^{\Delta t_n} e^{-\delta s}\lambda(s+t_n)(1-q)[c_{se}-c_{ap}(s+t_n)]\mathbb{P}(N_0^{(t_n)}(s) = x)ds. \tag{5.33}$$

Clearly, $B_n(x) = H_n(x) + S_n(x)$. Then for $x = 0$ it follows by relation (5.31) that the one period expected discounted costs is given by

$$B_n(0) = H_n(0) + S_n(0) = S_n(0) = \int_0^{\Delta t_n} e^{-\delta s}\lambda(s+t_n) \\ [q(c_{se}+c_{re}-c_a(s+t_n)) + (1-q)p(s+t_n)]ds. \tag{5.34}$$

Also by relations (5.30) and (5.33) we obtain

$$\Delta B_n(x) := B_n(x+1) - B_n(x) = \begin{cases} h\int_0^{\Delta t_n} e^{-\delta s}\mathbb{P}(N_0^{(t_n)} \leqslant x)ds \\ + \int_0^{\Delta t_n} e^{-\delta s}(1-q)\lambda(s+t_n)[c_{se} - c_{ap}(s+t_n)] \\ \mathbb{P}(N_0^{(t_n)}(s) = x)ds. \end{cases} \tag{5.35}$$

Hence $B_n(x)$ values can be computed starting with $B_n(0)$ and evaluating $\Delta B_n(x)$ for $x = 0, ..., x_U(\overline{x}, \overline{\tau})$.

5.5 Explicit Results with Piecewise Constant Arrival Rate Functions

In this section, we restrict ourselves to the case where the arrival rate function is piecewise constant. To construct a piecewise continuous arrival intensity function, for some integer $K \in \mathbb{N}$, let us introduce the time points $0 = a_1 < a_2.... < a_K < a_{K+1} = T$, the intervals $A_k = [a_k, a_{k+1})$ for $k = 1, ...K-1$ with $A_K = [a_K, a_{K+1}]$, and also the constants $\lambda_1, \ldots \lambda_K$. For $k \leqslant K$, the rate function is constant over A_k and its value is λ_k. With this notation, we have the representation

$$\lambda(t) = \sum_{k=1}^{K} \lambda_k 1_{A_k}(t), \qquad t \geqslant 0. \tag{5.36}$$

Lemma 35. *For the arrival intensity function λ in (5.36), the cumulative intensity function $\Lambda(t) = \int_0^t \lambda(u)\,du$ has the explicit form*

$$\Lambda(t) = \sum_{k=1}^{K} (\lambda_k t + \beta_k) 1_{A_k}(t), \qquad t \geqslant 0, \tag{5.37}$$

with the intercepts β_k *given by*

$$\beta_1 = 0 \quad \textit{and} \quad \beta_k = \sum\nolimits_{i=2}^{k} a_i(\lambda_{i-1} - \lambda_i), \quad \textit{for } k = 2, ..., K. \tag{5.38}$$

For the cost of procurement, cost of the alternative policy, and the penalty cost, we adopt the functions in Pourakbar et al. [2012] and we set $c(x) = c_p x$, $c_a(u) = c_a(0)e^{-\gamma u}$ with $c_a(0) > c_{se}$ and $p(u) = p$ with $p \geqslant c_{se}$. For a fixed static policy $(\bar{x}, \bar{\tau})$ with $\tau \in \mathcal{F}_0$ (for example $\bar{x} = 0$, $\bar{\tau} = 0$), we need to evaluate the function $g(x)$ in (3.36) for consecutive x values in order to find an upper bound $x_U(\bar{x}, \bar{\tau})$ on the order quantity; see Lemma 33. For $x = 0$ we have

$$g(0) = \int_0^T e^{-\delta u}\lambda(u) \min\{c_{se} + qc_{re}, c_a(0)e^{-\gamma u}\}du \tag{5.39}$$

and if $c_a(0)e^{-\gamma T} \geqslant c_{se} + qc_{re}$ we obtain by straightforward integration that

$$g(0) = (c_{se} + qc_{re})\delta^{-1} \sum\nolimits_{k=1}^{K} \lambda_k(e^{-\delta a_k} - e^{-\delta a_{k+1}}). \tag{5.40}$$

For the case $c_a(0)e^{-\gamma T} < c_{se} + qc_{re}$ and $c_a(0) > c_{se} + qc_{re}$ again by some standard and yet lengthier calculations, $g(0)$ has the following elementary expression

$$g(0) = \begin{cases} \begin{cases} \lambda_1 \delta^{-1}(c_{se}+qc_{re}) \left(1 - (\frac{c_{se}+qc_{re}}{c_a(0)})^{\delta\gamma^{-1}}\right) + \\ +c_a(0)\lambda_1(\delta+\gamma)^{-1} \left(\left(\frac{c_{se}+qc_{re}}{c_a(0)}\right)^{\delta\gamma^{-1}+1} - e^{-(\delta+\gamma)a_2}\right) \\ +c_a(0)(\delta+\gamma)^{-1}\sum_{k=2}^{K}\lambda_k \left(e^{-(\delta+\gamma)a_k} - e^{-(\delta+\gamma)a_{k+1}}\right) \end{cases} & i_* = 1 \\ \begin{cases} (c_{se}+qc_{re})\delta^{-1}\sum_{k=1}^{i_*-1}\lambda_k(e^{-\delta a_k} - e^{-\delta a_{k+1}}) \\ +\lambda_{i_*}(c_{se}+qc_{re})\delta^{-1}\left(e^{-\delta a_{i*}} - (\frac{c_{se}+qc_{re}}{c_a(0)})^{\delta\gamma^{-1}}\right) \\ +\lambda_{i_*}c_a(0)(\delta+\gamma)^{-1}\left(\left(\frac{c_{se}+qc_{re}}{c_a(0)}\right)^{\delta\gamma^{-1}+1} - e^{-(\delta+\gamma)a_{i_*+1}}\right) \\ +c_a(0)(\delta+\gamma)^{-1}\sum_{k=i_*+1}^{K}\lambda_k[e^{-(\delta+\gamma)a_k} - e^{-(\delta+\gamma)a_{k+1}}] \end{cases} & 2 \leqslant i_* \leqslant K-1 \\ \begin{cases} (c_{se}+qc_{re})\delta^{-1}\sum_{k=1}^{K-1}\lambda_k(e^{-\delta a_{k+1}} - e^{-\delta a_k}) \\ +\lambda_K(c_{se}+qc_{re})\delta^{-1}\left(e^{-\delta a_K} - (\frac{c_{se}}{c_a(0)})^{\delta\gamma^{-1}}\right) \\ +\lambda_n(c_{se}+qc_{re})(\delta+\gamma)^{-1}\left(\left(\frac{c_{se}+qc_{re}}{c_a(0)}\right)^{\delta\gamma^{-1}+1} - e^{-(\delta+\gamma)a_{K+1}}\right) \end{cases} & i_* = K. \end{cases} \tag{5.41}$$

with $i_* := \max\left\{1 \leqslant i \leqslant K+1 : a_i \leqslant \gamma^{-1}\ln\left(\frac{c_a(0)}{c_{se}+qc_{re}}\right)\right\}$.

Finally for $c_a(0) \leqslant c_{se}+qc_{re}$ and hence automatically $c_a(0)e^{-\gamma T} < c_{se}+qc_{re}$ it follows that

$$g(0) = (c_{se}+qc_{re})(\gamma+\delta)^{-1}\sum_{k=1}^{K}\lambda_k(e^{-(\delta+\gamma)a_k} - e^{-(\delta+\gamma)a_{k+1}}). \tag{5.42}$$

To evaluate $g(x)$ for other values of x, we use iteratively the relation $g(x) = g(x-1)+\Delta g(x-1)$ in which, by (3.36), we have

$$\Delta g(x-1) = c_p + c_{scr} - c_{scr}^{+}\int_0^T e^{-\delta u}\lambda_0(u)\mathbb{P}(N_0(u) = x-1)du. \tag{5.43}$$

For $x \in \mathbb{Z}_+$ and $t \geqslant 0$ define the function

$$G(x,t) := \int_t^\infty e^{-s}\frac{s^x}{x!}ds = e^{-t}\sum\nolimits_{j=0}^{x}\frac{t^j}{j!}, \tag{5.44}$$

and the constants

$$\beta_k^* := (\delta\lambda_k^{-1} + 1 - q)\beta_k, \quad \lambda_k^* := \delta + (1-q)\lambda_k, \quad k = 1, ..., K. \tag{5.45}$$

Also introduce for every $x \in \mathbb{N}$ and $k = 1, ..., K$ the function

$$H(x-1,k) := G(x-1, \lambda_k^* a_k + \beta_k^*) - G(x-1, \lambda_k^* a_{k+1} + \beta_k^*) \geqslant 0 \tag{5.46}$$

with a_k the points of discontinuity of the piecewise constant arrival intensity function λ listed in relation (5.36). In terms of the function H above, the integral term in (5.43) can be computed explicitly again by standard integration as

$$\int_0^T e^{-\delta u}\lambda_0(u)\mathbb{P}(N_0(u) = x-1)du = \sum\nolimits_{k=1}^{K} e^{\delta\beta_k\lambda_k^{-1}}\left(1 - \frac{\delta}{(1-q)\lambda_k + \delta}\right)^x H(x-1,k). \tag{5.47}$$

Once $x_U(\bar{x},\bar{\tau})$ is determined, for a given absolute/relative error $\epsilon \geqslant 0$, we have to determine the upper bound on the mesh size of the discrete set Θ; see Corollaries 5 and 6. For that, we need to evaluate $\| \lambda f_i \|_\infty$ for $i = 1, 2$ with f_i given by relation (5.17) and (5.18). For our particular choice of functions it follows that

$$\begin{aligned} f_1(u) &= (1-q)(c_{se} - c_{scr} - p - c_a(0)e^{-\gamma u}) \quad \text{and} \\ f_2(u) &= q(c_{se} + c_{re}) + (1-q)p - qc_a(0)e^{-\gamma u}, \quad \text{for } u \geqslant 0. \end{aligned} \tag{5.48}$$

Since $p \geqslant c_{se}$ and $c_{scr} \geqslant 0$ and hence $p \geqslant c_{se} - c_{scr}$ the function f_1 is non-positive and increasing, and f_2 is increasing. This implies

$$\begin{aligned} \| \lambda f_1 \|_\infty &= \max_{1\leqslant k\leqslant K} \{-\lambda_k f_1(a_k)\} \quad \text{and} \\ \| \lambda f_2 \|_\infty &= \max_{1\leqslant k\leqslant K} \lambda_k \max\{| f_2(a_k) |, | f_2(a_{k+1}) |\}, \end{aligned} \tag{5.49}$$

in terms of which $f_0(x_U(\bar{x},\bar{\tau}))$ is computed using (5.19). Having computed the upper bound on Δ as in either Corollary 5 or 6, we can set Δ equal to this upper bound and construct the discrete set $\Theta = \{t_0, ..., t_N\}$ as follows. Let $M_0 := 0$ and $M_k := \left\lceil \frac{a_{k+1}-a_k}{\Delta} \right\rceil, k = 1, ..., K$. For each $k = 1, ..., K$ introduce

$$\Delta_k = \frac{a_{k+1} - a_k}{M_k} \leqslant \Delta. \tag{5.50}$$

We define the sequence $t_n, n = 1, ..., N$ with $N := \sum_{j=1}^{K} M_j$ as

$$t_n = a_k + (n - \sum_{j=0}^{k-1} M_j)\Delta_k \qquad \text{for} \quad \sum_{j=0}^{k-1} M_j \leqslant n < \sum_{j=0}^{k} M_j, \quad k = 1, ..., K \tag{5.51}$$

and $t_N = T$. By construction, the mesh of the set Θ is less than or equal to Δ. Also, it follows that on every interval $[t_n, t_{n+1}]$ the arrival rate of the Poisson process is a constant. In particular, $N_0^{(t_n)}$ is a homogeneous Poisson process on $[t_n, t_{n+1}]$ with constant arrival rate λ_k for $\sum_{j=0}^{k-1} M_k < n \leqslant \sum_{j=0}^{k} M_j$, $k = 1, ..., K$. This simplifies the calculation of the Bellman operator and the expected one period discounted cost function B_n in (5.26). This is explained in the remainder of this section as the last step.

Lemma 36. *If the sequence t_n is given by relation (5.51) then for every $\sum_{j=0}^{k-1} M_j \leqslant n < \sum_{j=0}^{k} M_k, k = 1, ..., K$*

$$B_n(0) = [q(c_{se}+c_{re})+(1-q)p]\lambda_k\delta^{-1}(1-e^{-\delta\Delta_k})-qc_a(0)e^{-\gamma t_n}\lambda_k(\delta+\gamma)^{-1}\left[1 - e^{-(\delta+\gamma)\Delta_k}\right].$$

Proof. It follows by relation (5.34) that

$$B_n(0) = [q(c_{se}+c_{re})+(1-q)p]\int_0^{\Delta t_n} e^{-\delta s}\lambda(s+t_n)ds - qc_a(0)e^{-\gamma t_n}\int_0^{\Delta t_n} e^{-(\delta+\gamma)s}\lambda(s+t_n)ds.$$

Since for $\sum_{j=0}^{k-1} M_j \leqslant n \leqslant \sum_{k=0}^{k} M_k, k = 1,...K$ it follows by relation (5.51) that $\Delta t_n = \Delta_k$, $\lambda(s+t_n) = \lambda_k$ for all $s \leqslant \Delta t_n$, and the result follows. □

For $n_k = \sum_{j=0}^{k} M_j - 1$ we evaluate

$$B_{n_k}(0) = \begin{cases} [q(c_{se}+c_{re}) + (1-q)p]\lambda_k \delta^{-1}(1-e^{-\delta\Delta_k}) \\ \\ -q_{c_a} e^{-\gamma(a_{k+1}-\Delta_k)}\lambda_k(\delta+\gamma)^{-1}[1-e^{-(\delta+\gamma)\Delta_k}] \end{cases} \tag{5.52}$$

and for $\sum_{j=0}^{k-1} M_j < n < \sum_{j=0}^{k} M_j$ we compute

$$B_{n-1}(0) = B_n(0) - qc_a(0)\lambda_k(\delta+\gamma)^{-1}e^{-\gamma t_{n-1}}(1-e^{-\gamma\Delta_k})(1-e^{-(\delta+\gamma)\Delta_k}). \tag{5.53}$$

This gives us an easy and fast recursive procedure for updating $B_n(0)$ for $\sum_{j=0}^{k-1} M_j \leqslant n < \sum_{j=0}^{k} M_j$. To obtain $B_n(x)$ values for $x = 1, \ldots, x_U(\bar{x}, \bar{\tau})$, we can use the relation $B_n(x) = B_n(x-1)+\Delta B_n(x-1)$; see (5.35). In our case, ΔB_n values can be even more conveniently obtained using the second order difference terms $\Delta B_n(x) - \Delta B_n(x-1)$. To pave the way, we introduce for $\theta \geqslant 0, x \in \mathbb{Z}_+$

$$J_n(\theta, x) := \int_0^{\Delta t_n} e^{-\theta s}\mathbb{P}(N_0^{(t_n)}(s) = x)ds.$$

Since the sequence t_n, $n = 1, ..., N$ with $N = \sum_{j=0}^{K} M_j$ is given by relation (5.51) it follows for every $\sum_{j=0}^{k-1} M_j \leqslant n < \sum_{j=0}^{k} M_j, k = 1, ..., K$ that

$$J_n(\theta, x) = \int_0^{\Delta_k} e^{-(\theta+\lambda_k)s}\frac{(\lambda_k s)^x}{x!}ds = \lambda_k^{-1}\left(\frac{\lambda_k}{\theta+\lambda_k}\right)^{x+1}(1 - G(x, \Delta_k(\theta+\lambda_k))$$

with G defined in relation (5.44). The next result follows after a lengthy and tedious application of the chain rule for $\Delta B_n(0)$. The proof is skipped for conciseness, we only give the result.

Lemma 37. *If the sequence t_n is given by relation (5.51) and*

$$\theta_k := \frac{c_a(0)\lambda_k}{\delta+\gamma+\lambda_k}(1-e^{-(\delta+\gamma+\lambda_k)\Delta_k}), \quad k=1,...,K, \tag{5.54}$$

then for every $\sum_{j=0}^{k-1} M_j \leqslant n < \sum_{j=0}^{k} M_j$, $k=1,...,K$

$$\Delta B_n(0) = -\theta_k e^{-\gamma t_n} + \left(c_{se} - p + \frac{h+\delta(p-c_{se})}{\delta+\lambda_k}\right)(1-e^{-(\delta+\lambda_k)\Delta_k}). \tag{5.55}$$

Once again we use a recursive procedure for updating $\Delta B_n(0)$ for $\sum_{j=0}^{k-1} M_j \leqslant n < \sum_{j=0}^{k} M_j$. More precisely, applying Lemma 37 it follows for $n_k = \sum_{j=0}^{k} M_j - 1$ that

$$\Delta B_{n_k}(0) = -\theta_k e^{-\gamma(a_{k+1}-\Delta_k)} + \left(c_{se} - p + \frac{h+\delta(p-c_{se})}{\delta+\lambda_k}\right)(1-e^{-(\delta+\lambda_k)\Delta_k}) \tag{5.56}$$

while for $\sum_{j=0}^{k-1} M_j + 1 \leqslant n < \sum_{j=0}^{k} M_j$

$$\Delta B_{n-1}(0) = \Delta B_n(0) - \theta_k e^{-\gamma t_{n-1}}(1-e^{-\gamma\Delta_k}) \leqslant \Delta B_n(0). \tag{5.57}$$

Finally, by another application of the chain rule and (5.44), we obtain after some calculations the following expression for $\Delta B_n(x) - \Delta B_n(x-1)$ and $x \in \mathbb{N}$.

Lemma 38. *If the sequence t_n is given by relation (5.51) and we introduce for every $k=1,...,K$ the functions*

$$D_k(x) = \lambda_k^{-1} c_a(0)(\delta+\gamma)\left(\frac{\lambda_k}{\delta+\gamma+\lambda_k}\right)^{x+1}[1-G(x,\Delta_k(\delta+\gamma+\lambda_k))] \tag{5.58}$$

and

$$\overline{D}_k(x) = \lambda_k^{-1}(h + \delta(p - c_{se})) \left(\frac{\lambda_k}{\delta + \lambda_k}\right)^{x+1} [1 - G(x, \Delta_k(\delta + \lambda_k))] \tag{5.59}$$

then for every $\sum_{j=0}^{k-1} M_j \leqslant n < \sum_{j=0}^{k} M_j, k = 1, ..., K$ *and* $x \in \mathbb{N}$

$$\Delta B_n(x) - \Delta B_n(x-1) = \begin{cases} -e^{-\delta\Delta_k}\big[c_{se} - p - c_a(0)e^{-\gamma t_{n+1}}\big]e^{-\lambda_k\Delta_k}\frac{(\lambda_k\Delta_k)^x}{x!} \\ \\ +\overline{D}_k(x) + e^{-\gamma t_n}D_k(x). \end{cases} \tag{5.60}$$

Applying Lemma 38 it follows for $n_k = \sum_{j=0}^{k} M_j - 1$ and $x \in \mathbb{N}$

$$\Delta B_{n_k}(x) - \Delta B_{n_k}(x-1) = \begin{cases} -e^{-\delta\Delta_k}\big[c_{se} - p - c_a(0)e^{-\gamma a_{k+1}}\big]e^{-\lambda_k\Delta_k}\frac{(\lambda_k\Delta_k)^x}{x!} \\ \\ +\overline{D}_k(x) + e^{-\gamma t_n}D_k(x) \end{cases} \tag{5.61}$$

while for $\sum_{j=0}^{k-1} M_j + 1 \leqslant n < \sum_{j=0}^{k} M_j$

$$\Delta B_{n-1}(x) - \Delta B_{n-1}(x-1) = \begin{cases} \Delta B_n(x) - \Delta B_n(x-1) \\ \\ +c_a e^{-\gamma t_n}(1 - e^{-\gamma\Delta_k})e^{-\lambda_k\Delta_k}\frac{(\lambda_k\Delta_k)^x}{x!} + \\ \\ e^{-\gamma t_{n-1}}(1 - e^{-\gamma\Delta_k})D_k(x). \end{cases} \tag{5.62}$$

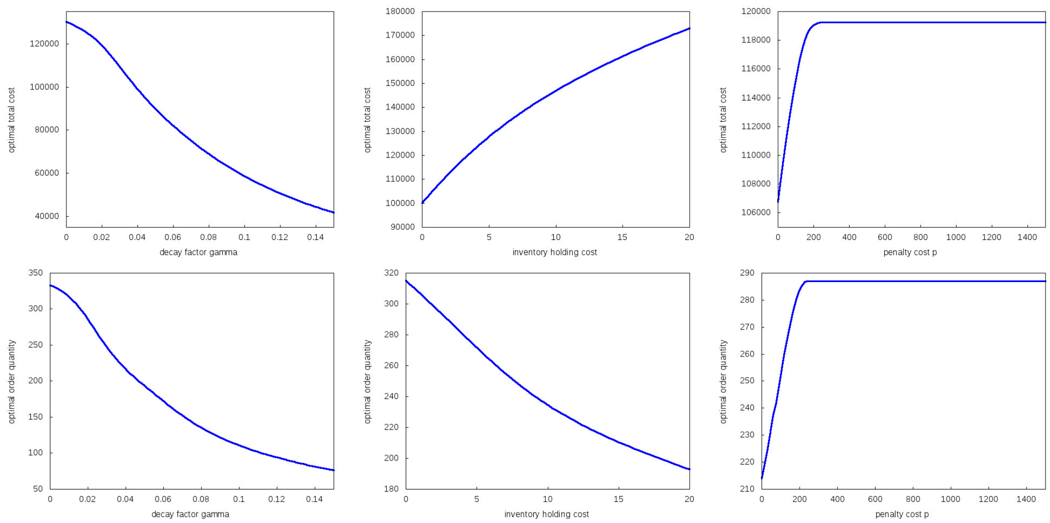
5.6 Numerical Examples

In this section, we give some numerical examples, carry out a sensitivity analysis over problem parameters, and compare our results with those produced by other policies. For that, we consider a setup where the arrival intensity function is piecewise continuous and we set the functions $c(x)$, $c_a(u)$ and $p(u)$ as in Pourakbar et al. [2012]. More precisely, we take $c(x) = c_p x$, $c_a(u) = c_a(0)e^{-\gamma u}$ with $c_a > c_{se}$ and $p(u) = p$ with $p \geqslant c_{se}$. For this case, the results are derived explicitly in Section 5.5 above. Due to these explicit results, it is possible to solve a problem instance within seconds (for our computational experiments, we used a computer with a 2.20 GHz processor). As the data in Pourakbar et al. [2012] come from a case study for a consumer electronics company as in Chapter 4.7, we adopt the similar cost parameters and we set up the base case scenario as in Table 5.1. The time horizon $[0, T]$ is split into three equal intervals over which the arrival rate is constant. With the notation of Section 5.5 we have $K = 3$, $a_k = \frac{(k-1)}{3}T$ for $k = 1, \ldots, 4$, and $A_k = [\frac{(k-1)}{3}T, \frac{k}{3}T]$ for $k = 1, 2, 3$. The constant arrival rate λ_k over A_k is taken as $\ell\beta^{k-1}$. The value of β is equal to 0.5 in the base case and ℓ is set so that the over all expected number of customer arrivals over $[0, T]$ is $10T$ (i.e., on average 10 customer requests per unit interval).

Table 5.1: Problem parameters for the base case scenario

T	c_{scr}	c_{se}	c_{re}	h	q	c_p	$c_a(0)$	γ	p	δ	β
66 (months)	30	30	20	3.25	0.5	225	645	0.02	1290	0.003	0.5

In the base case scenario, we set a relative error level of $\epsilon = 1/250$; see Corollary 6. For this tolerance level, the mesh Δ of our discretization set $\Theta = \{t_1, t_2, \ldots, T\}$ is no larger than 0.004. This gives a very fine discretization of the interval $[0, T]$. Starting with the boundary condition (5.25) at $t_n = T$, we iterate the Bellman equation (5.26) backwards to compute $V_n(x)$ values. Finally at $t_1 = 0$, we numerically solve the problem $v(\widehat{P}_\Theta) = \min_x c_p x + \widehat{\varphi}_\theta(x) = \min_x c_p x + V_0(x)$ to find the optimal initial order quantity x^*, and in the base case scenario this is equal to 287. With this initial inventory level, we serve customers using the repair-replacement policy until τ^* given in (5.29) at which time the value of continuing is equal to the value of scrapping. The first plot in the second row of Figure 5.2 illustrates the stopping region of the problem. The colored/shaded region (the upper right corner and the x-axis) is the set of points (x, t_n)'s for which $V_n(x) = c_{scr}x$. In plain words, it is optimal to stop at the first time point t_n at which the stochastic process (t_n, X_{t_n}) enters this region which X denoting the inventory process.



(a) (b) (c)

Figure 5.1: Sensitivity of the solution in the base case on the cost parameters γ, h, and p.

In Figure 5.1, we report graphically the dependence of the solution on the cost parameters γ, h, and p. The curves are obtained by changing one parameter at a time and keeping all others fixed in the base case scenario. In each panel of the figure, the upper curve is the minimal cost and the lower one is the optimal initial order quantity both as functions of the changing parameter. In panel (a) of Figure 5.1, we observe that both the optimal cost and the initial order quantity are decreasing in the decay factor γ of the price of the alternative policy. Both curves exhibit first concave and then convex behaviors. That is, for low values of γ, incremental increase in the decay factor (or equivalently slightly faster decrease in the price of the alternative policy) has higher effect compared to the case where there is already a significant pace of reduction in the price. Further increments still affect the cost and order quantity but the effect is less.

In panel (b), we see the dependence of the inventory holding cost h on the solution. As the holding cost increases, the optimal cost increases and we start with less items to control this increase. Similar to the one by γ, the marginal effect of the holding cost is decreasing as its value gets higher. The dependence on the penalty terms p

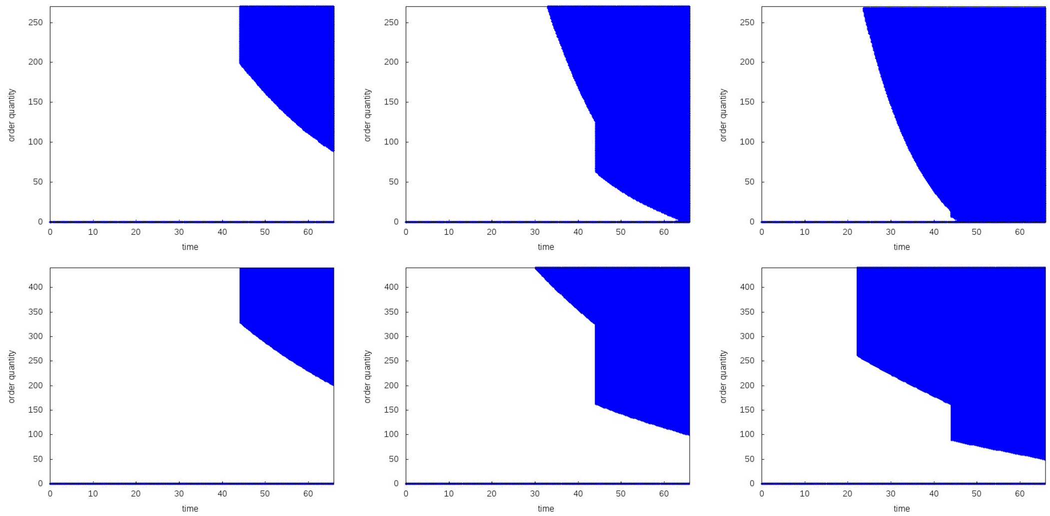
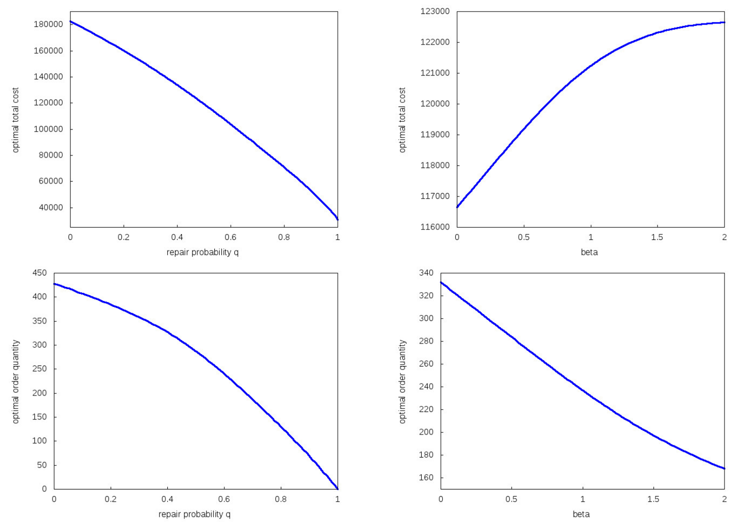


Figure 5.2: Stopping regions for different γ and h values obtained from the base case scenario.

is given in panel (c). As the penalty increases, the optimal cost naturally increases. We also order more to avoid high penalties. Yet, these effects/dependences become negligible after some value of p. We believe that this is because, in the optimal policy, the supplier can adjust the policy variables (x, τ) in order to reduce the probability of incurring a penalty to low levels (especially when the penalty is very high). If the inventory is depleted soon, an early switching decision can always be used as a solution. Although it is a static decision, the initial inventory also acts as a tool to control this cost. As a result, we see that additional increments in p does not effect the costs any further after a point. When we compare the y-axes of the plots in Figure 5.1, we note that the solution has higher sensitivity to γ and h than p. We would expect that these two parameters affect the stopping region of the problem as well. We indeed observe this in Figure 5.2, where we plot the changes in the stopping region as we change these two parameters. Recall that the shaded regions in the plots are the regions where we have $V_n(x) = c_{scr}x$ in the Bellman equation (5.26). The first row of the figure is for three different values of γ, and the second row is for three different values of h. As γ increases, the alternative policy becomes more attractive



(a) (b)

Figure 5.3: Optimal cost and order quantities as functions of q (panel (a)) and β (panel (b)) in the base case.

and as a result the stopping region gets larger. When the holding cost h increases, the repair-replacement policy becomes more expensive and it is more cost effective to switch to the alternative policy. Hence, we have larger stopping regions again.

In Figure 5.3, we illustrate the dependence of the solution on the demand parameters q and β. As the repair probability q increases, arriving items are more likely to be repairable. As a result, total cost decreases and the supplier starts with lower number of items. Note that in the extreme case with $q = 1$, there is no need to have any inventory and the cost is 30787. On the other hand, with $q = 0$, the initial order quantity is 428 and the cost is 182513 (almost six times more compared to the case with $q = 1$). These numbers indicate that the solution is indeed very sensitive to the quality of the sold items and this observation highlights the importance of the quality control efforts during the production phase. Panel (b) illustrates the effect of β. Recall that β is the growth factor in the arrival intensity over the intervals $[0, \frac{T}{3}], [\frac{T}{3}, \frac{2T}{3}], [\frac{2T}{3}, T]$ and the cumulative demand over $[0, T]$ is fixed at 10T. Hence, when $\beta > 1$ and gets higher, most of the demand arrive later (the opposite happens when $\beta < 1$ and gets lower). As β increases, the inventory holding cost component increases the total costs. In order to control the costs, the optimal policy then adjusts the initial inventory and starts with less items. This is what we observe in panel (b), and when we look at the y-axes of the plots in panel (b), we see that the optimal order quantity is more sensitive than the total costs.

Next, we compare the dynamic policy given by our optimal stopping formulation with other alternative policies in a number of different problem instances. The results are given in Table 5.2. The table reports in each problem instance the optimal initial order quantity and the switching time (if exists) along with the corresponding total expected cost for all policies. In the table, P denotes to the solution of our stopping problem. The first alternative P1 refers to the policy in which we never switch to the alternative method; that is; $\tau = T$. This problem is studied in Teunter and Fortuin [1999], Teunter and Fortuin [1998], and also in Section 4.3 above. In this problem, we simply solve the problem $\min_x c(x) + C(x,T)$. The second alternative policy P2 is a static policy in which the supplier can switch to the alternative policy before T and the (deterministic) switching time is the value of t solving the minimization problem $\min_{(x,t)\in\mathcal{F}_0} c(x) + C(x,t)$. This problem gives us the best static policy and it is solved in Section 4.4. These three policies are naturally ordered; P is the best, it is followed by P1 and then by P2. The table shows how and to what extent they differ from each other in different settings. Along with other results, the last two columns give the percentages

$$\text{P vs Pi} = 100 \times \frac{\text{cost given by Pi} - \text{cost given by P}}{\text{cost given by P}} \%$$

for both policies P1 and P2. In these problem instances, we observe that P1 and P2 yield on average 17.39% and 5.24% higher costs compared to P respectively. Also, P1 starts with 33.92% more items and P2 starts with 2.99% more items in the initial inventory, both compared to P again. These numbers show the importance of using an optimal dynamic policy in practice. Although lower compared to its improvement over P1, the cost improvement of P over P2 is also significant.

In the table, we note that as the price erosion factor γ of the alternative method increases (i.e., the alternative method of serving customers becomes more attractive over time), policies P and P2 significantly deviate from P1. They start with less number of items and their total costs drop remarkably. For example, when γ is very high (row with $\gamma = 0.1$), P and P2 start with less than half of the initial inventory of P1, and they have an expected cost which is less than half of the cost given by P1. In general, the policy P1 is not effected much by the c_a function (hence by the value of γ) as this cost is incurred (together with the penalty p) only when an item is not available and the arriving item is irreparable. Hence, there is some effect but it is negligible. Also note that the percentage cost improvement of P over P1 and P2 is increasing in γ. The option to make a dynamic switching decision proves more valuable when the price of the alternative policy decays faster. As the table illustrates, the effect of the parameter $c_a(0)$ is similar to that generated by γ as they both determine the cost of the alternative policy. As the penalty cost p increases, all policies start with more items to avoid high penalties. Since all policies adjust their initial order quantities accordingly, we do not observe a drastic change in the total expected costs. This is also observed in Figure 5.1 for the policy P alone. Here we also observe that the percentage cost reduction by P is higher with high values of p since the supplier can easily avoid high penalties by observing the inventory levels and taking the switching decision timely. Compared to the penalty p, we observe that the costs are more sensitive to the inventory holding cost term h. Clearly, starting with less items and switching to the alternative policy earlier can be used as a tool to control high inventory costs (as we scrap all the inventory upon switching). In the table, as h increases all costs increase and all policies start with less items. In return, the switching decision is made sooner. In the optimal static policy, the deterministic exit time is smaller. Similarly,

Table 5.2: Solutions of different problem instances for different policies

	P1		P2			P		P vs P1 (%)	P vs P2 (%)
	x	cost	x	tau	cost	x	cost		
γ									
0.005	339	132233.4	339	65.934	132197.6	327	128299.8	3.07	3.04
0.01	338	131848.2	338	65.934	131784.5	318	125944.6	4.69	4.64
0.02	337	131299	296	45.606	126469.9	287	119240.1	10.11	6.06
0.05	335	130664.7	201	22.506	95400.2	193	89839.4	45.44	6.19
0.1	334	130534.1	115	12.408	62673.7	110	58644.3	122.59	6.87
$c_a(0)$									
250	334	130604.7	176	19.404	87301.4	172	82035.6	59.20	6.42
322.5	335	130934.3	199	22.044	106980.8	194	101242.4	29.33	5.67
1290	339	131941.2	338	65.934	131829.3	322	126766.3	4.08	3.99
2580	342	132967.6	342	65.934	132875.2	337	131248.6	1.31	1.24
p									
322.5	307	123691.7	284	49.83	121398.6	287	119236.4	3.74	1.81
645	325	127778.7	290	47.058	124017.5	287	119238.3	7.16	4.01
2580	346	134401.2	301	44.418	128736.1	287	119240.9	12.71	7.96
5160	353	137171.6	307	44.088	130819.9	287	119241.5	15.04	9.71
h									
0.8125	343	111900.2	325	57.552	111330.5	308	105306.2	6.26	5.72
1.6125	341	118646.9	314	52.998	116801.7	302	110228.7	7.64	5.96
6.5	330	155959.1	262	37.488	142208.5	259	134162.4	16.25	6.00
13	317	201776.6	220	28.38	164158.8	219	154790.9	30.35	6.05
c_{scr}									
-30	338	130732.4	297	45.342	125775.6	287	119238.1	9.64	5.48
10	337	131113.5	296	45.408	126244.9	287	119239.5	9.96	5.88
60	336	131564.2	295	45.474	126795.3	287	119241.3	10.33	6.34
90	335	131826.8	295	45.738	127109.1	287	119242.1	10.55	6.60
q									
0.4	403	150495.1	335	40.986	141841.6	327	133738.3	12.53	6.06
0.6	270	112016.9	248	52.008	110027.4	240	103745.8	7.97	6.05
0.8	136	73036.1	136	65.934	73032.5	130	70792.6	3.17	3.16
1	0	30787.7	0	66	30787.7	0	30787.7	0.00	0.00
β									
0.25	338	125902.6	325	44.022	124053.3	309	117442.1	7.20	5.63
1	336	139706.8	238	45.606	128344.5	235	121536.7	14.95	5.60
1.5	336	144883.2	194	45.804	128595.2	192	122346.3	18.42	5.11
2	336	148130.7	162	45.87	128360.9	161	122578.1	20.85	4.72

in the dynamic policy, we observe that the stopping/switching region becomes larger as h increases. This was illustrated in Figure 5.2 for the policy P. The percentage cost reduction of P over the other two policies is also increasing in h. The effect is more visible when we compare P with P1. There is still some improvement but less when P is compared with P2. This is mainly because P2 also has the option to switch but can only do it statically. The cost of scrapping c_{scr} seems to be the parameter which has the lowest effect in the table. We believe that this is because all policies determine their initial order quantities by taking the demand information into account so that not many items are scrapped at the end. Unlike c_{scr}, the repair probability q has considerable effect in all policies. This was noted in Figure 5.3 for the policy P alone. Here, as q increases, all policies start with less items in the inventory. For low values of q, the initial inventories in P and $P2$ are relative lower than that in P1, and the differences between the expected revenues are more apparent. These gaps close as q increases. When q increases, policies behave similarly. In the case where $q = 1$, all policies give the same solution; they don't store any initial inventory and never switch to the alternative policy. Starting with no inventory is mainly because all products are repairable. P and P2 never switch to the alternative policy because repairing an item is always more cost effective in this setup; the lowest price for the alternative policy is $c_a(0)e^{-\gamma T} = 172.30$ whereas repairing an item costs $c_{se} + c_{re} = 50$. When we increase the value of γ to 0.07 to make the alternative policy more attractive, we see that both P and P2 start with zero inventory again but they switch to the alternative policy before T. In particular, P uses the same (static) switching time with P2 since both policies face the same uncertainty, all items are repairable with probability one and there is no inventory and no penalty costs involved.

Finally, as β increases, recall that we have the same expected cumulative demand over the horizon $[0, T]$ but customers are more likely to arrive later. The initial order quantity in $P1$ is not effected much by β since the initial inventory is the only (static) decision variable. However, as β increases, the inventory holding costs increase for the same initial inventory level, and P and P2 start with less items in the inventory in order to avoid high holding costs (as noted in Figure 5.3 for P alone). In return, total costs are less effected. In the table, the percentage cost improvement of P over P1 is more visible as β increases whereas the effect is less compared to P2. From $\beta = 0.25$ to $\beta = 2$ is a significant change, yet (unlike P1) neither the actual costs of P and P2 nor the percentage improvement of P over P2 vary much. This is mainly because both P and P2 can respond to changes in the time component of the demand, and P does it slight better as it is a dynamic policy.

Chapter 6

Concluding Remarks

In this study we focus on the so-called end-of-life inventory problem of a service part in its final phase. The final phase starts when the production of spare part terminates and lasts until the last service obligation expires. This phase is the longest phase in the service life cycle and the main challenge is the availability of service parts which are not produced anymore. Hence it is a challenge for companies to meet the demand during that phase and to face this challenge, companies use various tactics. In practice, one of the most popular tactics is to order a sufficient amount of service parts at the beginning of the final phase. However, due to the difficult statistical problem of giving a proper estimate of the demand for service parts, companies face a huge risk of a large number of left over parts at the end of the service obligation phase or on the other hand of not being able to satisfy the demand. This is the main difficulty in applying the above tactic. Moreover, a rapid development in technology and innovation imposes the parts to enter their final phases earlier. For example, consumer electronics parts start their final phase usually after one year of production, while the usual service obligations for such a product last for three to five years.

In Chapter 1, we consider service management and its importance in today's business world. The after-market became a profitable sector for companies and the operational problems of this sector received recently a lot of attention within the scientifc community. The literature in this so called service management expanded and can be classified into different categories. Hence in Chapter 1 we did this classification and also discuss the life cycle of service parts.

In Chapter 2, the literature on the end-of-life inventory problem is reviewed. As mentioned before, the end-of-life inventory problem deals with the control of inventory of service parts in the final phase and many researchers have studied this problem from different points of views. Some researchers have developed mathematical models describing the problem and depending on the complexity of the model have either used heuristic or exact mathematical solution procedures. Other researchers have paid more attention to the management side of this problem. Generally, the literature on the end-of-life inventory problem can be divided into a service-driven approach, a cost-driven approach or a forecasting based approach. In a service driven approach the aim is to optimize the service to the customers regardless of the cost incurred by the system. In a cost-driven approach one tries to identify the cost components of a given policy and by doing so identify that policy optimal with respect to the total costs incurred by the company during the final phase. And finally, in the forecasting based approach one ignores the production and inventory costs and only tries to estimate the demand for service parts. In Chapter 2 we reviewed the existing literature within this field. Since our research belongs to the cost driven approach we however did focus more on the existing literature in this subfield.

In Chapter 3, we introduce the end-of-life inventory problem with its cost structure. In the consumer electronic market, for example, thanks to a rapid development in technology, the price of new generation products decrease remarkably over years. This happens while the repair costs may stay constant over time. As a result, changing the current strategy to a more cost effective policy may be more appealing for companies. Instead of only using a repair policy we may combine this policy with an alternative policy to replace a defective item by a new generation model or offer to the customer a discount on this new generation model. In our model we assume that customers with defective items arrive according to a non-homogenous Poisson process and these customers request for repairing or replacing the items. Also in this model we not only consider the repair-replacement policy but also offer an alternative policy to cope with the demand in the final phase. In particular, at the beginning of the service phase we apply the repair-replacement policy until a (possibly random) time after which it becomes more cost effective to apply the alternative policy. After this so called switching time, we completely switch until the end of the service phase to the alternative policy. The incurred different costs of using both policies are calculated and the total expected cost is given by adding all these costs. This defines for each policy the objective function and its costs depends on the final order quantity and the (possibly stochastic) switching time. The switching time in general depends on the realizations of the arrival process of demands. Optimizing over these decision variables we obtain the optimal policy within this large class of feasible policies. The main contributions of Chapter 3 can be listed as follows:

- The demand for service parts arrive according to a non-homogenous Poisson process. During the repair-replacement phase of the considered policy two different stochastic processes count the number of non-repairable and repairable items. If during this phase the defective item is repairable we repair it at a certain repair cost and if not we need a service part. If such a service part is available we incur a service cost and if not we need to apply the alternative policy earlier then expected at a huge penalty cost. At the switching time we switch to the alternative policy and if at that time we still have inventory of service parts we discard this inventory and incur scrapping costs.

- The holding cost, repair cost, service cost, alternative cost and scrapping cost are calculated for each feasible policy and added up to obtain the total expected cost for that policy.

- Depending on the class of subpolicies (static or dynamic) we need to minimize the objective function with respect to the final order quantity and switching time. Within the class of static policies this results in solving a nonlinear optimization problem while for the larger class of dynamic policies we use dynamic programming techniques.

Chapter 4 deals in more detail with the subclass of static and closely related policies in the end-of-life inventory problem. Observe for all considered policies the manufacturer knows the optimal order quantity for service parts at the beginning of the final service phase. An important feature of a so-called static policy is that the manufacturer also knows the switching time at the beginning of the final service phase. Although such a policy is not optimal within the class of all polices this additional knowledge of knowing the switching time in advance is important in practice. In this chapter, four different policies are considered according to their restrictions on how the switching time is selected. The first policy simply assumes that during the service phase we never switch to the alternative policy and so we need to solve a one dimensional optimization problem in the order quantity. The second one selects the switching time at any time between the start and the end of the service phase. In fact, this is a natural extension of the first policy. The third one assumes that we switch to the alternative policy at the time the inventory level drops to zero during the service phase. And the last policy is a natural generalization of the third policy. It selects the switching time at a fixed time from the start or end of the service phase like in the second policy unless before that selected time the inventory level drops to zero. If this happens the alternative policy is initiated at the time the inventory level drops to zero. Deriving for all these policies the objective value we investigate the behavior of this function as a function of the order quantity keeping the switching time fixed. For almost all reasonable values of the cost parameters this one dimesional function is discrete convex in the order quantity and this enables us solving the optimization problem in an efficient way. For all the other values of the parameters we derive an upper bound on the optimal order quantity and apply a complete enumeration procedure to solve the problem. Our contributions in this chapter can be summarized as follows:

- We consider four sub classes of policies for the end-of-life problem. These policies differentiate in the way they select the switching time to the alternative policy.
- We study conditions under which the objective function is discrete convex in the order quantity. This enables us for each switching time to compute the optimal order quantity applying a simple first order condition.
- We provide an upper bound on the optimal order quantity and propose an enumeration based search algorithm for all the parameter selections for which the discrete convexity property cannot be shown.

Chapter 5 discusses the selection of the optimal dynamic policy for the end-of-life inventory problem. In such a policy the final order quantity is known at the beginning of the final phase, while the switching time is determined by the realizations of the arrival process of customers. In this chapter we solve this general problem by approximating the class of dynamic policies taking values at any point in time during the final service phase by the class of dynamic polices which only take values at a finite set of points within the final service phase. Solving the original problem is a difficult continuous optimal control problem while the approximated problem can be solved by standard Markov decision theory techniques. By replacing our original problem by an approximation we also incur an error and in this chapter we also give a bound on this error. Hence after deciding on the size of the error we first construct a finite subset of time points within the final service phase and apply to the subclass of dynamic policies taking only values in that predetermined finite set the Bellman optimality equations to compute the optimal dynamic switching time and final order quantity. In this chapter, we also study the special case of an arrival intensity function of a non-homogeneous Poisson arrival process being a a piece-wise constant function. The main contributions of this chapter are the following.

- We consider the general class of dynamic policies for the end-of life problem and propose a solution procedure for this problem replacing the set of all dynamic policies by a set of dynamic policies taking only a finite number of possible realized switching times. For this approximation we also compute the error.
- The optimal final order quantity is determined at the beginning of the final service phase while the switching time is a so-called stopping time of the stochastic arrival process. For each realization of this process we compute a so-called optimal policy table.
- We compare the optimal static policy with the optimal dynamic policy and try to understand these solutions.

Although the end-of-life inventory problem did arise from a practical application and as a consequence some heuristic ways were proposed to solve this problem its exact optimal solution has not been given in the literature. This thesis studies this problem and proposes an algorithm to solve this problem in its most general form. At the same time it also discusses the practical important class of static policies and how to determine among this class the optimal policy thereby improving existing heuristic procedures proposed in the literature.

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