ON THE EXISTENCE OF SOLUTIONS FOR INEXTENSIBLE STRING EQUATIONS

by

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ON THE EXISTENCE OF SOLUTIONS FOR INEXTENSIBLE STRING EQUATIONS

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La prima cosa bella.

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Abstract

In this thesis, we analyze existence of solutions for inextensible string equations. In particular, we have results in two directions.

On one hand, we find explicit traveling wave solutions for a system of hyperbolic conservation laws resulting from inextensible string equations via suitable change of variables. Then, we relate this solution with entropy and shock-wave solutions for which an established theory already exists.

On the other hand, we consider the problem with periodic boundary conditions and show local existence of solutions using well-studied results related to the wave equation.

Uzamayan Sicim Denklemlerine Ait Çözümlerin Varlığı Üzerine

Ayk Telciyan

Matematik, Yüksek Lisans Tezi, 2018

Tez Danışmanı: Dr. Öğr. Üyesi Yasemin Şengül Tezel

Anahtar Kelimeler: uzamayan sicim, hiperbolik korunum yasası, gezen dalgalar, periyodik sınır değerleri, sonuçların varlığı

Özet

Bu tezde uzamayan sicim denklemlerinin çözümlerinin varlığı analiz edilmektedir. Daha özel olarak, iki doğrultuda sonuçlar elde edilmektedir.

Öncelikle, uygun değişken değişimleri yaparak uzamayan sicim denklemlerinden elde edilen hiperbolik korunum yasası sistemleri için belirtik gezen dalga çözümleri bulunmaktadır. Daha sonra, bu çözümler varolan teoremler kullanarak entropi ve şok dalgası çözümleriyle ilişkilendirilmektedir.

Diğer yandan, problem periyodik sınır koşulları altında ele alınmakta ve dalga denklemleri hakkında bilinen sonuçlar kullanarak çözümlerin yerel varlığı gösterilmektedir.

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Ayk Telciyan

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CHAPTER 1

Introduction

1.1. Governing Equations

This is a study about the existence of solutions of inextensible string equation. The motion of the string is governed by the following differential system;

$$\begin{cases} \eta_{tt}(t,s) = (\sigma(t,s)\eta_s(t,s))_s + g, \ s \in \mathbb{R} \\ |\eta_s| = 1. \end{cases} \tag{1.1}$$

We have the constraint $|\eta_s| = 1$ coming from inextensibility of the string. In system (1.1), $\eta \in \mathbb{R}^3$ is the unknown position vector for the material point s at time t, g is the gravity constant (we will ignore it when it is convenient) and σ is the unknown scalar multiplier presented in the equation as tension satisfying

$$\sigma_{ss}(t,s) - |\eta_{ss}(t,s)|^2\sigma(t,s) + |\eta_{st}(t,s)|^2 = 0 \tag{1.2}$$

(see Section 2.2 for the derivation of (1.2) from (1.1)). We are given initial positions and velocities of the string as

$$\eta(0,s) = \alpha(s) \quad \text{and} \quad \eta_t(0,s) = \beta(s). \tag{1.3}$$

System (1.1) with (1.2) is the model of the motion done by a homogeneous, inextensible string with unit length.

There are several types of boundary conditions:

1. Two fixed ends: This is the most primitive case of the problem, which can be considered as the easiest case:

$$\eta(t,0) = \alpha(0) \text{ and } \eta(t,1) = \alpha(1) \tag{1.4}$$

2. Two free ends: This case can be seen as a sub-case of periodic boundary conditions. The difficulty of this case is that, since σ takes the value 0 for some t, it will be difficult for us to modify the system and having the hyperbolic conservation law. See Section 2.3 to understand the difficulty.

$$\sigma(t,0) = \sigma(t,1) = 0. \tag{1.5}$$

3. Periodic boundary conditions: This is the case that we will discuss in Chapter 4. One has to be careful about periodic boundary conditions. It is very easy to confuse that our solutions (σ, η) are periodic functions, but we are not interested periodic functions, periodic boundary conditions in 2-dimensions means a punctual equality in each boundary. This case can be thought as a general case, for instance it includes two free end case when $\sigma(t,0) = \sigma(t,1) = 0$:

$$\eta(t,s) = \eta(t,s+1) \text{ and } \sigma(t,s) = \sigma(t,s+1). \tag{1.6}$$

4. Whip boundary conditions (one end is free and one end is fixed): This is the most difficult case to study because there is a certain discontinuity and we do not have any information about derivatives of functions. Although, it is difficult, we see most examples of this condition in nature as bull whip, pendulum etc.:

$$\sigma(t,0) = 0 \text{ and } \eta(t,1) = 0. \tag{1.7}$$

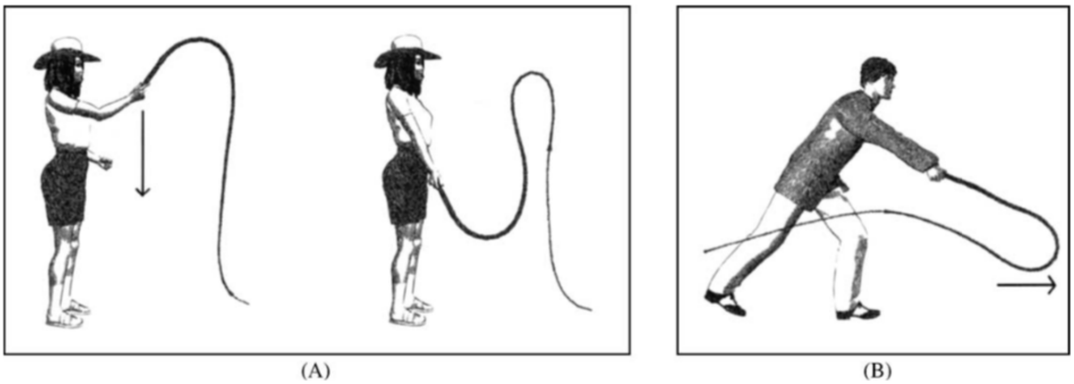

Figure 1.1: Two different cases of whip boundary conditions from Conway [1]

In Figure (1.1), (A) can be thought as periodic whip boundary conditions if the woman is doing the shown action repeatedly and equally and (B) is a regular action of whip boundary conditions, we see that one end is fixed by the hand of the man and the other one is moving.
Taking $s = 0$ domain $[s, s+1]$ becomes $[0,1]$. If we need to explain this better; as we said one must be careful with boundary conditions and specially with periodic boundary conditions, our solutions are repeating them in each boundary, we are not interested what is happening in $(s, s+1)$ interval. Taking $s = 0$, only important values for us become $\sigma(t,0)$, $\sigma(t,0)$, $\eta(t,1)$ and $\eta(t,1)$, since in each period they are equal, these are enough for us to use. Our integral that will be used in analysis is actually $\int_s^{s+1} F(t,s)ds$, for the reasoning, we take $s = 0$ and we will use $\int_0^1 F(t,s)ds$. In this thesis, we use subscript for derivatives (i.e. $u_x = \frac{du}{dx}$), C is a generic positive constant,

unless it is mentioned. The scalar product of two functions is given by $\beta\mu$ and $|\beta|$ is the Euclidean norm $\sqrt{\beta\beta}$, here $\beta, \mu \in \mathbb{R}^3$.

This thesis is consist of five chapters. In the remaining part of the first chapter, we give a long literature review about tools that we use throughout and then give some preliminary information. In Chapter 2, we set the problem, show conservation of energy and mention non-negativity of tension. To understand our modifications better Section 2.3 is very important where we obtain a new system from our system by change of variables. In Chapter 3, we find explicit traveling wave solution of modified system and we compare notions of solutions by passing to the limits in the parameters. In Chapter 4, we look for weak solutions to our equation with periodic boundary conditions, give some bounds for strong solutions which may be useful for future research and then we find some bounds for a modified system. Finally we show that under the condition of positivity of tension, our equation becomes linear wave equation with time dependent coefficient and then we state a result of existence of solution. Lastly, in Chapter 5 we mention our work briefly and speak off possible future research.

1.2. Literature Review

The analysis of the dynamics of inextensible string with different boundary conditions is one of the oldest applications of calculus. Due to its complexity, we still do not have proper results about its well-posedness. The first studies go back time of Galileo, Leibniz and Bernoulli. This problem with periodic boundary condition has never been studied before. One of the earliest successes in the calculus of variations was the demonstration that the inextensible string, hanging under gravity, would have the shape of a catenary (cf. [2]).

The most recent study on this topic is done by Y. Şengül and D. Vorotnikov [3] in 2016. They obtain a hyperbolic conservation law with discontinuous flux and the total variation wave equation, after some transformations which are admissible for all boundary conditions. Then, they work on a similar system which is not discontinuous. When they pass to the limit in this new system, they had to work on Young measures. The assumption of non-negativity of tension is crucial in [3]. After showing that defined energy is conserved, the principal of least action is used. They prove existence of generalized Young measure solutions. Moreover, details for the non-negativity of the tension for strong solutions is discussed.

In [4], Johnson deals with system of pendulum with a point mass attached vertically to the plane and he wants to find a condition to obtain a nontrivial periodic solution for the system of pendulum. His solution represents the angle between the local tangent vector to the string and downward vertical at a point and time. Another person who has periodicity in his results is Veiga [5], but he is interested in time periodic solutions to the nonlinear wave equation with $\eta(t,0) = \eta(t,1) = 1$ as boundary conditions. He wants to develop Greenberg's results in [6]. Where, the author works with a special form of σ and shows that there are some time periodic solutions under some constraints. Veiga assumes R as even functions of class $C^1(]-b,b[)\cap C^3(]-b,b[-\{0\})$ for some $b>0$ and shows the existence of time periodic solutions, where R is used in the special choice of $\sigma(\gamma)=|\gamma|^{m-1}\gamma(1+R(\gamma))$, here $m>1$ but m is chosen as 3 for physical meaning.

It is very common approach to studies of strings using chains, which is very thin material that is inextensible but completely flexible. Preston studies in [7] the motion of inextensible string with whip boundary conditions in the absence of gravity. He proves local existence and uniqueness in a weighted Sobolev space defined for the energy. In addition, he shows persistence of smooth solutions with a restriction. According to [7], V. Yudovich was interested in this problem and obtained some unpublished results. Preston in another article [8] studies the geometric aspects of the space of arcs parameterized by unit speed in the L^2-metric. He proves that the space of arcs is a submanifold of the space of all curves and the orthogonal projection exists but is not smooth, and as a consequence he gets a Riemannian exponential map that is continuous and even differentiable but not C^1.

Reeken approaches to the problem with chains in several articles [9–11]. In [9], he explains the difficulties of string equation without giving information about solutions, and mentions the situation for a non-positive tension and comments on possible solutions of the system. In [10, 11], Reeken uses whip boundary conditions for a classical solution, his results are for infinite string in $\mathbb{R}^3$. He proves local existence and uniqueness for initial data sufficiently close in H^{26} to the vertical solution (cf. [7]). Reeken's results are the only existence results for the string equation.

Preston and Saxton in [12] study geodesics of the H^1 Riemannian metric on the space of inextensible curves. This article is divided in two part; geometric analysis and analytical analysis. They use the results in [8] to show the geodesic equation is C^∞ in a Banach topology which implies that there is a smooth Riemannian exponential map. In addition, they give global-in-time solutions for a special case. They have an extra term in their partial differential equation, $-\alpha^2\eta_{ttss}$, and they work in the absence of gravity. The extra term changes the equation that tension satisfies. They give some informations for different values of α.

Another popular problem is the uniformly rotating inextensible string. Dickey [13] studies the two dimensional dynamic behavior of a geometrically exact inextensible string. He describes a variety of exact solutions and various asymptotic theories. Also, he mentions the similarities between the motion of the inextensible string and galactic motion, combines some theorems from mathematics with astrophysics. Kolodner [14] considers the rotations of a heavy string with one free endpoint. He shows that according to the more accurate non-linear theory, a string can rotate at any velocity and there are n distinct modes of rotation for an n dimensional system. Luning and Perry [15] construct two Picard-type iterative schemes and the sequences generated are proved to converge to a positive solution of that nonlinear boundary value problem. Also, they notice that iterative scheme can be used to solve the inverse problem of determining the angular velocity of the rotating string.

McMillen and Goriely in [16], studied one of the most interesting phenomena of whips. They have seen whips as unique objects due to the crack that they may produce. It is explained why this crack is a sonic bomb. Since it is an article regarding to an observation rather than pure mathematical analysis, they have added different parameters as the material of whip, the radius of whip etc.. In [16], we see wave type approaches to whips, they show by asymptotic analysis that a wave traveling along the whip increases its speed as the radius decreases. Also, there is a numerical scheme to support their experimental and mathematical results. They use the whip boundary conditions, and they give importance to the movement of the hand that moves the string, we see the different cases in Figure (1.2). In this paper, the angle is a variable, that has significant importance in equation, in Figure (1.2) (A), (B), (C) and (D) shows us different angles for the same action, which change their numerical results, but in view of analysis of mathematics they are similar.

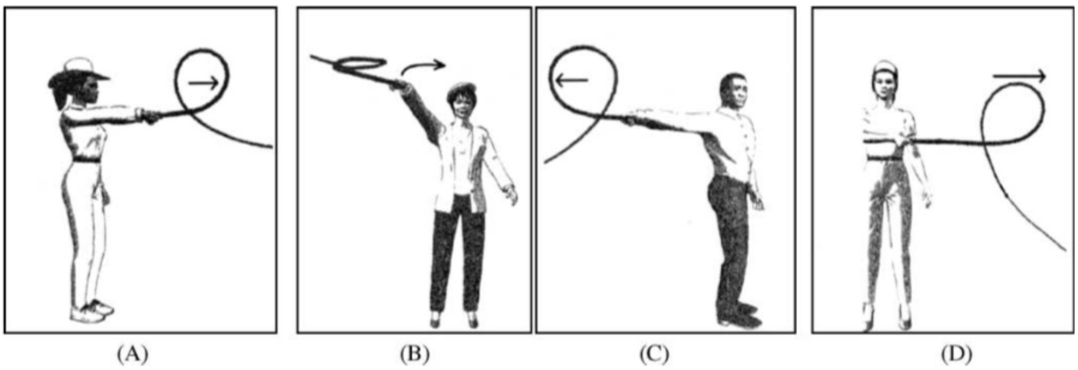

Figure 1.2: Different movements of the string for same case from Conway [1]

Some examples of applications of string equations can be mentioned as follows: Bernstein, Hall and Trent give an example of application of inextensible string equation with whip boundary condition in [17]. They study the production process of a crack produced by the tip of the whip which has higher speed than sound's and produces shock waves. Also, they discuss the speed differences of free end and fixed end and their mathematical structure. They give a mathematical solution assuming that a discontinuity in tension propagates down the whip. They use significant help from the photo cameras of their time, and their article contains many photos of the bull whip movement. Hanna and Santangelo [18] consider planar dynamics under the restriction that the spatially-dependent stress profile in the string is time independent, which results in a conservation law form for the string equation. They find an exact solution whose range of validity is time-dependent, limited to a distance function depending only on t from the free end, but combining the exact solution for the rest of the distance gives an error. Hanna and Santangelo [19] give a model for the growing structure including the amplification, change, and advection of slack in the presence of a steady stress field, validate their assumptions with numerical experiments. In [20], Serre gives a relaxed model for inextensible strings, he discusses two possible approaches to the problem; the relax constraint and the chain as the limit of a stiff elastic string. He says that both shows a concentration phenomena either tension in time or energy in space. Further examples in physics literature can be read in [21] by Wong and Yasui, and in [22] by McMillen, which are good surveys.

We would like to give some references from a crucial tool for us which is hyperbolic systems of conservation law. We believe that these references would be useful for future researchers. Constantine Dafermos has written one of the most important book [23] about hyperbolic conservation laws. In [23], he shows different hyperbolic systems and he gives many different approaches to possible problems. This book is also very nice from the point of view of physicists because he explains each problem by their physical meaning. Another nice book is by Alberto Bressan [24]. He focuses on the one-dimensional Cauchy problem. He explains each possible way to approach to different kind of problems and he gives examples which make everything more understandable.

Freistühler [25] gives existence, uniqueness and stability results for conservation law system which is $u_t + (u\phi(|u|))_x = 0$ where t is time, x is spatial variable, $u : \mathbb{R}^2_+ := \mathbb{R} \times [0, \infty) \to \mathbb{R}^n$ is vector-valued solution of this system. Here $n \geq 2$ which is the dimension of the system. He uses Wagner's results in [26] to show the existence. Freistühler gives slightly more general existence result than Lui and Wang [27]. They have a similar system with $\phi|u| = 1$. They show the existence of the solution to the system by using random of choice method. In both articles polar coordinates are used and also relation between the system, entropy solutions and shock waves are used.

In [28], Freistühler and Plaza study the system of equations

$$U_t - \nabla_x V = 0$$
$$V_t - \mathrm{div}_x \sigma(U) = 0$$

with $\mathrm{curl}_x U = 0$ where t is non-negative, x is d-dimensional spatial variable, U is local deformation gradient, V is local velocity and $\sigma(U)$ is stress. This paper considers an ideal non-thermal elastic medium described by a stored-energy function W. The article provides a normal modes determinant that characterizes the local-in-time linear and nonlinear stability of such patterns. It is studied specially the case that W has two local minimizers U_A, U_B which can coexist via a static planar phase boundary.

Di Perna in [29] studies the 2-dimensional system

$$U_t + F(U)_x = 0$$

where U and F are smooth nonlinear mapping from $\mathbb{R}^2$ to $\mathbb{R}^2$. He follows the article [30] of Lax and he gives the admissibility condition on solution. He needs to define and work with entropy tools because he knows that weak solutions are not uniquely determined by their initial data. In this article, he uses some results of Glimm from [31] that Cauchy problem with arbitrary initial data have small total variation, this allows him to use approximating methods to construct solutions. He approaches to the problem as the limit of a sequence of piecewise constant approximating solutions. Each vector-field of these approximations are exact weak solutions but they are approximating solution in the sense that entropy condition is only satisfied modulo an error term.

In [32], Takaaki Nishida studies the system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t - \left(\frac{a}{v}\right)_x &= 0 \end{aligned}$$

where a is a non-negative constant. This is the equation of gas dynamics, u is the speed of gas and v is the specific volume. Nishida shows the global existence of the weak solution for the Cauchy problem using modified Glimm's difference scheme [31]. According to Nishida's theorem, the L^∞ norm of these weak solutions may increase unboundedly with time. After that Bakhavalov in [33] extends these results, he identified a class of 2×2 systems

$$U_t + F(U)_x = 0$$

with $U = U(u_1, u_2)$ and $F(U) = F(f_1(U), f_2(U))$. Frid [34] knowing these results, has studied a similiar system and he has shown the existence of a global periodic entropy solution of his system

$$\begin{aligned} v_t - u_x &= 0 \\ u_t - p(v)_x &= 0 \end{aligned}$$

where p is a smooth function and satisfying some conditions. He has shown also that existing solution belongs to class $L^\infty \cap BV_{loc}(\mathbb{R} \times \mathbb{R}_+)$. He uses Glimm's scheme [31] to obtain this result.

1.3. Preliminaries

1.3.1. Sobolev Spaces

In this chapter, we give some well-known definitions and theorems. This chapter is mostly taken from [35], unless it is mentioned.

Definition 1.3.1 *Let $f : U \to \mathbb{R}^n$ be a continuous function. It is called locally Lipschitz if for each $x_0 \in U$, there exist constants $M > 0$ and $\delta > 0$ such that $|x - x_0| < \delta$*

$$|f(x) - f(x_0)| \leq M|x - x_0|.$$

The set of Lipschitz continuous functions is denoted by $Lip(U;\mathbb{R}^n)$, where $U \subset \mathbb{R}^n$.

L^p spaces are crucial for us, here we give its definition.

Definition 1.3.2 *For domain $U \subset \mathbb{R}^n$, the function space $L^p(U)$ is defined as*

$$L^p(U) = \left\{ f : U \to \mathbb{R} \Big| \int_U |f(x)|^p dx < \infty \right\}.$$

Now, we give one of the most important property of L^p spaces.

Lemma 1.3.1 *$L^p(U)$ is a Banach space with this norm $||f||_p^p = \int_U |f(x)|^p dx$.*

We need the following definition to be able to define Sobolev spaces.

Definition 1.3.3 *Let $f, F \in L^p(U)$ and α be the multi-index. We say that f is the α^{th} weak derivative of F if it satisfies*

$$\int F D^\alpha \phi dx = (-1)^{|\alpha|} \int f \phi dx, \; \forall \phi \in C_0^\infty(U)$$

where $C_0^\infty(U)$ is the space of infinitely-differentiable functions that are identically 0 outside a compact subset of U. In this case, we denote f by $F^{(\alpha)}$. Here, $\alpha = (\alpha_1, \alpha_2, ..., \alpha_n) \in \mathbb{N}_n$,$|\alpha| = \Sigma_{i=1}^n \alpha_i$ and $D^\alpha = \partial_{x_1}^{\alpha_1} \partial_{x_2}^{\alpha_2} ... \partial_{x_n}^{\alpha_n}$.

Here is the definition of the Sobolev space;

Definition 1.3.4 *The Sobolev space of index (k,p) is $W^{k,p}(U) = \{f^{(\alpha)} \in L^p(U) \big| |\alpha| \leq k\}$ where $f^{(\alpha)}$ denotes the α^{th} weak derivative of f.*

Following theorem is well-known and very useful for our analysis.

Theorem 1.3.2 *The Sobolev space $W^{k,p}(U)$ is a Banach space with this norm*

$$||f||^p_{W^{k,p}(U)} = \sum_{|\alpha| \leq k} ||f^{(\alpha)}||^p_{L^p}(u).$$

1.3.2. Hyperbolic Conservation Laws

This section contains some definitions about hyperbolic conservation laws, which is the most important tool for us, this chapter is taken from [24]

Definition 1.3.5 *A conservation law in one dimension is a first-order differential equation of the form*

$$U_t + F(U)_x = 0. \tag{1.8}$$

Here U is the conserved quantity while F is the flux, x is the spatial variable and t is the time variable.

Since the solution to (1.8) is difficult to find, we define the weak derivative as follows;

Definition 1.3.6 *The weak solution U of (1.8) satisfies*

$$\int\int \{U\phi_t + F(U)\phi_x\} dxdt = 0. \tag{1.9}$$

Here ϕ is the test function (i.e $\phi \in C_0^\infty(\mathbb{R})$). The equation (1.8) is the way to represent the $n \times n$ system of conservation law of the form

$$\begin{cases} (U_1)_t + (F_1(U_1, U_2, ..., U_n))_x = 0 \\ \vdots \\ (U_n)_t + (F_n(U_1, U_2, ..., U_n))_x = 0 \end{cases} \tag{1.10}$$

The following definition will lead us to make modification and to study with a similar system that has this property.

Definition 1.3.7 *If $A(U) = DF(U)$ is the $n \times n$ Jacobian matrix of the map F at the point U, the system can be written in the quasilinear form*

$$U_t + A(U)U_x = 0, \tag{1.11}$$

this form is also called non-divergent form. We say that this system is strictly hyperbolic if every matrix $A(U)$ has n real, distinct eigenvalues, say $\lambda_1(U) < \lambda_2(U) < ... < \lambda_n(U)$.

CHAPTER 2

Setting of the Problem

In this chapter, we give important tools for our problem. The calculations done in this chapter are used on the analysis of existence of solutions.

2.1. Energy Conservation

Now, we show that energy does not change by time for strong solutions, this has led us to use conservation laws.

Proposition 2.1.1 (Proposition 2.5 [3]) *Let (η, σ) be a regular solution of (1.1) with each boundary condition. Then the total energy does not change by time, furthermore it is also conserved in absence of gravity.*

Proof: Firstly, let us define kinetic energy and potential energy as

$$K(t) =: \frac{1}{2}\int_0^1 |\eta_t|^2 ds \text{ and } P(t) =: -\int_0^1 g\eta ds.$$

Using equation (1.1) and defining the total energy as $E(t) = K(t) + P(t)$ we have

$$\begin{aligned} \frac{dE(t)}{dt} &= \frac{d(K(t)+P(t))}{dt} \\ &= \int_0^1 \eta_t\eta_{tt} - g\eta_t ds = \int_0^1 \eta_t(\eta_{tt} - g)ds = \int_0^1 \eta_t(\sigma\eta_s)_s ds \end{aligned}$$

Now using $|\eta_s|^2 = 1$,which implies that $\eta_s\eta_{st} = 0$, and using integration by parts on the right-hand side, we are left with

$$\frac{dE(t)}{dt} = \sigma\eta_s\eta_t\Big|_{s=0}^{s=1} - \int_0^1 \sigma\eta_s\eta_{ts}ds = \sigma(t,1)\eta_s(t,1)\eta_t(t,1) - \sigma(t,0)\eta_s(t,0)\eta_t(t,0).$$

We see that these terms vanish using chosen boundary conditions, which means that there is no change in energy by time.
Also, we can show that energy is conserved in the absence of gravity (i.e. $g = 0$). For this, multiply (1.1) by η_t and take the integral with respect to the spatial variable to obtain

$$\int_0^1 \eta_{tt}\eta_t ds = \int_0^1 \eta_t(\eta_s\sigma)_s ds$$

We see that the term on the left-hand side can be written as time derivative of a function and we use integration by parts on right-hand side to get

$$\frac{d}{dt}\int_0^1 \frac{|\eta_t|^2}{2}ds = \eta_t\eta_s\sigma\Big|_{s=0}^{s=1} - \int_0^1 \eta_{st}\eta_s\sigma ds.$$

The right hand side of this equality is 0; since the first term vanishes due to boundary conditions and the second term is 0 by $\eta_{st}\eta_s = 0$. So,

$$\frac{d}{dt}E(t) = \frac{d}{dt}K(t) = 0.$$

This means that energy does not change by time also in the case of absence of gravity. $\square$

2.2. Non-negativity of Tension

We write the derivatives of the constraint $|\eta_s|^2 = 1$ as

- $$\frac{d|\eta_s|^2}{ds} = \frac{d}{ds}1 \Rightarrow \eta_{ss}\eta_s = 0 \tag{2.1}$$
- $$\frac{d^2|\eta_s|^2}{dt^2} = \frac{d^2}{dt^2}1 \Rightarrow \eta_{st}\eta_{st} + \eta_s\eta_{stt} = 0 \tag{2.2}$$

Now, multiplying (1.1) by η_s and using (2.1), we get

$$\eta_s\eta_{tt} = \sigma_s + \eta_s g \tag{2.3}$$

Differentiating (2.3) with respect to the spatial variable and then combining it with (2.2), we obtain

$$\sigma_{ss} - (\eta_{tt} - g)\eta_{ss} + |\eta_{st}|^2 = 0.$$

Using the fact that $\eta_{tt} - g = \eta_s\sigma_s + \eta_{ss}\sigma$, we find the equation of tension as

$$\sigma_{ss}(t,s) - |\eta_{ss}(t,s)|^2\sigma(t,s) + |\eta_{st}(t,s)|^2 = 0.$$

Now, we will give the non-negativity of σ. This fact is very important for our analysis.

Proposition 2.2.2 (Proposition 2.4 [3]) *Let (η, σ) be regular solution of (1.1) and (1.2) . For all boundary conditions $\sigma \geq 0$ for all t.*

2.3. Obtaining a Conservation Law

In this chapter, we transform our partial differential equation into a system of two equations. Hereafter in this thesis, we take $g = 0$. We start by putting $\kappa := \sigma\eta_s$, then using non-negativity of σ that we obtained in previous section, we write $\sigma = |\kappa|$ and $\eta_s = \frac{\kappa}{|\kappa|}$. Now, our system (1.1) is in this form;

$$\begin{cases} \eta_{tt} = \kappa_s \\ \eta_s = \dfrac{\kappa}{|\kappa|}. \end{cases}$$

We do another change of variables by putting $\upsilon := \eta_t$ and our syste becomes the following

$$\begin{cases} \upsilon_t = \kappa_s \\ \upsilon_s = \left(\dfrac{\kappa}{|\kappa|}\right)_t. \end{cases} \tag{2.4}$$

This system (2.4) is mentioned and studied implicitly by Dafermos in [23]. We see that in Chapter 7.1, Dafermos gives many types of hyperbolic conservation laws, and (2.4) is similar to the Equation 7.1.14 in indicated chapter. As, we do in Chapter 3 and 4, one must swap the spatial and the time variables to able to obtain general system of conservation law (1.8).

We will use system (2.4), in our analysis for traveling waves and existence of weak solutions with periodic boundary conditions. Since we want to use the theory of hyperbolic conservation laws, we will swap our time and spatial variable. Once we have such a system in form of

$$\beta_t + F(\beta)_s = 0,$$

where $\beta = (\kappa, \upsilon)$ is our solution, we will modify the system to have a hyperbolic conservation law. Here F is a 2×2 matrix. Swapping s and t, and writing F in non-divergence form (i.e. $F(\beta)_x = B(\beta)\beta_x$ where B is 2×2 matrix), we have

$$\beta_t + B(\beta)\beta_s = 0$$

with

$$B = \begin{pmatrix} 0 & -1 \\ p'(\kappa) & 0 \end{pmatrix}.$$

Notice that it is not easy to deal with $p(\kappa)_s = \left(\dfrac{\kappa}{|\kappa|}\right)_s$, since its derivative may not exist. We will do some modifications in following chapters and we will explain again why. Moreover, we want to use hyperbolic systems, but with this difficult derivative we do not have a hyperbolic system. Here, eigenvalues are either 0 or undefined. To overcome this problem of derivative, we will assume $\sigma \neq 0$, but it is natural; $\frac{\kappa}{|\kappa|} = \eta_s$ and $|\eta_s| = 1$, these imply that $\kappa \neq 0$ and this implies automatically $\sigma \neq 0$ by definition of κ.

Now, we try to explain our modifications to have a hyperbolic system of conservation law. We start by reminding the general form of hyperbolic systems (1.8)

$$U_t + F(U)_x = 0.$$

Also, we know by the definition of hyperbolic system of conservation law that $F(z)$ has to have distinct and real eigenvalues of (1.8) to be called a hyperbolic system, where $z = (z_1, z_2, ..., z_n)$. In our case, (2.4) is 2-dimensional. Also, we will use the non-divergence form of (1.8):

$$U_t + B(U)U_x = 0. \tag{2.5}$$

$B(U)U_x$ and $F(U)_x$ are 2×2 matrices.

Hereafter, we exchange t and s variables in (2.4) to have a system in form (1.8) and we obtain;

$$\begin{cases} \upsilon_s - \kappa_t = 0 \\ \upsilon_t - \left(\dfrac{\kappa}{|\kappa|}\right)_s = 0. \end{cases} \tag{2.6}$$

The system is called 2-dimensional p-system, here $p(x) = \dfrac{x}{|x|}$. In system (2.6), the second line is the equation of motion and the first one is the compatibility condition. One can write

$$\beta_{tt} - (p(\beta_s))_s = 0 \text{ in } \mathbb{R} \times (0, \infty).$$

Here, writing $\upsilon =: \beta_s$ and $\kappa = \beta_t$, we can obtain (2.6). Also, having our system (2.6) in form of (1.8), we have

$$F(\beta) = (-\upsilon, -p(\kappa)),$$

where $z = (z_1, z_2)$. In the rest of thesis, we will do different kind of modifications, to use system (2.6) without any problem.

CHAPTER 3

Traveling Wave Solutions

In this chapter, we show existence of a traveling wave solution of our system. We make also a comparison of the results of systems of hyperbolic conservation laws in [35] about entropy solutions and shock wave solutions of [36]. One must be careful that traveling wave solutions are particular solutions of the system. First of all, we give some definitions.

Definition 3.0.1 ([35]) *Let $u(x,t)$, a function of two variables, be a solution of a partial differential equation. A particular solution u of the form*

$$u(x,t) = v(x-ct) \quad (x \in \mathbb{R}, t \in \mathbb{R})$$

is called a traveling wave solution, where c is velocity and v is the wave profile.

Now remember that our system (2.4) is

$$\begin{cases} \upsilon_t - \kappa_s = 0 \\ \upsilon_s - \left(\dfrac{\kappa}{|\kappa|}\right)_t = 0. \end{cases}$$

We will try to find traveling wave solutions of (2.4). We know that a system of n-dimensional conservation law is written in this form for a solution U; for the traveling wave solutions we will have $\beta = \beta(t,s)$. Now, we convert β into a single variable function by transformation

$$\beta(t,s) = \mu(s-ct),$$

here $c \in \mathbb{R}$ is the velocity. For our new system (2.6), we have

$$\beta_t + B(\beta)\beta_s = 0,$$

where

$$B(z) = \begin{pmatrix} 0 & -1 \\ -p'(z_1) & 0 \end{pmatrix}.$$

The eigenvalues of $B(z)$ are $\lambda_1 = -p'(z_1)^{1/2}$ and $\lambda_2 = p'(z_1)^{1/2}$. Now, we need to check $p'(z_1)$ for the strict hyperbolicity condition. Having $p(x) = \dfrac{x}{|x|}$, we find its derivative as

$$p'(x) = \begin{cases} \text{undefined} & \text{if } x = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Since we do not have two distinct and real eigenvalues, we cannot use the theory of hyperbolic conservation law. To avoid this problem, we need to modify system (2.6). Let $\delta > 0$ be constant, then we can have a new system by modifying the equation of motion as

$$\begin{cases} \upsilon_s - \kappa_t = 0 \\ \upsilon_t - \left(\frac{\kappa}{\sqrt{\delta+|\kappa|^2}}\right)_s = 0. \end{cases} \tag{3.1}$$

Here, $p(x) = \frac{x}{\sqrt{\delta+|x|^2}}$ and its derivative becomes $p'(x) = \frac{\delta}{(\delta+|\kappa|^2)^{3/2}}$, which is always positive (i.e. $p' > 0$), which was our goal to modify. As δ approaches to 0, system (3.1) converges to (2.6). Here, since δ is a very small constant, adding it will not change anything physically.

Now, we will start to show the existence of traveling wave solution explicitly for system (3.1). Just to avoid the confusion, since we will have δ and ϵ in modified system, instead of β, we will write $\beta_{\delta,\epsilon}$. We start by adding a viscous term to the equation of motion to get, for $\epsilon > 0$,

$$\begin{cases} \partial_s \upsilon_{\delta,\epsilon} - \partial_t \kappa_{\delta,\epsilon} = 0 \\ \partial_t \upsilon_{\delta,\epsilon} - \partial_s\left(\frac{\kappa_{\delta,\epsilon}}{\sqrt{\delta+|\kappa_{\delta,\epsilon}|^2}}\right) = \epsilon \partial_{ss} \upsilon_{\delta,\epsilon}. \end{cases} \tag{3.2}$$

Theorem 3.0.1 *For $\delta > 0$ and $\epsilon > 0$, there exists a traveling wave solution $\mu_{\delta,\epsilon} = (\xi_{\delta,\epsilon}, \gamma_{\delta,\epsilon})$ for the system (3.2).*

Proof: We start to solve explicitly the system (3.2) and we make the transformations $\beta_{\delta,\epsilon}(t,s) = \mu_{\delta,\epsilon}\left(\frac{s-ct}{\epsilon}\right)$, $\upsilon_{\delta,\epsilon} = \gamma_{\delta,\epsilon}$ and $\kappa_{\delta,\epsilon} = \xi_{\delta,\epsilon}$, here μ is the traveling wave profile of β, we use $(\cdot)' = \frac{d}{da}$ where $a = \left(\frac{s-ct}{\epsilon}\right)$, we obtain

$$\begin{cases} \gamma'_{\delta,\epsilon} + c\xi'_{\delta,\epsilon} = 0 \\ \gamma''_{\delta,\epsilon} + c\gamma'_{\delta,\epsilon} + \left(\dfrac{\delta}{(\delta + \xi^2_{\delta,\epsilon})^{3/2}}\right)\xi'_{\delta,\epsilon} = 0. \end{cases} \tag{3.3}$$

Now, we try to solve (3.3) explicitly. Taking the derivative of the first equation in (3.3), we obtain $\gamma''_{\delta,\epsilon} = -c\xi''_{\delta,\epsilon}$ and $\gamma'_{\delta,\epsilon} = -c\xi'_{\delta,\epsilon}$, then substituting into second equation of (3.3), we end up with

$$\xi''_{\delta,\epsilon} - \left(\frac{\delta}{c}\Big(\frac{1}{(\delta+\xi^2_{\delta,\epsilon})^{3/2}}\Big) - c\right)\xi'_{\delta,\epsilon} = 0 \tag{3.4}$$

We can write that

$$\xi'_{\delta,\epsilon} = G(\xi_{\delta,\epsilon}) \text{ where } G(z) = \int \left(\frac{\delta}{c}\Big(\frac{1}{(\delta+z^2)^{3/2}}\Big) - c\right)dz, \text{ for } z = \xi_{\delta,\epsilon}.$$

Firstly, we find $G(z)$ as

$$G(z) = \int \left(\frac{\delta}{c}\Big(\frac{1}{(\delta+z^2)^{3/2}}\Big) - c\right)dz$$

Let $h(z) = \int H(z)dz$ so that

$$H(z) = \int \frac{\delta}{c}\Big(\frac{1}{(\delta+z^2)^{3/2}}\Big)dz \tag{3.5}$$

Now, use substitution $z = \sqrt{\delta}\tan u$ so that $u = \arctan\left(\frac{z}{\sqrt{\delta}}\right)$. Now substituting this into (3.5), we have

$$\int \frac{\sqrt{\delta}\sec^2 u}{(\delta\tan^2 u+\delta)^{3/2}}du = \int \frac{\sqrt{\delta}\sec^2 u}{\sqrt{\delta}\delta\sec^3 u}du = \frac{1}{\delta}\int \frac{1}{\sec u}du = \frac{1}{\delta}\int \cos u du.$$

and finally it gives

$$h(u) = \frac{1}{\delta}\sin u + k_1.$$

Here, k_1 is constant, and without loss of generality, we take $k_1 = 0$. Now, we need to go back to the solution with z by using $u = \arctan\left(\frac{z}{\sqrt{\delta}}\right)$ and we find that

$$h(z) = \frac{z}{c(\delta+z^2)^{1/2}}.$$

Finally, by the definition of $G(z)$ we find that

$$G(z) = \frac{z}{c(\delta+z^2)^{1/2}} - cz$$

Now, we can solve

$$z' = G(z). \tag{3.6}$$

Writing $(z)' = \frac{dz}{da}$, we see that, (3.6) is a separable differential equation. We have

$$\int \frac{dz}{\left(\frac{z}{c(\delta+z^2)^{1/2}} - cz\right)} = \int da \tag{3.7}$$

then

$$\int \frac{dz}{\left(\frac{z}{c(\delta+z^2)^{1/2}} - cz\right)} = a + k_2, \tag{3.8}$$

here k_2 is a constant and again we can take it as 0. Let $\int F(z)dz = f(z)$ so that

$$F(z) = \frac{1}{\dfrac{z}{c(\delta + z^2)^{1/2}} - cz}. \tag{3.9}$$

Start by integrating both sides of (3.9) with respect to z, then we have

$$f(z) = \int \frac{dz}{\dfrac{z}{c(\delta + z^2)^{1/2}} - cz}.$$

Firstly, for simplicity multiply the denominator and the numerator by c. Now, we substitute $u = \sqrt{\delta + z^2}$ then $du = \frac{\xi}{\sqrt{\delta+z^2}}dz$ which can also be written as $udu = zdz$ and $u^2 - \delta = (z)^2$. Using all we have

$$f(u) = c\int \frac{u^2}{(1 - c^2u)(u^2 - \delta)}du. \tag{3.10}$$

To calculate (3.10), we will use the partial fractions as follows

$$\frac{u^2}{(1 - c^2u)(u^2 - \delta)} = \frac{Au + B}{u^2 - \delta} + \frac{C}{1 - c^2u}, \tag{3.11}$$

and we find $A = \dfrac{\delta c^2}{1 - \delta c^4}$, $B = \dfrac{\delta}{1 - \delta c^4}$ and $C = \dfrac{1}{1 - \delta c^4}$. Now, we can write

$$\int \frac{u^2}{(1 - c^2u)(u^2 - \delta)}du = \frac{1}{1 - c^4\delta}\int \delta c^2\frac{u}{u^2 - \delta} + \delta\frac{1}{u^2 - \delta} + \frac{1}{1 - c^2u}du.$$

Notice that we have to use another partial fraction method to the second term on right hand side to able to integrate it easily and we end up with

$$f(u) = \frac{c}{1 - \delta c^4}\Big[\frac{\delta c^2}{2}\ln|u^2 - \delta| + \frac{\sqrt{\delta}}{2}\Big(\ln\Big|\frac{u - \sqrt{\delta}}{u + \sqrt{\delta}}\Big|\Big) - \frac{1}{c^2}\ln|1 - c^2u|\Big]. \tag{3.12}$$

Substituting back $u = \sqrt{\delta + z^2}$ and z into (3.12), we end up with

$$f(\xi_{\delta,\epsilon}) = \frac{c}{1 - \delta c^4}\Big[\frac{\delta c^2}{2}\ln\xi_{\delta,\epsilon} + \frac{\sqrt{\delta}}{2}\Big(\ln\Big|\frac{\sqrt{\delta + \xi_{\delta,\epsilon}^2} - \sqrt{\delta}}{\sqrt{\delta + \xi_{\delta,\epsilon}^2} + \sqrt{\delta}}\Big|\Big) - \frac{1}{c^2}\ln\Big|1 - c^2\sqrt{\delta + \xi_{\delta,\epsilon}^2}\Big|\Big]. \tag{3.13}$$

Finally, having (3.13), we have the solution of (3.4) as

$$\frac{c}{1 - \delta c^4}\Big[\frac{\delta c^2}{2}\ln\xi_{\delta,\epsilon} + \frac{\sqrt{\delta}}{2}\Big(\ln\Big|\frac{\sqrt{\delta + \xi_{\delta,\epsilon}^2} - \sqrt{\delta}}{\sqrt{\delta + \xi_{\delta,\epsilon}^2} + \sqrt{\delta}}\Big|\Big) - \frac{1}{c^2}\ln\Big|1 - c^2\sqrt{\delta + \xi_{\delta,\epsilon}^2}\Big|\Big] = \frac{s - ct}{\epsilon}. \tag{3.14}$$

Using that $\gamma' = -c\xi'$, we may write that

$$\frac{-1}{c}\frac{d\gamma}{da} = \frac{d\xi}{da},$$

then integrating both sides with respect to a, we have

$$\frac{-1}{c}\gamma = \xi + b^*,$$

where b^* is an integration constant. Without loss of generality, take $b^* = 0$. Finally, we have

$$\frac{c}{1-\delta c^4}\left[\frac{\delta c^2}{2}\ln\left|\frac{-\gamma_{\delta,\epsilon}}{c}\right| + \frac{\sqrt{\delta}}{2}\left(\ln\left|\frac{\sqrt{\delta+\frac{\gamma_{\delta,\epsilon}^2}{c^2}}-\sqrt{\delta}}{\sqrt{\delta+\frac{\gamma_{\delta,\epsilon}^2}{c^2}}+\sqrt{\delta}}\right|\right) - \frac{1}{c^2}\ln\left|1-c^2\sqrt{\delta+\frac{\gamma_{\delta,\epsilon}^2}{c^2}}\right|\right] = \frac{s-ct}{\epsilon}. \tag{3.15}$$

So, we find a traveling wave solution $\mu_{\delta,\epsilon} = (\xi_{\delta,\epsilon}, \gamma_{\delta,\epsilon})$. □

Now, since we see how $\mu_{\delta,\epsilon} = (\xi_{\delta,\epsilon}, \gamma_{\delta,\epsilon})$ depends on ϵ and δ explicitly, we look at limits as $\epsilon \to 0$ and $\delta \to 0$. Later in Sections 3.1 and 3.2, we will compare these limits with other notions of solutions.

Now as ϵ approaches to 0, we find that the right-hand sides of (3.14) and (3.15) are

$$\lim_{\epsilon\to 0}\frac{s-ct}{\epsilon} = \begin{cases} -\infty, & \text{if } s < ct \\ \infty, & \text{if } s > ct. \end{cases} \tag{3.16}$$

Left-hand sides of (3.14) and (3.15) have to have the same values of (3.16). Taking the limits for ϵ approaches to 0, we find

$$\lim_{\epsilon\to 0}\xi_{\delta,\epsilon}\Big(\frac{s-ct}{\epsilon}\Big) = \begin{cases} 0, & \text{if } s < ct \\ 0, & \text{if } s > ct. \end{cases} \tag{3.17}$$

and

$$\lim_{\epsilon\to 0}\gamma_{\delta,\epsilon}\Big(\frac{s-ct}{\epsilon}\Big) = \begin{cases} 0, & \text{if } s < ct \\ 0, & \text{if } s > ct. \end{cases} \tag{3.18}$$

Here, for $|a| \to \infty$, $\xi_{\delta,\epsilon} = 0$ and $\gamma_{\delta,\epsilon} = 0$ gives us ∞ and we may assume some conditions on c to find exact sign of infinity with respect to right hand-side knowing that δ is positive, we have (3.17) and (3.18).
Now, let δ approaches to 0 in (3.14), since δ is not there we do not indicate it as subscript, we find

$$\frac{-1}{c}\ln|1-c^2\xi_\epsilon| = \frac{s-ct}{\epsilon},$$

then

$$\xi_\epsilon(a) = \frac{1}{c^2} - \frac{1}{c^2}e^{-ca}. \tag{3.19}$$

and using $\gamma_\epsilon' = -c\xi_\epsilon'$, we find

$$\gamma_\epsilon(a) = k^* + \frac{1}{c}e^{-ca} \tag{3.20}$$

where k^* is an integration constant.

3.1. Shock Wave Solutions

In this section, we want show the existence of a shock wave solution. To show this, we will use the theorem in [36] by Conlon.

Theorem 3.1.2 ([36]) *There is a constant $\epsilon(m, M) > 0$ depending only on m, M such that the shock wave with $U_l = 0$ and $|u| \leq \epsilon$ satisfies the strictly entropy condition if and only if there is a traveling wave solution of $\beta_t + B(\beta)\beta_s = \epsilon\beta_{ss}$ contained in an ϵ neighborhood of 0 which joins 0 to U_r. The traveling wave is unique if it exists. (Here M is the bound for $B(\beta)''$, which is automatically bounded by convexity and $m > 0$ guarantees the distinct eigenvalues (i.e.$|\lambda_1 - \lambda_2| > m$)).*

Let us give some definitions to understand the theorem better

Definition 3.1.2 ([35]) *The system (1.8) with*

$$g = \begin{cases} U_l, & \text{if } x < 0 \\ U_r, & \text{if } x > 0 \end{cases} \tag{3.21}$$

This is called Riemann's problem. U_r and U_l are given vectors and called right and left initial states.

Now, we will give some definitions related to shock waves.

Definition 3.1.3 ([36]) *A weak solution $U(x, t)$ of (1.8) is called a shock wave if it has the form*

$$U = \begin{cases} U_r, & \text{if } x < ct \\ U_l, & \text{if } x > ct. \end{cases} \tag{3.22}$$

In this case U_r, U_l, c are related to by the Rankie-Hugoniot equation

$$F(U_r) - F(U_l) = c[U_r - U_l].. \tag{3.23}$$

Also, we would like to define the limits for $U_\epsilon(s, t) = V\left(\frac{s-ct}{\epsilon}\right)$ and $\lim_{a\to-\infty} = U_l$ and $\lim_{a\to\infty} = U_r$, where $a = \left(\frac{s-ct}{\epsilon}\right)$. Futhermore, If U_ϵ is a shock wave then (U_r, U_l, c) satisfies (3.23) and $\frac{dV}{da} = F(V) - F(U_l) - c[V - U_l]$. Here, U_ϵ which is the solution of $U_t + F(U)_x = \epsilon U_{xx}$ converges to the shock wave solution of $U_t + F(U)_x = 0$.

Now, we will find explicit solution for following system, and to avoid the confusion we write β as β_ϵ

$$\begin{cases} \partial_s \upsilon_\epsilon - \partial_t \kappa_\epsilon = 0 \\ \partial_t \upsilon_\epsilon - \partial_s \left(\dfrac{\kappa_\epsilon}{|\kappa_\epsilon|}\right) = \epsilon \partial_{ss} \upsilon_\epsilon. \end{cases} \tag{3.24}$$

It is easy to notice that, in (3.24) we do not have δ. System (3.24) is not a hyperbolic system, since it does not have distinct and real eigenvalues, but we have been curious about its solutions, also we will show that the traveling wave solutions that we have found converges to system (3.24) as $\delta \to 0$.

Again, we choose ξ_ϵ and γ_ϵ as traveling wave profiles of κ_ϵ and υ_ϵ respectively, and again $\beta_\epsilon = (\upsilon_\epsilon, \kappa_\epsilon)$, $\beta_\epsilon(t,s) = \mu_\epsilon\left(\frac{s-ct}{\epsilon}\right)$ these two imply $\mu_\epsilon = (\xi_\epsilon, \gamma_\epsilon)$, we call $a = \frac{s-ct}{\epsilon}$. After, we do these transformations, we obtain a system of second order differential equation, which is

$$\begin{cases} \gamma_\epsilon' + c\xi_\epsilon' = 0 \\ \gamma_\epsilon'' + c\gamma_\epsilon' + p'(\xi_\epsilon)\xi_\epsilon' = 0. \end{cases}$$

Here, we ignore that for some a, we may have $\xi_\epsilon = 0$, and now we can say that $p'(\xi_\epsilon) = 0$, and the system becomes

$$\begin{cases} \gamma_\epsilon' + c\xi_\epsilon' = 0 \\ \gamma_\epsilon'' + c\gamma_\epsilon' = 0. \end{cases} \tag{3.25}$$

We will find explicit solutions for our new system (3.25).

$$\gamma_\epsilon'' + c\gamma_\epsilon' = 0$$

here, since we are looking for nontrivial solutions, we assume that $\gamma_\epsilon = e^{ra}$ and we obtain the characteristic equation $r^2 + cr = 0$, here roots are $r_1 = 0$ and $r_2 = -c$ and the solution is

$$\gamma_\epsilon(a) = b_1 + b_2 e^{-ca}. \tag{3.26}$$

Now, using $\gamma_\epsilon(a)$, we can find $\xi_\epsilon(a)$ from the first line of (3.25), and

$$\xi_\epsilon(a) = b_3 - \frac{b_2}{c} e^{-ca}. \tag{3.27}$$

Here, b_1, b_2 and b_3 are constants. Now, we have the solutions of (3.24) and (3.3). We know that as δ approaches to 0, (3.3) must converge to (3.24). To have this, we may choose $b_3 = \frac{1}{c^2}$,$b_2 = \frac{-1}{c}$ and $b_1 = k^*$ in (3.27) and we have the same solutions. So, we see that system (3.3) has a traveling wave solution which is the result in Theorem 3.1.2

Also, we need to know what happens when ϵ approaches to 0.

$$\lim_{\epsilon \to 0} \gamma_\epsilon\left(\frac{s-ct}{\epsilon}\right) = \begin{cases} \infty, & \text{if } s < ct \\ k^*, & \text{if } s > ct \end{cases}$$

and

$$\lim_{\epsilon \to 0} \xi_\epsilon\left(\frac{s-ct}{\epsilon}\right) = \begin{cases} -\infty, & \text{if } s < ct \\ \frac{1}{c^2}, & \text{if } s > ct. \end{cases}$$

$\mu_\epsilon = (\xi_\epsilon, \gamma_\epsilon)$ are shock wave solutions, so we have the following theorem

Theorem 3.1.3 *There exists a shock wave solution to system* (2.4).

Proof: (3.27) and (3.26) are shock wave solutions. $\square$

3.2. Entropy Solutions

In this section, we use the theory from [35] about entropy solutions for hyperbolic system of conservation laws and then we will compare the previous theorem of Conlon.

Definition 3.2.4 ([**35**]) *Two smooth functions* $\Phi : \mathbb{R}^n \to \mathbb{R}$ *is called an entropy for the system* (1.8)*,with entropy flux* $\Psi : \mathbb{R}^n \to \mathbb{R}$*,if*

$$D\Phi(U) \cdot DF(U) = D\Psi(U), \;\; U \in \mathbb{R}.$$

Definition 3.2.5 ([**35**]) *A weak solution* U *of* (1.8) *is called entropy admissible solution if it satisfies the following inequality*

$$\Phi(U)_t + \Psi(U)_x \leq 0 \tag{3.28}$$

in the distributional sense, for every pair (Φ, Ψ)*, where* Φ *is a convex entropy for* (1.8) *and* Ψ *is the corresponding entropy flux.*

We have given a general definition of entropy condition. There are entropy conditions for special cases which are derived from the general one, but first, we need other definitions to define Liu entropy criterion.

Definition 3.2.6 ([**35**]) *Let* $U_r \in S_k(U_l)$ *for some* $k \in \{1, ..., m\}$*, Lui's entropy criterion is as following*

$$\begin{cases} c(z, U_l) > c(U_r, U_l) \text{ for each } z \text{ lying} \\ \text{on the curve } S_k(U_l) \text{ between } U_r \text{ and } U_l. \end{cases}$$

Now, we will show the existence of traveling wave solutions using entropy criteria Let $\beta(t, s) = \mu\left(\frac{s-ct}{\epsilon}\right)$ and substitute $\upsilon\left(\frac{s-ct}{\epsilon}\right) = \gamma$ and $\kappa\left(\frac{s-ct}{\epsilon}\right) = \xi$, which implies that $\mu(a) = (\xi, \gamma)$. Substituting the traveling wave profiles into (3.2), we have

$$\begin{cases} \gamma' + c\xi' = 0 \\ \gamma'' + c\gamma' + p'(\xi)\xi' = 0 \end{cases} \tag{3.29}$$

where, $(\frac{d}{da} =')$. Since we are looking for the solutions of the system (3.1), we need to know what happens to these functions as ϵ goes to 0. So, we define the limits

$$\lim_{a \to \infty} \mu = \mu_r, \quad \lim_{a \to -\infty} \mu = \mu_l, \quad \lim_{a \to \pm\infty} \mu' = 0. \tag{3.30}$$

Here, subindexes l and r are used to show the waves going left and right. Now we integrate system (3.29) from $-\infty$ to ∞ using (3.30) to get;

$$\begin{cases} \gamma + c\xi = \gamma_l + c\xi_l = \gamma_r + c\xi_r \\ \gamma' = c(\gamma_l - \gamma) + (p(\xi_l) + p(\xi)) = c(\gamma_r - \gamma) + (p(\xi_r) + p(\xi)). \end{cases} \tag{3.31}$$

Now, we solve the system (3.31) for c and we obtain

$$c^2 = \frac{p(\xi_r) - p(\xi_l)}{\xi_r - \xi_l}. \tag{3.32}$$

Suppose hereafter $\xi_r > \xi_l$, since $p' > 0$ we may take $c > 0$.

Evans in [35] claims that (3.29) with (3.30) has a traveling wave solution if and only if Liu's entropy criterion holds. Start by eliminating γ in (3.31) and we have

$$\xi' = \frac{p(\xi) - (\xi_l)}{c} - c(\xi - \xi_l) \text{ or } \xi' = p(\xi) - (\xi_l) - c^2(\xi - \xi_l). \tag{3.33}$$

Call $g(\xi) =: \xi'$, we can calculate easily that $g(\xi_l) = 0$ and $g(\xi_r) = 0$. Thus, in order that (3.33) has a solution with $\lim_{a\to\infty} \xi = \xi_r$ and $\lim_{a\to-\infty} \xi = \xi_l$, we require

$$g(z_1) > 0 \text{ for } \quad \xi_l < z_1 < \xi_r. \tag{3.34}$$

Briefly explaining this requirement, since $p' > 0$ and $\xi_l < \xi_r$, there must be a point for all z on the curve $S_k(\mu_l)$ between βl and βr, here $S_k(\mu_l)$ is the shock set. Now, we can write for all z on the curve that

$$\frac{p(z_1) - p(\xi_l)}{z_1 - \xi_l} > \frac{p(\xi_r) - p(\xi_l)}{\xi_r - \xi_l}, \tag{3.35}$$

and (3.35) is exactly Liu's entropy criterion. So, it has a traveling wave solution solution.

Now, we give some definitions

Definition 3.2.7 ([35]) *U is called an integral solution of $U_t + F(U)_x = 0$ provided the equality*

$$\int_0^\infty \int_{-\infty}^\infty \Big(UV_t + F(U)V_x\Big)dxdt = 0$$

holds for all test functions V. Here $U \in L^\infty(\mathbb{R}^n \times (0,\infty); \mathbb{R}^m)$ and $V : \mathbb{R} \times [0,\infty) \to \mathbb{R}^m$ is smooth and has compact support.

Definition 3.2.8 ([35]) *U is called an entropy solution of (1.8) provided U is an integral solution and U satisfies the (3.28) for each entropy/entropy flux pair (Φ, Ψ).*

Remember that, we found the explicit solution of (3.2) and we had

$$\lim_{\epsilon\to 0} \xi\Big(\frac{s - ct}{\epsilon}\Big) = 0$$

and

$$\lim_{\epsilon\to 0}\gamma\Big(\frac{s-ct}{\epsilon}\Big)=0.$$

$\mu_{\delta,\epsilon}$ was the traveling wave profile of the $\beta_{\delta,\epsilon}$, and which solves of

$$\partial_t\beta_{\delta,\epsilon}+\partial_s F(\beta_{\delta,\epsilon})-\epsilon\partial_{ss}\beta_{\delta,\epsilon}=0. \tag{3.36}$$

Let us further suppose $\{\beta_{\delta,\epsilon}\}_{0<\epsilon\leq 1}$ is uniformly bounded in L^∞ and as ϵ approaches to 0, $\beta_{\delta,\epsilon}$ converges to β_δ. Before, giving the theorem, we choose our entropy/entropy flux pairs for $z=(z_1,z_2)$ as

$$\Phi(z)=\frac{z_2^2}{2}+\sqrt{z_1^2+\delta}\text{ and }\Psi(z)=-\frac{z_1}{\sqrt{z_1+\delta}}z_2. \tag{3.37}$$

It is easy to notice that Φ is convex and they satisfy (3.28) as required.
Now, we prove the main theorem of this section.

Theorem 3.2.4 ([35]) *The function β_δ is an entropy solution of*

$$\partial_t\beta_\delta+\partial_s F(\beta_\delta)=0.$$

Proof: Let $\epsilon\to 0$ in solution (3.2),we have the results (3.17) and (3.18). Also, in (3.36), we let $\epsilon\to 0$, which give us exactly the same results of solutions of (3.2). Now, using the definition 3.2.8 and definition 3.2.5, we see that our entropy/entropy flux pairs (3.37) satisfy the (3.28), having this fact we can say that β_δ is an entropy solution of (3.36). □

Now, we will give a remark, which is crucial for comparing entropy solution and shock wave solution.

Remark 3.2.1 *The reason for us not to be able to pass to the limit as $\delta\to 0$ in $\mu_{\delta,\epsilon}$ is because of the fact that dependence on δ comes from the modified system of conservation laws rather than as a variable of μ. Moreover, once we pass to the limit in ϵ in order to satisfy necessary conditions for the existence of entropy solutions, we have to have that μ converges to 0. Hence, we do not see any δ dependence anymore. This does not contradict with the fact that an entropy solution could still converge to a shock-wave solution when the corresponding hyperbolic system of conservation laws is converging to a non-hyperbolic one. However, we cannot see the relation of entropy solution and the shock wave mentioned by Conlon when we pass to the limit in our analysis when δ converges to 0. See Figure 3.1 for the relations of solutions.*

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\beta_{\delta,\epsilon} = (\upsilon_{\delta,\epsilon}, \kappa_{\delta,\epsilon}) \arrow[r, "\delta \to 0"'] \arrow[d, "\epsilon \to 0"'] & \beta_{\epsilon} = (\upsilon_{\epsilon}, \kappa_{\epsilon}) \arrow[d, "\epsilon \to 0"] \\
\begin{array}{c}\beta_{\delta} = (\upsilon_{\delta}, \kappa_{\delta}) \\ \text{(Entropy solution)}\end{array} \arrow[r, "\text{Remark 3.2.1}", "\delta \to 0"'] & \begin{array}{c}\beta = (\upsilon, \kappa) \\ \text{(Shock wave solution)}\end{array}
\end{tikzcd}
$$

Figure 3.1: Scheme of results and relations

CHAPTER 4

Weak Solutions with Periodic Boundary Conditions

In this chapter, we find conditions satisfied by strong solutions, we give some bounds for a new approximating system and finally, with the help of following proposition we convert our string equation into linear wave equation with a time dependent coefficient.

Proposition 4.0.1 *Assume that $\sigma(t,s)$ is different than 0 at boundaries, then $\sigma > 0$ for all t.*

Proof: By Proposition 2.2.2, we know that $\sigma \geq 0$. Given that $\sigma(t,0) = \sigma(t,1) \neq 0$, using the minimum principle which tells that a function takes its minimum at the boundaries otherwise it is a constant, we find that $\sigma > 0$. □

Remark 4.0.1 *In this chapter, using Proposition 4 we exclude the two free end case such that it is a sub-case of periodic boundary conditions.*

Now, we give initial values for system (2.4) with periodic boundary conditions

$$\kappa(t,0) = \kappa(t,1), \kappa_s(t,0) = \kappa_s(t,1), \upsilon(t,0) = \upsilon(t,1).$$

We can write that our initial values become

$$\frac{\kappa}{|\kappa|}(0,s) = \eta_s(0,s) = \alpha_s(s)$$

and

$$\upsilon(0,s) = \eta_t(0,s) = \beta(s).$$

4.1. Some Equalities Satisfied by Solutions

In this section, we try to obtain bounds by multiplying (1.1) with η, η_s, η_t and η_{tt},and we take $g = 0$. We use integration by parts on the spatial variables and then we use periodic boundary conditions. In the calculations, we use $|\eta_s|^2 = 1, \eta_s\eta_{ss} = 0, \eta_s\eta_{st} = 0$ and $\eta_s\eta_{stt} + \eta_{st}\eta_{st} = 0$. These equalities are important for us, because when we use integration by parts we often obtain similar terms.

- Multiplying (1.1) by η and integrating with respect to s variable from 0 to 1:

$$\begin{aligned}
\int_0^1 \eta_{tt}\eta ds &= \int_0^1 (\sigma\eta_s)_s\eta ds \\
&= \sigma\eta_s\eta\Big|_{s=0}^{s=1} - \int_0^1 \sigma\eta_s\eta_s ds \\
&= -\int_0^1 \sigma ds
\end{aligned}$$

Now, we use the Fundamental Theorem of Calculus (FTC) on time variable and integrate by parts

$$\begin{aligned}
-\int_0^1 \sigma ds &= \int_0^1 \int_0^t \eta_{\tau\tau}\eta d\tau ds \\
&= \int_0^1 \left(\eta_t\eta\Big|_{\tau=0}^{\tau=t} - \int_0^t |\eta_\tau|^2 d\tau\right) ds \\
&= \int_0^1 \left(\eta_t(s,t)\eta(s,t) - \alpha\beta - \int_0^t |\eta_\tau|^2 d\tau\right) ds.
\end{aligned}$$

- Multiplying (1.1) by η_t and integrating with respect to s variable from 0 to 1:

$$\begin{aligned}
\int_0^1 \eta_{tt}\eta_t ds &= \int_0^1 (\sigma\eta_s)_s\eta_t ds \\
&= \sigma\eta_s\eta_t\Big|_{s=0}^{s=1} - \int_0^1 \sigma\eta_s\eta_{st} ds.
\end{aligned}$$

We see that the term on left hand side is the time derivative of a function, we may write it as;

$$\frac{1}{2}\frac{d}{dt}\int_0^1 |\eta_t|^2 = 0.$$

It was shown in the previous chapter that energy does not change by time.

- Multiplying (1.1) by η_s and integrating with respect to s variable from 0 to 1:

$$\begin{aligned}
\int_0^1 \eta_{tt}\eta_s &= \int_0^1 (\sigma\eta_s)_s\eta_s ds \\
&= \sigma\eta_s\eta_s\Big|_{s=0}^{s=1} - \int_0^1 \sigma\eta_s\eta_{ss} ds \\
&= 0
\end{aligned}$$

Now, we use FTC and integration by parts with respect to time variable

$$
\begin{aligned}
0 &= \int_0^1 \int_0^t \eta_{\tau\tau}\eta_s d\tau ds \\
&= \int_0^1 \Big(\eta_s\eta_\tau\Big|_{\tau=0}^{\tau=t} - \int_0^t \eta_\tau\eta_{s\tau} d\tau\Big) ds \\
&= \int_0^1 \Big(\eta_s\eta_\tau\Big|_{\tau=0}^{\tau=t} ds - \int_0^1 \int_0^t \frac{d}{ds}\Big(\frac{|\eta_\tau|^2}{2}\Big) d\tau\Big) ds \\
&= \int_0^1 \eta_s\eta_\tau\Big|_{\tau=0}^{\tau=t} ds - \int_0^t \frac{|\eta_\tau|^2}{2} dt
\end{aligned}
$$

- Multiplying (1.1) by η_{tt} and integrating with respect to s variable from 0 to 1:

$$
\begin{aligned}
&\int_0^1 \eta_{tt}\eta_{tt} ds = \int_0^1 (\sigma\eta_s)_s\eta_{tt} ds \\
&\int_0^1 |\eta_{tt}|^2 ds = \sigma\eta_s\eta_{tt}\Big|_{s=0}^{s=1} - \int_0^1 \sigma\eta_s\eta_{tts} ds = \int_0^1 |\eta_{st}|^2\sigma ds
\end{aligned}
$$

Using (1.2), we end up with

$$
\int_0^1 |\eta_{tt}|^2 ds = \int_0^1 |\eta_{ss}|^2\sigma^2 - \sigma\sigma_{ss} ds.
$$

4.2. Approximating System

In this section, we will give bounds for our approximating system, this system is similar to the system (3.2), but we will not have δ, all the modifications are done by ϵ. Let $\epsilon \in (0,1]$ be a constant and consider auxiliary problem, remember that $\kappa(t,s) = \sigma(t,s)\eta_s$ and $\upsilon(t,s) = \eta_t(t,s)$;

$$
\upsilon_s - \kappa_t = 0 \tag{4.1a}
$$

$$
\upsilon_t - \Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big)_s = \epsilon\upsilon_{ss} \tag{4.1b}
$$

$$
\kappa(t,0) = \kappa(t,1) \tag{4.1c}
$$

$$
\Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big)\Big| = \alpha_s \tag{4.1d}
$$

$$
\upsilon(t,0) = \upsilon(t,1) \tag{4.1e}
$$

$$
\upsilon(0,s) = \beta \tag{4.1f}
$$

$$
\upsilon_s(s,0) = \upsilon_s(s,1) \tag{4.1g}
$$

Here, we added the viscous term on only to (4.1b) and the equality (4.1g) is added for technical reasons and it vanishes as ϵ approaches to 0. Now, we need an existence theorem for our new system (4.1).

Theorem 4.2.2 *Let $\alpha, \beta \in C^3([0,1];\mathbb{R}^3)$,$\alpha_s(0) = \alpha_1$,$\alpha_{ss}(0) = \alpha_{ss}(1)$,$\beta_s(0) = \beta_s(1)$, then there exist a unique solution (υ, κ).*

Proof: Under periodic boundary conditions, choosing $\alpha(s)$ and $\beta(s)$ smooth, we obtain the result in the same way as in the proof of Theorem 4.2 in [3] $\square$

Hereafter, we assume that

$$|\alpha_s(s)| \leq 1 \text{ for } 0 \leq s \leq 1$$

that

$$\alpha|_{s=0} = \alpha|_{s=1}$$

and

$$\int_0^1 |\alpha|^2(s)ds + \int_0^1 \frac{1}{2}|\beta|^2 ds \leq C \tag{4.2}$$

here C is a constant.
Multiply (4.1b) by υ and we have

$$\upsilon\upsilon_t = \upsilon\Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big) + \epsilon\upsilon\upsilon_{ss}.$$

Now integrate from 0 to 1 with respect to the spatial variable

$$\int_0^1 \upsilon\upsilon_t ds = \int_0^1 \upsilon\Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big)_s ds + \epsilon\int_0^1 \upsilon\upsilon_{ss} ds.$$

Start by integrating by parts the terms on the right-hand side

$$\int_0^1 \upsilon\upsilon_t ds = -\int_0^1 \upsilon_s\Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big) ds - \epsilon\int_0^1 \upsilon_s\upsilon_s ds,$$

using (4.1a), we get

$$\int_0^1 \upsilon\upsilon_t ds = -\int_0^1 \Big(\frac{\kappa\kappa_t}{\sqrt{\epsilon + |\kappa|^2}}\Big) ds - \epsilon\int_0^1 \upsilon_s\upsilon_s ds. \tag{4.3}$$

Rewriting (4.3), we have

$$-\epsilon\int_0^1 \upsilon_s\upsilon_s ds = \int_0^1 \upsilon\upsilon_t ds + \int_0^1 \Big(\frac{\kappa\kappa_t}{\sqrt{\epsilon + |\kappa|^2}}\Big) ds \tag{4.4}$$

$$= \frac{1}{2}\frac{d}{dt}\int_0^1 |\upsilon|^2 ds + \frac{d}{dt}\int_0^1 \sqrt{\epsilon + |\kappa|^2} ds \tag{4.5}$$

Define a new energy as

$$E_\epsilon(t) = \frac{1}{2}\int_0^1 |\upsilon|^2 + \int_0^1 \sqrt{\epsilon + |\kappa|^2} ds. \tag{4.6}$$

Take the time derivative of (4.6), and we have

$$\big(E_\epsilon(t)\big)_t = \int_0^1 \upsilon\upsilon_t ds + \int_0^1 \Big(\frac{\kappa\kappa_t}{\sqrt{\epsilon + |\kappa|^2}}\Big) ds.$$

Using (4.5), we can write

$$\big(E_\epsilon(t)\big)_t = -\epsilon \int_0^1 \upsilon_s \upsilon_s ds \leq 0.$$

We find (4.6) for $t = 0$, which is initial energy, as

$$E_\epsilon(0) = \frac{1}{2}\int_0^1 |\beta|^2 + \int_0^1 \sqrt{\alpha^2 + \epsilon} ds,$$

and this is bounded due to (4.2). Therefore,

$$\frac{1}{2}\int_0^1 |\upsilon_s|^2 ds \leq C. \tag{4.7}$$

Now, we define some new functions, which are going to simplify our calculations

$$\eta(t,s) = \alpha(s) + \int_0^t \upsilon(r,s) dr \tag{4.8}$$

$$\xi(t,s) = \int_0^s \frac{\kappa(t,w)}{\sqrt{\epsilon + |\kappa(t,w)|^2}} dw \tag{4.9}$$

We start by multiplying (4.1b) by ξ and integrating with respect to s from 0 to 1

$$\int_0^1 \upsilon_t \xi ds = \int_0^1 \Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big)_s \xi ds + \epsilon \upsilon_{ss} ds,$$

using integration by parts and substituting (4.9) and (4.1a) we get

$$\begin{aligned}
\int_0^1 \upsilon_t \xi ds &= \int_0^1 -\Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big)\Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big) ds - \int_0^1 \kappa_t \Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big) ds \\
&= -\int_0^1 \frac{\kappa^2}{\epsilon + |\kappa|^2} ds - \int_0^1 \kappa_t \Big(\frac{\kappa}{\sqrt{\epsilon + |\kappa|^2}}\Big) ds \\
&= -(1 - (\arctan \frac{1}{\sqrt{\epsilon}})\sqrt{\epsilon}) - \frac{d}{dt}\int_0^1 (\sqrt{|\kappa|^2 + \epsilon}) ds.
\end{aligned}$$

We may bound the last equation; the first term is a scalar and the second term is bounded by $E_\epsilon(t)$, so we have

$$\int_0^1 \upsilon_t \xi ds \leq C \Rightarrow \int_0^1 \upsilon_t \Big[\int_0^s \frac{\kappa(t,w)}{\sqrt{\epsilon + |\kappa(t,w)|^2}} dw\Big] ds \leq C.$$

4.3. Wave Equation Approach to Inextensible String Equation

Firstly, remember the equation of tension (1.2) is given by

$$\sigma_{ss}(t,s) - |\eta_{ss}(t,s)|^2 \sigma(t,s) + |\eta_{st}(t,s)|^2 = 0,$$

and the model for inextensible strings is (1.1).

We make an analysis on transformations in Section 2.3 using periodic boundary conditions. We have shown that $\sigma > 0$; if $\kappa = \eta_s \sigma$ and $|\kappa| = \sigma$, then we have $\eta_s = \dfrac{\kappa}{|\kappa|}$. We know that $|\eta_s|^2 = 1$, which comes from inextensibility of the string, so $\eta_s \neq 0$, using this fact we may find its derivatives as

$$(\eta_s)_s = \frac{d}{ds}\left(\frac{\kappa}{|\kappa|}\right) = 0 \text{ and } (\eta_s)_t = \frac{d}{dt}\left(\frac{\kappa}{|\kappa|}\right) = 0.$$

If we plug them into (1.2), we obtain

$$\sigma_{ss} = 0. \tag{4.10}$$

Integrating twice (4.10), we find

$$\sigma = g(t) + h(t)s, \tag{4.11}$$

here $g(t)$ and $h(t)$ are single variable functions coming from integration. Now, we use the periodic boundary conditions (1.6) and we find that

$$\sigma(t,0) = g(t) = g(t) + h(t) = \sigma(t,1) \text{ and } 0 = h(t).$$

Hence, $\sigma(t,s) = g(t)$. Now, the spatial derivatives of σ are 0, using this we may write our equation as following

$$\eta_{tt} = g(t)\eta_{ss}, \tag{4.12}$$

which is linear wave equation with a time dependent coefficient.

Now, we can rewrite our system as

$$\begin{cases} \eta_{tt} = g(t)\eta_{ss} \\ \eta(0,s) = \alpha(s) \text{ and } \eta_t(0,s) = \beta(s) \\ \eta(t,0) = \eta(t,1) \end{cases} \tag{4.13}$$

Theorem 4.3.3 *Assume that $g(t) \in C^1(0,\infty)$ is non decreasing Lipschitz continuous functions such that $g^{-1}(a) = 0$, $\lim_{t\to 0} g(t) = a_1$ and $\lim_{t\to\infty} g(t) = a_2$ where $a = \dfrac{a_1 + a_2}{2}$ also assume that $\alpha, \beta \in C^2([0,1])$. Then there exists a local solution $\eta \in ([0,1] \times [0,\infty))$ for (4.13).*

Proof: See [37]. □

CHAPTER 5

Conclusions

In this thesis, we analyzed the existence of solutions for the inextensible string equation. which consist of a nonlinear partial differential equation and a compatibility condition ensuring the lenght of the string is fixed. Using the equation of motion and this inextensibility condition we showed that tension satisfies an ordinary differential equation. We showed that energy does not change by time which led us to use system of conservation laws. By some change of variables, we have transformed our partial differential equation into a 2×2 system of conservation law. We wanted to make this system hyperbolic in order to use notion of shock wave and entropy solutions. In this thesis, tension σ is very important, since it may cause discontinuities in systems. In order to tackle this problem, we showed non-negativity of it and in Chapter 4, assuming only that it is different than 0 at boundaries.

In Chapter 3, we modified our system to obtain a hyperbolic system and added a viscosity term to show the existence of explicit traveling wave solutions. Then by passing to limits in parameters, we relate our traveling wave solution with shock wave solutions of Conlon [36] and entropy solutions of Evans [35].

In Chapter 4, we considered an other modified system to obtain a hyperbolic conservation law and showed some bounds on tension and displacement. These bounds might be careful for further research. Finally in this chapter, assuming that tension is nonzero only at the boundary, we prove positiveness of it and use this property to reduce our model to wave equation with time dependent coefficient.

Further research could be conducted on the global existence of solutions with periodic boundary conditions. Also, investigating a case where two free-ends boundary conditions are not excluded would be another interesting area of research. Looking at the asymptotic behavior of solutions as t goes to infinity can also be considered as an open problem associated with any type of boundary conditions.

Bibliography

[1] A. Conway, *The Bullwhip Book.* Greenery Press, 2000.

[2] D. H. R. Courant, *Methods of Mathematical Physics*, vol. 2 of *10.* Interscience Publishers, 2 ed., 1962.

[3] Y. Şengül and D. Vorotnikov, "Generalized solutions for inextensible string equations," *J. Differential Equations*, vol. 262, pp. 3610–3641, 2017.

[4] H. Johnson, "The existence of a periodic solution of a vibrating hanging string," *SIAM J. Appl. Math.*, vol. 16(5), pp. 1048–1058, 1968.

[5] B. d. V. Hugo, "On the existence of branches of time-periodic solutions to the nonlinear vibrating string equations," *Nonlinear Differential Equations and Applications - NoDEA*, vol. 1, pp. 125–148, 1994.

[6] J. Greenberg, "Smooth and time-dependent solutions to the quasilinear wave equation," *Arch. Rat. Mech. Anal.*, vol. 60, pp. 29–50, 1975.

[7] S. Preston, "The motion of the whips and chains," *J. Differential Equations*, vol. 251, pp. 504–550, 2011.

[8] S. Preston, "The geometry of whips," *Ann. Global Anal. Geom.*, vol. 41, pp. 281–305, 2012.

[9] M. Reeken, "The equation of a motion of chain," *Math Z.*, vol. 155, pp. 219–237, 1977.

[10] M. Reeken, "Classical solutions of the chain equation 1," *Math Z.*, vol. 165, pp. 143–169, 1979.

[11] M. Reeken, "Classical solutions of the chain equation 2," *Math Z.*, vol. 166, pp. 67–82, 1979.

[12] S. Preston and R. Saxton, "An H^1 model for inextensible strings," *Disc. and Cont. Dynamical Systems - Series A*, vol. 33, pp. 2065–2083, 2013.

[13] R. Dickey, “Dynamic behavior of the inextensible string,” *Quart. Appl. Math*, vol. 62(1), pp. 135–161, 2004.

[14] I. Kolodner, “Heavy rotating string- a nonlinear eigenvalue problem,” *Comm. Pure Appl. Math*, vol. 8(3), pp. 395–408, 1955.

[15] C. Luning and W. Perry, “Iterative solutions of a non-linear boundary value problem for a rotating string,” *Internat. J. non-linear Mech.*, vol. 19(1), pp. 83–92, 1984.

[16] T. McMillen and A. Goriely, “Whip waves,” *Physica D-nonlinear Phenomena - PHYSICA D*, vol. 184, pp. 192–225, 10 2003.

[17] B. Bernstein, D. Hall, and H. Trent, “On the dynamics of a bull whip,” *J. Acoust. Soc. Am*, vol. 30(12), pp. 1112–1115, 1958.

[18] J. A. Hanna and C. Santangelo, “At the end of a moving string,” 09 2012.

[19] J. Hanna and C. Santangelo, “Slack dynamics on an unfurling string,” *Phys. Rev. Lett*, vol. 109, p. 134301, 2012.

[20] D. Serre, “Un modle relax pour les cbles inextensibles,” *Modl. Math. Anal. Numr.*, vol. 25(4), pp. 465–481, 1991.

[21] C. Wong and K. Yasui, “Falling chains,” *Amer. J. Phys.*, vol. 74(6), pp. 490–496, 2006.

[22] T. McMillen, “On the falling (or not) of the folded inextensible string.” unpublished, 2005.

[23] C. Dafermos, *Hyberbolic Conservation Laws in Continuum Physics*, vol. 323 of *A series of Comprehensive Studies in Mathematics*. Springer, 2 ed., 2000.

[24] A. Bressan, *Hyperbolic Systems of Conservation Laws: The One-dimensional Cauchy Problem*. Oxford Lecture Series in Mathe, Oxford University Press, 2000.

[25] F. Freistühler, “On the cauchy problem for a class of hyperbolic systems of conservation laws,” *J. Differential Equations*, vol. 112, pp. 170–178, 1994.

[26] D. Wagner, “Equivalance of the euler and lagrangian equations of gas dynamics for weak solutions,” *J. Differential Equations*, vol. 68, pp. 118–136, 1987.

[27] T. Lui and C. Wang, “On a nonstrictly hyperbolic system of conservation laws,” *J. Differential Equations*, vol. 57, pp. 1–14, 1985.

[28] F. Freistühler and R. Plaza, “Normal modes and nonlinear stability behavior of dynamic phase boundaries,” *Arch. Rational Mech. Anal.*, vol. 186, pp. 1–24, 2007.

[29] R. Diperna, “Global existence of solutions to nonlinear hyperbolic systems of conservation laws,” *J. Differential Equations*, vol. 20, pp. 187–212, 1976.

[30] P. Lax, “Hyperbolic systems of conservation laws 2,” *Comm. Pure Appl. Math.*, vol. 10, pp. 537–566, 1957.

[31] J. Glimm, “Solutions in the large for nonlinear hyperbolic conservation laws,” *Comm. Pure Appl. Math.*, vol. 18, pp. 697–715, 1965.

[32] T. Nishida, “Global solutions for an initial boundary value problem of a quasilinear hyperbolic system,” *Proc. Japan Acad.*, vol. 44, pp. 642–646, 1968.

[33] N. Bakhvalov, “The existence in the large of a regular solution of a quasilinear hyperbolic systems,” *USSR Comp. Math. and Math.Phys.*, vol. 10, pp. 205–219, 1970.

[34] H. Frid, “Periodic solutions of conservation laws constructed through glimm scheme,” *Trans. Amer. Math. Soc.*, vol. 353, pp. 4529–4544, 2001.

[35] L. Evans, *Partial Differential Equations.* Graduate studies in mathematics, American Mathematical Society, 2010.

[36] J. Conlon, “A theorem in ordinary differential equations with an application to hyperbolic conservation laws,” *Advances in Mathematics*, vol. 35, pp. 1–18, 1980.

[37] A. Chambolle and F. Santosa, “Control of the wave equation by time-dependent coefficient,” *ESAIM: Control, Optimisation and Calculus of Variations*, vol. 8, pp. 375–392, 10 2002.