ON A DYNAMIC PRICING MODEL WITH A POSSIBILITY TO EXIT THE MARKET

by
ELİFNAS ERTEKİN

Submitted to the Graduate School of Engineering and Natural Sciences
in partial fulfillment of the requirements for the degree of
Master of Science

Sabancı University
July 2018

ON A DYNAMIC PRICING MODEL WITH A POSSIBILITY TO EXIT THE MARKET

Approved by:

Prof. Dr. J.B.G. Frenk ____________

(Thesis Supervisor)

Assoc. Prof. Dr. Semih Onur Sezer ____________

(Thesis Co-supervisor)

Assoc. Prof. Dr. Abdullah Daşcı ____________

Assoc. Prof. Dr. Kemal Kılıç ____________

Assist. Prof. Dr. Firdevs Ulus ____________

Date of approval: 30-07-2018

ABSTRACT

ON A DYNAMIC PRICING MODEL WITH A POSSIBILITY TO EXIT THE MARKET

ELİFNAS ERTEKİN

Industrial Engineering, Master of Science Thesis, July 2018

Thesis Supervisors: Prof. Dr. J.B.G. Frenk, Assoc. Prof. Dr. Semih Onur Sezer

Keywords: Dynamic Programming, Non-homogeneous Poisson Process, Optimal Pricing Policy

Taking pricing decisions over time is an important tool to maximize profit in revenue management. In most of the literature with dynamic pricing and stochastic demand, costs are considered as fixed components independent of the pricing policy. Due to the fact, exiting the market is not included as an option in these models. Next to revenue through sales, in this thesis we comprise inventory holding cost which leads staying in the market to be costly. Therefore, we consider the possibility to exit the market before the season ends. In particular, we deal with the problem of selling a seasonal product in a retail store over a finite sales season. Initial order quantity is also a decision variable; hence, we consider ordering cost per item. During the season, inventory replenishment or backlogging is not allowed. In continuous time demand model which is our proposed model, Poisson sales process is assumed with arrival rate function depending on both the time of arrival and the price of the product. At predetermined decision moments known at the beginning, the supplier has to decide either staying in the market and adjusting the price or exiting the market and selling the leftover inventory at a certain salvage value. We formulate both our proposed model and discrete time demand model by dynamic programming techniques. Static version of our proposed model is also provided. For numerical experiments, we investigate the sensitivity of the optimal pricing policy with respect to different problem parameters of a given base scenario.

ÖZET

PAZARDAN ÇEKİLME OLANAĞI İLE DİNAMİK FİYATLANDIRMA MODELİ

ELİFNAS ERTEKİN

Endüstri Mühendisliği, Yüksek Lisans Tezi, Temmuz 2018

Tez Danışmanları: Prof. Dr. J.B.G. Frenk, Doç. Dr. Semih Onur Sezer

Anahtar Kelimeler: Dinamik Programlama, Homojen Olmayan Poisson Süreci, Optimal Fiyatlandırma Politikası

Fiyatlandırma kararları almak gelir yönetiminde önemli bir uygulama olarak değerlendirilmektedir. Ancak dinamik fiyatlandırma ve stokastik talep süreci içeren birçok makalede, maliyetler kullanılan fiyatlandırma politikasından bağımsız sabit bir öğe olarak sayılmaktadır. Bu demek oluyor ki, bu modellerde pazardan çekilmek bir seçenek olarak kabul edilmemektedir. Bu tezde, pazarda kalmanın maliyet yaratmasına sebep olan zaman ve adet birimi başına envanter tutma maliyetini dikkate aldık. Buna bağlı olarak, önceden belirlenmiş karar anlarında pazardan çekilme olasılığına yer verdik. Ayrıntılı olarak, bir perakende mağazasında sezonluk bir ürünün sınırlı bir zaman diliminde satılması ve en yüksek geliri sağlayan fiyatlandırma politikasının belirlenmesi sorununu ele alıyoruz. Yalnızca satış sezonu başlangıcında sipariş verilebilmektedir ve dolayısıyla ürün başına sipariş maliyeti modelde yer almaktadır. Satış sezonu içerisinde envanter yenileme veya geciktirilmiş talebin karşılanma ihtimalini modelimize dahil etmedik. Talebin Poisson sürecine göre gerçekleştiğini ve varış sıklığı fonksiyonunun hem zamana hem de ürünün fiyatına bağlı olduğunu kabul ettik. Önceden belirlenmiş karar anlarında, satıcı iki durumdan birini seçmelidir: pazarda kalmak ve belirli bir fiyat listesinden ürün için optimal fiyatı ayarlamak veya pazardan çekilmek ve kalan envanteri belirli bir kurtarma değerinde satmak. Belirtilen modeli dinamik programlama algoritması kullanarak formüle ettik ve çeşitli problem parametrelerinde duyarlılık analizi ile sayısal bir çalışma gerçekleştirdik.

to my mother and beloved Mert, Ç.

ACKNOWLEDGMENTS

First and foremost, I would like to express my deepest gratitude to my thesis advisors Prof. Dr. J.B.G. Frenk and Assoc. Prof. Dr. Semih Onur Sezer for their precious guidance throughout my research. This thesis would not have been possible without their support and motivation.

I owe a great debt of gratitude to Assist. Prof. Dr. Andrei Sleptchenko for teaching me Python programming language and assisting me with the problems I have encountered. Additionally, it is a pleasure to thank all the other professors at Sabancı University for enhancing my knowledge and experience. I am also appreciative to my thesis committee members; Assoc. Prof. Dr. Abdullah Daşcı, Assoc. Prof. Dr. Kemal Kılıç and Assist. Prof. Dr. Firdevs Ulus.

Very special thanks to full stack developer Uğurcan Emre Ataş for his endless friendship and supporting me to overcome all the computational difficulties throughout this period. I also want to thank my dear friend Sevde Karataş for encouraging and exhilarating me whenever I needed.

Last but not least, I am indebted from the bottom of my heart to my mother and my fiancé Mert Çelik for their immense devotion. I feel the presence of their support and reassurance all the time. I also want to thank my other family members for their love and belief in me.

Contents

List of Figures

List of Tables

Chapter 1

Introduction

In this chapter, we first give a short explanation of our proposed stochastic dynamic pricing model for a seasonal product and after, we make a review on the most important dynamic pricing models which appeared in the literature. A classical example of our proposed pricing model is the sales of fashion clothes during a given season and the way a supplier should react on changing market conditions by adapting the price or leaving the market. In our model, customers are assumed to be *myopic*. Contrary to *strategic* customers, *myopic* customers buy a product as soon as the offered price falls below the price they are willing to pay. These customers do not anticipate on the expected future pricing strategy of the supplier as *strategic* customers. Along the same line, most of the models in the literature comprise *myopic* customers and deal with the problem of selling a *perishable* product during a finite horizon. Although our model deals with a *seasonal* product, there is not a big difference between perishable and seasonal product types. Both products can only be sold during a short period and also the demand for both products decreases over time due to deterioration for perishable products or loss of popularity for seasonal products. However in all stochastic dynamic pricing models for myopic customers, only revenue is considered and the costs are regarded as fixed components independent of the used pricing policy. Therefore staying in the market does not create additional cost and leaving the market is not an option. After a short explanation of our model in the first section and its comparison with the existing models in the literature, in the second section we will explain in more detail our stochastic dynamic pricing model. Since we include inventory holding costs per item per unit of time, staying in the market creates costs; therefore, we also consider the option to leave the market and sell the existing leftover inventory before the end of the season.

1.1 Literature Review

In this thesis we deal with the problem of selling a *seasonal* product during a sales season of finite length. During this sales season, customers arrive according to a Poisson process with an arrival rate function depending on the time and the price of the product. We include inventory holding costs in our model which leads staying in the market to be costly. Therefore, we not only try to adjust the prices for the product at certain points in time but at the same time, we consider the possibility to exit the market. The main objective is to maximize the expected profit. This is a common situation for seasonal products such as fashion clothing in retail stores. Also during the sales season neither inventory replenishment nor backlogging is allowed. Only at the end of the horizon or when deciding to quit the market, the (possibly remaining) leftover items can be sold to outside suppliers (such as outlet stores) at a given salvage value. Next to setting prices optimally or quitting the market, the initial order quantity is also a decision variable in our model; therefore, we consider ordering cost per item at the beginning of the season.

In the literature review to be discussed in this section, we encounter a lot of pricing models for a *perishable* product. In general, a seasonal product is not the same as a perishable product since a seasonal product does not face deterioration; however, seasonal products have similar characteristics as perishable products. Firstly, they have to be sold within a short sales season and secondly, as for perishable products suffering from deterioration, the demand for seasonal products is decreasing as time progresses. In our model, as already observed demand of potential customers is assumed to be a non-homogeneous Poisson process with arbitrary arrival rate function depending on time and price. Since the selling season is relatively short, we do not consider in this thesis *demand learning* or the behavior of customers who anticipate on the future expected price policy of the supplier. These are so-called *strategic* customers and next to demand learning, this is a completely different line of research and outside the scope of this thesis. Instead, we assume that we are dealing with so-called *myopic* customers of which we know beforehand the arrival rate function of the stochastic Poisson arrival process as a function of time and price. These potential customers arrive singly, i.e. no group arrivals are allowed, and they decide to buy the product or not depending on their willingness to pay for a certain offered price. The maximum price that a potential customer desires to pay for a product is known as the reservation price of that customer and in general this reservation price is a random variable. In this paper we assume that the random reservation price has an arbitrary cumulative distribution function which is the same for each potential customer. In our computational section, we need to select a given parametric family of cumulative distribution functions (CDFs) and as in [1] and [20], we assume for computational convenience that this family is the exponential family of CDFs characterized by one parameter. Also in our model, the season is divided into a finite number of time periods. At the beginning of each time period, the supplier has to decide to either adjust the price from a given set of prices or quit the market and sell the remaining inventory at a certain salvage value. We refer to those times as decision moments. As far as we know, the possibility to exit the market is not considered before in the pricing literature for models with stochastic demand. Since inventory cost are not included in all of these papers, exiting the market is not an option due to no additional costs of staying in the market. If staying in the market is costly, it might be a good strategy to exit the market. And if the supplier chooses to stay in the market, he selects the optimal price from a given known price set to maximize the expected revenue. That selected price will be fixed until the next decision moment. At the end of the season, the supplier will certainly exit the market and sell the possibly remaining products at a certain salvage value.

It has always been a practice to influence profits by adjusting prices. Since nowadays online sales increase rapidly and setting different prices can be done easily on the internet, selecting a proper pricing strategy has gained an extreme importance. Especially, taking pricing decisions over time (so-called *dynamic pricing*) became crucial in revenue management. This line of research is also called by some authors *yield management*. For more information on the different models used in pricing and the main assumptions of these models, we refer the reader to the book of Talluri and Van Ryzin (2004) and the literature surveys of Elmaghraby and Keskinocak (2003) and Simchi-Levi et al. (2004). In the remaining part of this section, we examine some of the existing literature in this area of research. Pricing models can be distinguished in terms of considered sales process: deterministic or random. Accordingly, the first type of models are called *deterministic models* and the second ones *stochastic models*.

Most of the deterministic demand models comprise either holding or purchase cost. For example, Khedlekar and Shukla (2013) study a perishable product with deterioration rate and having a so-called logarithmic demand rate function depending both on price of the product and time of buying. It is assumed that sufficient initial inventory is available to satisfy the demand. Next to classical inventory holding cost, they also include a fixed cost of changing the price and exclude procurement costs. They aim to maximize profit under the restriction that n price changes occur at n equally spaced points in time within a finite horizon. Liu et al. (2014) also examine perishable products together with the temperature of the warehouse. The temperature influences the quality of the product and therefore its value, and adjusting the temperature is costly. Holding cost is also included in their model and they assume a linear demand function. They aim to find the optimal temperature and maximize profit under a continuous time pricing policy. Pyke et al. (2005) also decide on the price of a perishable product in each period to maximize the profit. In their model, stockpile and consumption effects are taken into account and the optimal pricing policies for both finite and infinite horizon models with a discount factor are studied. They analyze pricing policies where all the major effects, such as cost or demand function, are stationary and linear or one of the major effects can also be non-linear. Netessine (2006) approaches dynamic pricing policy in a different way. In the first part of his paper, price changes are not considered as decision variables but order of prices and timing of price changes are decision variables. In the second part of the paper; pricing, timing and inventory decisions can be made jointly and several results about the impact of the inventory decision on pricing and timing are provided.

Unlike deterministic demand models, we rarely or never encounter cost components in stochastic demand models. In most of these stochastic models, the size of the initial inventory is not a decision variable but given, and the other possible costs are fixed and independent of the used pricing policy. For example, Van Ryzin and Gallego (1994) assume the demand process is a Poisson process only depending on the price. Dynamic pricing decisions can be made throughout the period with the aim of maximizing expected revenue and they derive an explicit solution for the exponential demand function. In another part, they consider a discrete price set of prices to select from and determine an upper bound on expected revenue by considering a deterministic model. Feng and Gallego (1995) also assume that the demand arrival process is a homogeneous Poisson process only depending on the price. Different from Van Ryzin and Gallego (1994), their objective is to maximize expected revenue with an optimal timing of price changes. They try to determine optimal switching times with time thresholds depending on the number of items on hand. In one part, only an increase in the present price is allowed and a dynamic programming algorithm is used to solve the problem. Feng and Xiao (2000) generalize the model of Feng and Gallego (1995) and they provide the optimal solution in analytical form. Among stochastic and homogeneous arrivals approaches, Lin (2004) differs from the others because he assumes that customers arrive one after another, in other words customers arrive sequentially. During the horizon, the supplier can select a different price for each customer and the objective is to achieve maximum expected revenue with dynamic pricing. The horizon ends at the moment no inventory is left or no more customers show up. Both a fixed number of sequentially arriving customers and a stochastic number of customers are considered in the paper with different parametric distributions selected for the random number of customers. In the last part, a lower and upper bound for the optimal expected revenue is studied for Poisson arrivals and a numerical example is given. Chatwin (2000) allows pricing decisions to be taken at any time during the selling season. Again the demand process is a homogeneous Poisson process and the objective is to determine the optimal continuous pricing policy with maximizing the expected revenue. Apart from the literature cited above, we also encounter many articles with non-homogeneous Poisson process arrivals. For example, Bitran and Mondschein (1997) examine the dynamic pricing problem for two different cases; prices can be updated at any point in time, or prices can be changed at certain fixed times during the finite horizon. An example of a pricing policy which is associated with initial inventory and its sensitivity with respect to the variance of the reservation price distribution is also given. Moreover, Feng and Gallego (2000) address the optimal timing of the price change problem with a given set of prices for perishable items. In one part, they consider the Markovian case based on a deterministic dynamic pricing literature and aim to maximize expected revenue. Zhao and Zheng (2000) improve the results presented by Van Ryzin and Gallego (1994). It is assumed that the demand process is a non homogeneous Poisson process and all cost components are independent of the pricing policy. They aim to find the maximum expected revenue with a given feasible price set. Also a numerical example is given where the continuous time pricing problem is approximated by discretizing it to a finite number of equally spaced decision moments. Bitran et al. (1998) study a different case which involves periodic pricing for retail chains and so they consider more than one store. Prices are kept the same in all the stores during the season and the objective is to maximize total discounted expected revenue. They derive heuristics to find the approximate solution for two different cases; no inventory transfer is allowed between stores and inventory transfer can be done. They also contrast their results with the data obtained from a retail chain in Chile.

Apart from the literature cited above, there are also interesting articles which are a little beyond of the scope of this thesis. For example, Şen (2013) proposes two different heuristics to approximate the optimal solution of the dynamic pricing policy. First, he proposes a heuristic which is effective for finding optimal prices with the help of a dynamic programming formulation and the second one is based on resolving a deterministic formulation of the problem continuously. Aviv and Pazgal (2005) study dynamic pricing policy for partially observed Markov decision process in order to maximize expected revenue. On the other hand, reader can gather information about pricing policies where strategic customers are considered by Du and Chen (2017) and the articles referenced therein. Phillips et al. (2015) assume a setting where headquarters determine a price list and establish limits for price customization depending on their objective. Besides, salespeople can negotiate the price based on the discretionary authority granted to them within the customization limits. They investigate this kind of a customized pricing model by using a data set acquired from an automotive lender and state several empirical outcomes. Tang et al. (2012) analyze the news-vendor problem from a different point of view by interpolating the fundamental inventory problem with dynamic pricing decisions. Various costs are included in the model and objective is to find the maximum expected profit.

In the last part of the literature review, we also include articles on dynamic pricing with different objectives. Frenk et al. (2017) study a product with short life cycle considering two models. In the first model, the supplier stays in the market until τ or the time when inventory finishes (which happens first). In the second model, the supplier decides on τ at the beginning of horizon and no exiting allowed until the end, also the supplier faces penalty cost per unit of unsatisfied demand. They include procurement cost per item, holding cost per item per unit of time and salvage value for each leftover item. It is assumed that demand process is a non homogeneous Poisson process and the price function is given beforehand. They aim to maximize the expected profit by determining optimal order quantity and optimal stopping time. Zhang and Weatherford (2017) regard rooms in a hotel as separate resources and they indicate that applying dynamic pricing for the hotel industry can be treated as a network revenue management problem. They provide different heuristic approaches in order to solve the dynamic programming formulation. The sales horizon is divided into finite time periods and they test their heuristics on a real data received from a hotel. Chen et al. (2017) study a dynamic pricing model where demand depends both on the current price and the reference price which is gathered by weighting past prices exponentially. They aim to maximize the total profit by making price decisions in each period over a finite horizon. Two pricing strategies are considered; the reference price effect is not included in the first pricing strategy and a solution for the model is provided. In the second pricing strategy, seasonality effects are not included and the model can be solved via dynamic programming. Chen and Gallego (2018) provide a dynamic pricing procedure to maximize the total surplus of consumers and the revenue of the firm during the sales period. They refer to properties derived in Van Ryzin and Gallego (1994) and indicate that maximizing this welfare policy has similar features as maximizing revenue. In their model they assume that arrival process is a Poisson process with an arrival rate only depending on price. In the next section, we discuss in more detail our considered model.

1.2 A Pricing Model with Inventory Costs

In this section, we explain our pricing model in more detail. In particular, we consider a pricing model over a finite horizon for a given product and next to pricing decisions at certain moments in time, we include the possibility to stop the sales of that product and leave the market. As observed in our literature review in the previous section, most pricing models in the literature do not include the possibility of leaving the market since they do not include any costs related to staying in the market. Thereby, we consider a supplier taking these decisions at the selected times $0 = \tau_0 < \tau_1 < ... < \tau_N = T$ with T denoting the known length of the selling horizon. As an example, we mention that the parameter T represents the duration of a season during which a particular collection of clothes are sold and the decision moments τ_n, $n = 0, ..., N$ are the times at which the supplier reconsiders his price of that particular product or stops the sales of that product and leaves the market.

At the start of the season, the supplier can only order once of this particular product from the manufacturer and so the first decision to be taken at time 0 is whether an order should be placed and if so what would be the order size. It is assumed that the procurement costs are given by the function c with $c(x)$ denoting the cost of ordering x items. In case an order is placed, the second decision is how to set the price of this product up to the first upcoming decision moment $\tau_1 \leq T$. The set $\mathcal{P}$ of prices which the supplier can select from is either a finite set $p_1 < ... < p_J$ of increasing prices or an interval $[c, p_{\max}]$ with finite $p_{\max}$. The range of these prices are determined by the supplier. Next to revenue due to sales, the supplier also faces inventory holding costs. We will specify the inventory holding cost both in a discrete and continuous time demand setting. At the first upcoming decision moment τ_1, the supplier either decides to stop the sales of the product and sells the remaining inventory at a salvage value θ per unit or selects a new price from the same set $\mathcal{P}$ of feasible prices. The parameter θ can be positive (salvage revenue) or negative (salvage cost). If the supplier decides at time τ_1 not to stop the sales, he faces the same decision at the first upcoming second decision moment τ_2. Finally at time T at the end of the season, the supplier for sure stops selling the product. As an example, the supplier may decide to take these decisions at the end of every week in a season lasting several weeks. Clearly the supplier will only enter the market at time 0 by ordering the product if his expected profit will be positive.

The pricing model explained above can be distinguished as *discrete* or *continuous time* demand model according to the selection of demand process for this particular product. Since sales clearly depend on the demand process, we can consider either accumulated demand in a given period (discrete time demand model) or demand generated by a continuous arrival process of customers (continuous time demand model).

If we consider the discrete time demand model, we denote by the random variable $D_n(p)$ the accumulated sales within the time interval $(\tau_n, \tau_{n+1}]$ if the price in this interval equals p. To assure that this discrete time demand problem is Markovian in the current inventory level and the time index we need to assume that the random variables $D_n(p)$, $n = 0, ..., N-1$ are independent but not necessarily identical distributed. This enables us to solve this problem by stochastic dynamic programming. It is also assumed in this discrete time demand model that within the interval $(\tau_n, \tau_{n+1}]$, we only incur inventory costs of the leftover items at time τ_{n+1} and it is given by $h_n \geq 0$ per leftover item.

If we consider the continuous time demand model, we assume that the cumulative sales process for a given price $p \in \mathcal{P}$ is given by a non homogeneous Poisson process N_p with bounded arrival rate function $t \mapsto \lambda(t, p)$ depending both on the time a product is bought and on the price. This means that the arrival process of customers is given by non homogeneous Poisson process and each customer buys exactly one product. In general we can model this accumulated demand process by an increasing Levy process (see [5] for the definition of such a process) but we will not pursuit this approach in this thesis. To denote the dependence of the probability law of the arrival process on the price p, we use the subscript p in N_p. In the continuous time setting, we additionally assume that the inventory costs are given by h per item per unit of time. A very special important instance is given by an arrival rate function with no time component but only a price component and such a arrival rate function only depends on the selected price p. In this case we assume that the interest in the product does not decrease over time and a buying decision only depends on the price. If this holds, we can also derive some nice properties of the optimal policy. In the most general case, due to the decreasing interest in the product over time, it is natural to assume for any $t > 0$ that the function $p \mapsto \lambda(p, t)$ is decreasing and for any $p \in \mathcal{P}$ the function $t \mapsto \lambda(t, p)$ is decreasing. In the above formulation, the continuous time demand model is again Markovian in the current inventory level and the time index; and again we can solve this problem by stochastic dynamic programming. Both ways of solving the continuous and discrete time model will be discussed in the next chapter.

Chapter 2

Solving The Dynamic Pricing Model Using DP

In this chapter, we propose in the first two sections a generic dynamic programming (DP) approach to solve the proposed continuous and discrete time demand pricing model. Since this approach needs some additional modifications to compute the optimal objective value and optimal policy, we discuss in the third section an efficient way of evaluating on a computer the different cost and revenue components. In the fourth section, we also give a procedure to compute an upper bound on the optimal order quantity for both models and by computing this upper bound beforehand, we only need to evaluate a finite number of different states in our dynamic programming formulation. In the fifth section, we show by means of a numerical example that the optimal to go function in the DP formulation is not always discrete concave. This property in the literature also holds for the model where we are not allowed to leave the market. This implies that we cannot use a special purpose solution procedure to identify the optimal policy and optimal objective value. Using our constructed upper bound to identify the optimal solution, we need to perform a complete enumeration over a finite number of states. It also indicates that the optimal policy might not belong to a special subclass. Finally in the last section of this chapter, we derive an intuitively appealing property of the optimal policy related to leaving the market under some special conditions on the pricing behavior of myopic customers.

2.1 The Bellman Optimality Equations for the Continuous Time Demand Model

To write down in a compact way the dynamic programming equations, we first consider the continuous time demand setting and introduce the first order difference operator

$$\Delta\tau_n = \tau_{n+1} - \tau_n, n = 0, ..., N-1 \tag{2.1}$$

and the shifted stochastic process $N_p^{(n)} = \{N_p^{(n)}(t) : t \geq 0\}$ given by

$$N_p^{(n)}(t) := N_p(t + \tau_n) - N_p(\tau_n). \tag{2.2}$$

Since the stochastic process N_p is a non-homogeneous Poisson process, it follows by the independent and non-stationary increments property of a Poisson process (see [5]) that the shifted stochastic process $N_p^{(n)}$ counting the number of arrival in the interval $[\tau_n, \tau_n + t]$ is again a non-homogeneous Poisson process with arrival rate function $(t, p) \to \lambda(t + \tau_n, p)$. To formulate the dynamic programming equations, we introduce for $n = 1, ..., N$ the functions $V_n : \mathbb{Z}_+ \to \mathbb{R}$ with $V_n(x)$ denoting the maximum expected incremental revenue that the supplier collects from time τ_n on-wards given the inventory level x at time τ_n. At time $\tau_N = T$ the season ends and we have the natural boundary condition

$$V_N(x) = \theta x. \tag{2.3}$$

The parameter θ can be positive (salvage revenue) or negative (salvage cost). At each of the intermediate decision moments, we either will leave the market and stop the sales or we continue selling. By introducing the function $z^+ := \max\{z, 0\}$, $z \geq 0$, we have for $n = 1, ..., N-1$, the recursive dynamic programming equation

$$V_n(x) = \max\{\theta x, U_n(x)\} \tag{2.4}$$

where

$$U_n(x) = \sup_{p_n \in \mathcal{P}}\{\mathbb{E}[r_n(x, p_n) + V_{n+1}((x - N_{p_n}^{(n)}(\Delta\tau_n))^+)]\} \tag{2.5}$$

with $r_n(x, p)$ the random revenue in period $(\tau_n, \tau_{n+1}]$ and $N_p^{(n)}(\Delta\tau_n)$ the total demand in period $(\tau_n, \tau_{n+1}]$ if at time τ_n the price p is selected. In the above equation, the index n refers to the state of the process at time τ_n and not to the number of periods still to go. In this formulation, the region of feasible prices is always the same. It is easy to remove that restriction but we will not analyze this more general model. To explain the above relations, we observe that the function U_n in relation (2.5) for $n = 1, ..., N-1$ gives the optimal expected revenue of staying in the market at time τ_n and the price (if it exists) attaining the supremum gives the optimal price to be selected in $(\tau_n, \tau_{n+1}]$. In relation (2.4), we simply compare the immediate reward for exiting the market with the optimal value of continuing by determining the best price at time τ_n. Finally, to find the optimal initial inventory level and the initial price (from the given feasible set $\mathcal{P}$) at time $\tau_0 = 0$, the supplier needs to solve the problem

$$v(P) = \sup_{x \in \mathbb{Z}_+} \{U_0(x) - c(x)\} \tag{P}$$

with

$$U_0(x) = \sup_{p_0 \in P} \{\mathbb{E}[r_0(x, p_0) + V_1((x - N_{p_0}(\tau_1))^+)]\} \tag{2.6}$$

and p_0 the selected price from the set $\mathcal{P}$ at $\tau_0 = 0$.

As already observed the random variable $r_n(x, p)$, $n = 0, ..., N-1$ in relation (2.5) and (2.6) represent the random revenue in period $(\tau_n, \tau_{n+1}]$ having selected price p in that time interval. To write down these one period costs we introduce the stopping time $\sigma_x^{(n)}, n = 0, ..., N-1$ of the stochastic process $N_p^{(n)}$ given by

$$\begin{aligned} \sigma_x^{(n)} &= \inf\{t \geq 0 : N_{p_n}(t + \tau_n) - N_{p_n}(\tau_n) \geq x\} \\ &= \inf\{t \geq 0 : N_{p_n}^{(n)}(t) \geq x\}. \end{aligned} \tag{2.7}$$

Using the definition of the above stopping time it is obvious that the random one period revenue $r_n(x, p)$ within $(\tau_n, \tau_{n+1}]$ observing at time τ_n inventory level x and setting the price equal to p in the time interval $(\tau_n, \tau_{n+1}]$ equals

$$r_n(x, p) = pN_p^{(n)}(\Delta\tau_n \wedge \sigma_x^{(n)}) - h \int_{\tau_n}^{\tau_{n+1}} (x - N_p^{(n)}(u))^+ du \tag{2.8}$$

with

$$\Delta\tau_n \wedge \sigma_x^{(n)} = \min\{\Delta\tau_n, \sigma_x^{(n)}\}.$$

By relation (2.8), it is obvious for $n = 0, ..., N-1$ that

$$\mathbb{E}(r_n(0, p)) = 0. \tag{2.9}$$

Hence by relation (2.5), it follows that $U_n(0) = 0$ for every $n = 1, ..., N-1$ and so by relation (2.4)

$$V_n(0) = 0 \tag{2.10}$$

for every $n = 1, ..., N$. Using the above equations, we need to apply the following dynamic programming algorithm to determine the optimal objective value and optimal policy of the pricing model in the continuous time demand setting.

Generic dynamic programming algorithm

- **Step 1.** Evaluate

  $$V_N(x) = \theta x,\ x \in \mathbb{Z}_+$$

  and go to Step 2.

- **Step 2.** For every $n = N-1$ going downwards every time one unit to $n = 1$ evaluate for every $x \in \mathbb{Z}_+$ the values

  $$U_n(x) = \sup_{p_n \in \mathcal{P}} \{\mathbb{E}[r_n(x, p_n) + V_{n+1}((x - N_{p_n}^{(n)}(\Delta\tau_n))^+)]\}$$

  and set

  $$V_n(x) = \max\{\theta x, U_n(x)\}$$

  and go to Step 3.

- **Step 3.** Evaluate for every $x \in \mathbb{Z}_+$ the value

  $$U_0(x) = \sup_{p_0 \in P_0} \{\mathbb{E}[r_0(x, p_0) + V_1((x - N_{p_0}(\tau_1))^+)]\}$$

  and go to Step 4.

- **Step 4.** Solve the optimization problem

  $$\upsilon(P) = \sup_{x \in \mathbb{Z}_+} \{U_0(x) - c(x)\}.$$

Since in the above dynamic programming algorithm we need to evaluate $V_n(x)$ for every $x \in \mathbb{Z}_+$, this procedure cannot be executed on a computer. To solve this problem we need to compactify the state space and derive an upper bound on the optimal order quantity. This will be the topic of Section 2.4. Also in the above algorithm, a precise description of how to calculate the expected one period revenues $\mathbb{E}(r_n(x,p)), n = 0, ..., N-1$ is still missing and this will be discussed in the next section. A simpler version of the model is given by the one in which only prices can be changed at decision moments and the possibility of leaving the market is not allowed. We call this model as no stopping model (NSM) and numerical results can be found in Chapter 4. In this case, the dynamic programming equations are given by

$$U_N(x) = \theta x,\ x \in \mathbb{Z}_+ \tag{2.11}$$

$$U_n(x) = \sup_{p_n \in \mathcal{P}} \{\mathbb{E}[r_n(x,p_n) + U_{n+1}((x - N_{p_n}^{(n)}(\Delta\tau_n))^+)]\} \tag{2.12}$$

and for this model, we obtain

$$v(P) = \sup_{x \in \mathbb{Z}_+} \{U_0(x) - c(x)\}. \tag{2.13}$$

Since it is tempting to conjecture that both optimal to go functions U_0 are discrete concave, we will show in Section 2.5 by means of a numerical example that this conjecture is not true for both models.

Although not discussed in this thesis, we can also analyze the model in which a decision to change the price can be taken at any moment during the season. To formulate this model, let $\mathbb{F}$ be the natural filtration generated by the arrival process $\mathbf{N}$ of customers and denote by $\boldsymbol{\tau} \leq T$ the stopping time with respect to this filtration to leave the market. Denote now by $\mathbf{P} = \{\mathbf{P}(t) : t \geq 0\}$ the piece-wise constant $\mathbb{F}$-measurable random price process satisfying $c \leq \mathbf{P}(t) \leq p_{\max} < \infty$. Now our continuous time optimal control problem is given by

$$U_0(x) := \sup_{\boldsymbol{\tau} \in \mathbb{F}, \boldsymbol{\tau} \leq T, \mathbf{P} \in \mathbb{F}} \mathbb{E}\left(\int_0^{\boldsymbol{\tau} \wedge \boldsymbol{\sigma}_x} \mathbf{P}(t) d\mathbf{N}_{\mathbf{P}}(t) - h\int_0^{\boldsymbol{\tau}} (x - N_{\mathbf{P}}(u))^+ du\right), x \in \mathbb{Z}_+$$

and

$$v(P) = \sup_{x \in \mathbb{Z}_+} \{U_0(x) - c(x)\}.$$

Since it is assumed that the function $(p, t) \mapsto p\lambda(p, t)$ is uniformly bounded, we can create a finite number of decision points at which we can change the price or leave the market within $[0, T)$ and this reduces to the model we discuss in this thesis. Clearly this restriction lowers the optimal objective value but it is possible using similar techniques as in [16] and $(p, t) \mapsto p\lambda(p, t)$ being uniformly bounded on $[c, p_{\max}] \times [0, T]$ to bound this error. We will not discuss this model since in practice decisions to change the price are taken at time moments decided beforehand.

2.2 The Bellman Optimality Equations for the Discrete Time Demand Model

By the same arguments, the dynamic programming equations for the discrete time demand setting can be derived. As before, for $n = 1, ..., N$ the functions $V_n : \mathbb{Z}_+ \to \mathbb{R}$ with $V_n(x)$ denote the maximum expected incremental revenue that the supplier collects from the end of period n at time τ_n until the end of the season given the inventory level is x at the end of period n. At time $\tau_N = T$ the season ends and we have again the natural boundary condition

$$V_N(x) = \theta x. \tag{2.14}$$

In the same manner, for $n = 1, ..., N-1$ we have the recursive dynamic programming equation

$$V_n(x) = \max\{\theta x, U_n(x)\} \tag{2.15}$$

where

$$U_n(x) = \sup_{p_n \in \mathcal{P}}\{\mathbb{E}[r_n(x, p_n) + V_{n+1}((x - D_n(p))^+)]\}. \tag{2.16}$$

In the above equation, the index n refers to the state of the pricing process at time τ_n and not to the number of periods still to go. Also in relation (2.16) the random variable $r_n(x, p)$ represents the random revenue in the time interval $(\tau_n, \tau_{n+1}]$ with price p in that same interval. Since the total demand in a time interval is either continuous or discrete, we also assume that x is either continuous or discrete. Finally, to find the optimal order quantity and the initial price (from the given feasible set $\mathcal{P}$) at time $\tau_0 = 0$, the supplier needs to solve the problem

$$\upsilon(P) = \sup_{x \in \mathbb{Z}_+}\{U_0(x) - c(x)\} \tag{2.17}$$

with

$$U_0(x) = \sup_{p_0 \in P}\{\mathbb{E}[r_0(x, p_0) + V_1((x - D_1(p))^+)]\}$$

and p_0 the selected price from the set $\mathcal{P}$ at $\tau_0 = 0$. Since we only measure the cost of inventory at the end of the period, it is obvious that the random one period revenue in $(\tau_n, \tau_{n+1}]$ observing at time τ_n inventory level x and setting the price equal to p equals

$$r_n(x, p) = p(D_n(p) \wedge x) - h_n(x - D_n(p))^+ \tag{2.18}$$

with

$$D_n(p) \wedge x := \min\{D_n(p), x\}.$$

An alternative way to write down the DP equation in relation (2.16) is by introducing the function

$$G_n(x) = V_n(x) - h_n x$$

and writing for $n = 1, ...N - 1$ the dynamic programming equation

$$U_n(x) = \sup_{p_n \in \mathcal{P}}\{\mathbb{E}(pD_n(p) \wedge xp) + G_{n+1}((x - D_n(p))^+)]\}. \tag{2.19}$$

Clearly by relation (2.18), it is obvious for $n = 0, ..., N - 1$ that

$$\mathbb{E}(r_n(0, p)) = 0. \tag{2.20}$$

Hence by relation (2.15), it follows that $U_n(0) = 0$ for every $n = 1, ..., N - 1$ and so by relation (2.15)

$$V_n(0) = 0 \tag{2.21}$$

for every $n = 1, ..., N$. Again we can now list a generic dynamic programming algorithm for the discrete time demand setting. Since this is similar to the continuous time demand setting, it is left for the reader. Finally, to model in more detail a continuous random variable $D_n(p)$, the simplest way to assume

$$D_n(p) = d_n(p)\epsilon_n \tag{2.22}$$

with ϵ_n, $n \in \mathbb{N}$ non-negative continuous independent random variables having expectation 1. This model is called the multiplicative demand model and it is discussed in [28] and [29]. For this model, it is obvious that $d_n(p)$ is the first moment of the random demand in period n at price p and so, $d_n(p)$ represents the expected demand for the product at price p. Possible choices in mathematical economics for the demand function d_n are listed in [18] and [29]. The most common forms of the arrival rate function used in the literature are linear, exponential and logit functions (see also [7] and [24]).

2.3 The Expected One Period Revenues for the Continuous and Discrete Time Demand Model

In this section, we first evaluate the expected one period revenues $\mathbb{E}(r_n(k,p))$, $n = 0, ..., N-1$ for the continuous and discrete time demand setting. By relation (2.8), we know for $n = 0, ..., N-1$

$$\mathbb{E}(r_n(0,p)) = 0. \tag{2.23}$$

Also for every $x \in \mathbb{N}$, it follows using (2.23) that

$$\mathbb{E}(r_n(x,p)) = \sum_{k=0}^{x-1} \Delta_x \mathbb{E}(r_n(k,p)) \tag{2.24}$$

with $\Delta_x \mathbb{E}(r_n(k,p))$ denoting the first order difference operator given by

$$\Delta_x \mathbb{E}(r_n(k,p)) = \mathbb{E}(r_n(k+1,p)) - \mathbb{E}(r_n(k,p)), k \in \mathbb{Z}_+. \tag{2.25}$$

Hence by relation (2.24), the computation of $\mathbb{E}(r_n(x,p))$ is reduced to the computation of the first order difference operator. For this first order difference operator, one can show the following result.

Lemma 1 *For every $k \in \mathbb{Z}_+, n = 0, ..., N-1$ and $p \in \mathcal{P}$*

$$\Delta_x \mathbb{E}(r_n(k,p)) = \begin{cases} p - p\sum_{j=0}^{k} \mathbb{P}(N_p^{(n)}(\Delta\tau_n) = j) \\ -h\sum_{j=0}^{k} \int_0^{\Delta\tau_n} \mathbb{P}(N_p^{(n)}(s) = j)ds. \end{cases} \tag{2.26}$$

with $N_p^{(n)}$ a non-homogeneous Poisson process having arrival intensity function $(t,p) \mapsto \lambda(t+\tau_n, p)$.

Proof. It follows by relation (2.7) and (2.8) that for $n = 0, ..., N-1$ and $k \in \mathbb{Z}_+$

$$\begin{aligned} r_n(k,p) &= pN_p^{(n)}(\Delta\tau_n \wedge \sigma_k^{(n)}) - h\int_0^{\Delta\tau_n}(k - N_p^{(n)}(s))^+ ds \\ &= pN_p^{(n)}(\Delta\tau_n \wedge \sigma_k^{(n)}) - h\int_0^{\Delta\tau_n}(k - N_p^{(n)}(s \wedge \sigma_k^{(n)}))ds \\ &= -hk\Delta\tau_n + pN_p^{(n)}(\Delta\tau_n \wedge \sigma_k^{(n)}) + h\int_0^{\Delta\tau_n} N_p^{(n)}(s \wedge \sigma_k^{(n)})ds \end{aligned} \tag{2.27}$$

with $N_p^{(n)} = \{N_p^{(n)}(t) : t \geq 0\}$ the non-homogeneous Poisson process defined in relation (2.2) having arrival rate function $(t,p) \to \lambda(t + \tau_n, p)$. To simplify the expression in relation (2.27) we observe for every $0 \leq s \leq \Delta\tau_n$ and using relation (2.7) that

$$N_p^{(n)}(s \wedge \sigma_k^{(n)}) = \sum_{j=1}^{k} 1_{\{\sigma_j^{(n)} \leq s\}}.$$

This implies by relation (2.27) that

$$r_n(k,p) = -hk\Delta\tau_n + p\sum_{j=1}^{k} 1_{\{\sigma_j^{(n)} \leq \Delta\tau_n\}} + h\sum_{j=1}^{k}\int_0^{\Delta\tau_n} 1_{\{\sigma_j^{(n)} \leq s\}} ds$$

and so for every $k \in \mathbb{Z}_+$

$$\begin{aligned} r_n(k+1,p) - r_n(k,p) &= -h\Delta\tau_n + p1_{\{\sigma_{k+1}^{(n)} \leq \Delta\tau_n\}} + h\int_0^{\Delta\tau_n} 1_{\{\sigma_{k+1}^{(n)} \leq s\}} ds \\ &= -h\Delta\tau_n + p1_{\{N_p^{(n)}(\Delta\tau_n) \geq k+1\}} + h\int_0^{\Delta\tau_n} 1_{\{N_p^{(n)}(s) \geq k+1\}} ds \\ &= p1_{\{N_p^{(n)}(\Delta\tau_n) \geq k+1\}} + h\int_0^{\Delta\tau_n} 1_{\{N_p^{(n)}(s) \leq k\}} ds. \end{aligned}$$

This shows using the definition of the difference operator given in relation (2.25) that

$$\Delta_x(\mathbb{E}(r_n(k,p) = p\mathbb{P}(N_p^{(n)}(\Delta\tau_n) \geq k+1) + h\int_0^{\Delta\tau_n} \mathbb{P}(N_p^{(n)}(s) \leq k) ds$$

and the result follows. □

To compute the fist order difference operator $\Delta_x(\mathbb{E}(r_n(k,p))$ for any $k \in \mathbb{Z}_+$, we use the following iterative procedure. By Lemma 1 it follows for $k = 0$ that

$$\Delta_x(\mathbb{E}(r_n(0,p)) = p - p\mathbb{P}(N_p^{(n)}(\Delta\tau_n) = 0) - h\int_0^{\Delta\tau_n} \mathbb{P}(N_p^{(n)}(s) = 0) ds. \tag{2.28}$$

Moreover, having computed $\Delta_x\mathbb{E}(r_n(k-1,p))$ for some $k \in \mathbb{N}$ we obtain by Lemma 1 that

$$\Delta_x\mathbb{E}(r_n(k,p)) = \begin{cases} \Delta_x\mathbb{E}(r_n(k-1,p)) - \mathbb{P}(N_p^{(n)}(\Delta\tau_n) = k) \\ -h\int_0^{\Delta\tau_n} \mathbb{P}(N_p^{(n)}(s) = k) ds. \end{cases} \tag{2.29}$$

Since in our computational section we need to evaluate relation (2.29), the next result shows for the selected arrival rate function how to simplify these calculations. Although a more general result appeared in [17], for completeness we list the next lemma and give a short proof.

Lemma 2 *Let N be a non-homogeneous Poisson process with a piece-wise continuous arrival rate function β, and ψ a differentiable function. Then for every $k \in \mathbb{Z}_+$ and $\tau \leq T$ we have*

$$\int_0^{\tau} \psi(u)\beta(u)\mathbb{P}(N(u) = k)du = \begin{cases} \int_0^{\tau} \psi'(u)\mathbb{P}(N(u) \leq k)du \\ \\ +\psi(0) - \psi(\tau)\mathbb{P}(N(\tau) \leq k). \end{cases} \tag{2.30}$$

Proof. It is well known (see for example [25]) for a non-homogeneous Poisson process with an piece-wise continuous arrival rate function β that, for every $k \in \mathbb{Z}_+$, the function $\varphi(u) := \mathbb{P}(N(u) \leq k)$, for $u \geq 0$, is differentiable and satisfies

$$\varphi'(u) = -\beta(u)\,\mathbb{P}(N(u) = k)$$

with the initial condition $\varphi(0) = \mathbb{P}(N(0) \leq k) = 1$. Then, the chain rule gives

$$\begin{aligned} \psi(\tau)\varphi(\tau) - \psi(0) &= \int_0^{\tau} \psi'(u)\varphi(u)\,du + \int_0^{\tau} \psi(u)\varphi'(u)\,du \\ &= \int_0^{\tau} \psi'(u)\varphi(u)\,du - \int_0^{\tau} \psi(u)\beta(u)\,\mathbb{P}(N(u) = k)\,du \end{aligned}$$

from which relation (2.30) follows after re-arranging the terms. □

We next consider the following important class of arrival rate functions. Let $0 = a_1 < a_2 < ... < a_{m+1} = T$ and consider the arrival intensity function

$$\lambda(t, p) = \sum\nolimits_{i=1}^{m} \lambda_i(1 - F_i(p))1_{A_i}(t) \tag{2.31}$$

with $A_i = [a_i, a_{i+1}), i = 1, ..., m - 1$ and $A_m = [a_m, a_{m+1}]$. This means that on the intervals A_i, $i = 1..., m+1$ the arrival rate of potential customers is constant and the CDF of their so-called reservation prices might change from interval to interval. Also in this section, we assume that a subset of the decision moments

$$0 = \tau_0 < \tau_1 < .... < \tau_N = T$$

occurs at each time $a_i, i = 1, ..., m$ that a change in the demand occurs. Under this assumption, we introduce for every $1 \leq n \leq N-1$ the value

$$i(n) := \min\{i \in \{1, ..., m\} : \tau_n \geq a_i\} \tag{2.32}$$

representing the interval A_i containing the decision τ_n.

For the above choice of an arrival rate function it follows by taking $\psi(u) = 1$ in Lemma 2 that for every $k \in \mathbb{Z}_+$ and $n = 0, ..., N-1$

$$\begin{aligned} & \lambda_{i(n)}(1 - F_{i(n)}(p)) \int_0^{\Delta\tau_n} \mathbb{P}(N_p^{(n)}(s) = k) ds \\ = \ & 1 - \mathbb{P}(N_p^{(n)}(\Delta\tau_n) \leq k) \\ = \ & 1 - e^{-\lambda_{i(n)}\Delta\tau_n(1-F_{i(n)}(p))} \sum_{i=0}^{k} \frac{(\lambda_{i(n)}\Delta\tau_n(1-F_{i(n)}(p)))^i}{i!}. \end{aligned} \tag{2.33}$$

For the arrival rate function in relation (2.31), the next simplified result for the one period difference operator of the one period expected revenues follows immediately applying relation (2.33) and Lemma 1.

Lemma 3 *If the arrival rate function is given by relation (2.31), then for every* $k \in \mathbb{Z}_+, n = 0, ..., N-1$ *and* $p \in \mathcal{P}$

$$\Delta_x \mathbb{E}(r_n(k,p)) = \begin{cases} p - p\sum_{j=0}^{k} \mathbb{P}(N_p^{(n)}(\Delta\tau_n) = j) \\ -h\sum_{j=0}^{k} \lambda_{i(n)}^{-1}(1 - F_{i(n)}(p))^{-1}[1 - \mathbb{P}(N_p^{(n)}(\Delta\tau_n) \leq j)] \end{cases} \tag{2.34}$$

with $i(n)$ *listed in relation (2.32).*

Proof. Apply Lemma 1 and relation (2.33). □

In the discrete time demand setting we observe by relation (2.18)

$$\mathbb{E}(r_n(x,p)) = (p + h_n)\mathbb{E}(D_n(p) \wedge x) - h_n x. \tag{2.35}$$

Since in the discrete time demand setting the random variable can be an integer valued or continuous random variable, we first calculate the expected one period revenues for the demand represented by a continuous random variable given in relation (2.22). In this case it follows

$$D_n(p) \wedge x = d_n(p)\epsilon_n \wedge x = d_n(p)(\epsilon_n \wedge x d_n(p)^{-1}). \tag{2.36}$$

Note for any continuous non-negative random variable Y and $y > 0$ it follows that

$$Y \wedge y = \int_0^y 1_{\{Y>u\}} du$$

and so by Fubini's theorem we obtain

$$\mathbb{E}(Y \wedge y) = \int_0^y \mathbb{E}(1_{\{Y>u\}})du = \int_0^y (1 - F(u))du \tag{2.37}$$

with F the continuous CDF of the random variable Y. Applying now relation (2.37) to relations (2.35) and (2.36), we obtain

$$\begin{aligned} E(r_n(x,p)) &= (p + h_n)d_n(p) \textstyle\int_0^{xd_n(p)^{-1}} (1 - F(u))du - h_n x \\ &= (p + h_n)d_n(p)F_e(xd_n(p)^{-1}) - h_n x \end{aligned} \tag{2.38}$$

with $F_e(x)$ denoting the equilibrium CDF of the random variable ϵ_n given by

$$F_e(x) := \frac{1}{\mathbb{E}(\epsilon_n)} \int_0^x 1 - F(u)du = \int_0^x (1 - F(u))du$$

(use $\mathbb{E}(\epsilon_n) = 1$).

In case the random demand $D_n(p)$ is integer valued, we introduce as before the difference operator

$$\Delta_x(\mathbb{E}(r_n(x,p))) = \mathbb{E}(r_n(x+1,p)) - \mathbb{E}(r_n(x,p)), x \in \mathbb{Z}_+$$

and it follows by relation (2.18)

$$\Delta_x(\mathbb{E}(r_n(x,p))) = (p + h_n)\mathbb{P}(D_n(p) \geq x + 1) - h_n$$

and for most used CDFs, this can be easily calculated. Using now relation (2.24), we can easily compute the expected one period revenues. In the next section, we will derive an upper bound on the optimal order quantity and this will compactify our state space of the dynamic programming formulation.

2.4 An Upper Bound on the Optimal Order Quantity

To solve the optimization problem (P) on a computer, we need to bound the state space of the dynamic programming model (see the Bellman optimality equations for the continuous and discrete time demand model discussed in the previous sections) and construct an upper bound on the optimal order quantity. To derive such an upperbound we consider the same problem with no inventory costs and denote by $\overline{V}_n(x)$ the maximum expected incremental revenue of this problem from time τ_n up to time τ_N given at time τ_n we observe inventory level x and by $\overline{U}_0(x)$ the optimal expected revenue after ordering x items. Since inventory costs are zero and so staying in the market does not create any additional cost but due $p > \theta$ only additional revenue it is not optimal to leave the market. Due to the lack of inventory costs it is obvious for every $1 \leq n \leq N$ and $x \in \mathbb{Z}_+$

$$V_n(x) \leq \overline{V}_n(x) \tag{2.39}$$

and

$$U_0(x) \leq \overline{U}_0(x). \tag{2.40}$$

By a similar reasoning as for the model with inventory costs, it follows that the Bellman optimality equations of the model without inventory costs are given by

$$\overline{V}_N(x) = \theta x \tag{2.41}$$

and for $1 \leq n \leq N-1$

$$\overline{V}_n(x) = \sup_{p_n \in \mathcal{P}} \mathbb{E}[\overline{r}_n(x, p_n) + \overline{V}_{n+1}((x - N_{p_n}^{(n)}(\Delta\tau_n))^+)]\}. \tag{2.42}$$

Now for $n = 0$ it follows that

$$\overline{U}_0(x) = \sup_{p_0 \in \mathcal{P}} \{\mathbb{E}[\overline{r}_0(x, p_0) + \overline{V}_1((x - N_{p_0}(\tau_1))^+)]\}. \tag{2.43}$$

and we need to solve

$$\overline{v}(Q) = \sup_x \{\overline{U}_0(x) - cx\} \tag{Q}$$

In this particular case without inventory costs, the one period revenues are given by

$$\overline{r}_n(x, p_0) = p_0 N_{p_0}^{(n)}(\Delta\tau_n \wedge \sigma_x^{(n)}).$$

Setting as before $\Delta\tau_n = \tau_{n+1} - \tau_n, n = 0, ..., N-1$ and introducing the optimization problems

$$\upsilon(Q_n) = \sup_{p\in\mathcal{P}} \left\{ (p-\theta)\int_0^{\triangle\tau_n} \lambda(s+\tau_n, p)ds \right\} \qquad (Q_n)$$

one can verify the following result.

Lemma 4 *For every $x \in \mathbb{Z}_+$ and $\theta \geq 0$ it holds*

$$\overline{U}_0(x) \leq \sum_{n=0}^{N-1} \upsilon(Q_n) + \theta x.$$

Proof. We will first show by induction that

$$\overline{V}_n(x) \leq \sum_{i=n}^{N-1} \upsilon(Q_n) + \theta x \qquad (2.44)$$

for every $1 \leq n \leq N$. Clearly by relation (2.41) the upper bound holds for $n = N$. Suppose now it holds for $n = m+1, m = 1, ..., N-1$ and so

$$\overline{V}_{m+1}(x) \leq \sum_{i=m+1}^{N-1} \upsilon(Q_i) + \theta x. \qquad (2.45)$$

By relation (2.42), we then obtain using relation (2.45) that

$$\begin{aligned} \overline{V}_m(x) &= \sup_{p_m\in\mathcal{P}}\{\mathbb{E}(\overline{r}_m(x,p) + \overline{V}_{m+1}((x - N^m_{p_m}(\Delta\tau_m))^+)\} \\ &\leq \textstyle\sum_{i=m+1}^{N-1} \upsilon(Q_n) + \sup_{p_m\in\mathcal{P}}\{\mathbb{E}(\overline{r}_m(x,p) + \theta(x - N^{(m)}_{p_m}(\Delta\tau_m))^+]\}. \end{aligned}$$

Since

$$(x - N^{(m)}_{p_m}(\Delta\tau_m))^+ = x - N^{(m)}_{p_m}(\Delta\tau_m \wedge \sigma^{(m)}_x)$$

this shows that

$$\overline{r}_m(x,p) + \theta(x - N^{(m)}_{p_m}(\Delta\tau_m))^+ = \theta x + (p-\theta)N^{(m)}_{p_m}(\Delta\tau_m)$$

and so

$$\begin{aligned} &\sup_{p_m\in\mathcal{P}(p)}\{\mathbb{E}(\overline{r}_m(x,p_m) + \theta(x - N^{(m)}_{p_m}(\Delta\tau_m))^+)\} \\ \leq\ & \theta x + \sup_{p_m\in\mathcal{P}_m(p)}\{(p_m-\theta)\mathbb{E}(N^{(m)}_{p_m}(\Delta\tau_m \wedge \sigma^{(n)}_x))\} \\ \leq\ & \theta x + \upsilon\left(Q_m\right). \end{aligned}$$

Hence it follows that

$$\overline{V}_m(x) \leq \sum\nolimits_{i=m}^{m} \upsilon(Q_i) + \theta x$$

and we have shown by induction that relation (2.44) holds. Applying now relation (2.43), we finally obtain by a similar argument that

$$\overline{U}_0(x) \leq \sum\nolimits_{i=0}^{N-1} \upsilon(Q_i) + \theta x$$

and the result is verified. □

Using Lemma 4, the main result of this section is easy to verify.

Lemma 5 *An optimal order quantity exists for the optimization problem* (P) *and any optimal order quantity is bounded above by* $x_U = \left\lceil \frac{\sum_{n=0}^{N-1} \upsilon(Q_n)}{c-\theta} \right\rceil$.

Proof. Since $p \geq c > \theta$ it follows by Lemma 4 that

$$U_0(x) - cx \leq \overline{U}_0(x) - cx \leq \sum\nolimits_{n=0}^{N-1} \upsilon(Q_n) + (\theta - c)x.$$

This shows for every $x > x_U$ that

$$U_0(x) - cx < 0.$$

Since $\upsilon(P) \geq 0$ it must follow that

$$\upsilon(P) = \max\nolimits_{x \leq x_U} \{U_0(x) - cx\}$$

and we have shown the result. □

By the above result, we have to evaluate the optimal value functions $V_n(x)$ for every $x \leq x_U$ as well as $U_0(x)$ for every $x \leq x_U$ to solve optimization problem (P). Also it is clear by this result that

$$\upsilon(P) = \max\nolimits_{x \leq x_U} \{U_0(x) - cx\}. \tag{2.46}$$

Hence, we need to apply the following improved dynamic programming algorithm. Observe we already know that $V_n(0) = 0$.

Implementable dynamic programming algorithm

- **Step 1.** Solve for $n = 0$ until $N - 1$ the optimization problems

$$\upsilon(Q_n) = \sup_{p\in\mathcal{P}} \left\{(p-\theta)\int_0^{\Delta\tau_n} \lambda(s+\tau_n, p)ds\right\}$$

  and compute

$$x_U = \left\lceil \frac{\sum_{n=0}^{N-1} \upsilon(Q_n)}{c-\theta} \right\rceil$$

  and go to Step 2.

- **Step 2.** For every $x = 0$ up to x_U evaluate

$$V_N(x) = \theta x \tag{2.47}$$

  and go to Step 3.

- **Step 3.** For every $n = N - 1$ down to 1 evaluate for every $x = 0, ..., x_U$

$$\begin{aligned} U_n(x) &= \sup_{p_n\in\mathcal{P}_n(p)}\{\mathbb{E}(r_n(x,p_n) + V_{n+1}((x - N_{p_n}^{(n)}(\Delta\tau_n))^+)\} \\ &= \sup_{p_n\in\mathcal{P}}\left\{\mathbb{E}(r_n(x,p_n)) + \sum_{j=0}^{x-1}\mathbb{P}(N^{(n)}(\Delta\tau_n) = j)V_{n+1}(x-j)\right\} \end{aligned} \tag{2.48}$$

  and

$$V_n(x) = \max\{\theta x, U_n(x)\}$$

  Also record for every x the optimal $p_n^* = p_n(x)$ which achieves the above maximum and go to Step 4.

- **Step 4.** For $n = 0$ evaluate for every $x = 0, ..., x_U$ the value function

$$\begin{aligned} U_0(x) &= \sup_{p_0\in\mathcal{P}}\{\mathbb{E}(r_0(x,p_0) + V_1((x - N(\tau_1))^+)\} \\ &= \sup_{p_0\in\mathcal{P}}\left\{\mathbb{E}(r_0(x,p_0)) + \sum_{j=0}^{x-1}\mathbb{P}(N(\tau_1) = j)V_1(x-j)\right\} \end{aligned} \tag{2.49}$$

  and go to Step 5.

- **Step 5.** Evaluate
$$x_{opt} = \max_{1 \le x \le x_U} \{U_0(x) - cx\} \tag{2.50}$$
and compute
$$\upsilon(P) = U_0(x_{opt}) - cx_{opt}.$$
If $\upsilon(P) \le 0$, do not enter the market. If $\upsilon(P) > 0$, use the optimal constructed table to derive the optimal policy.

Once we have derived the optimal policy, we can construct the optimal stopping sets in a graphical figure and apply the optimal policy in a practical situation. An interesting question now is whether we will always reduce the price and give reduction in the optimal policy table. In Chapter 4, we give by means of a numerical example that this is not always the case. To evaluate for our particular arrival rate function in relation (2.31), the value $\sum_{n=0}^{N-1} \upsilon(Q_n)$ used in Step 1 of our algorithm, we observe

$$\begin{aligned} \sum_{n=0}^{N-1} \upsilon(Q_n) &= \sum_{n=0}^{N-1} \sup_{c \le p \le p_{\max}} \{(p-\theta) \int_0^{\Delta\tau_n} \lambda_{i(n)}(1 - F_{i(n)}(p))ds \\ &= \sum_{n=0}^{N-1} \lambda_{i(n)} \Delta\tau_n \sup_{c \le p \le p_{\max}} (p-\theta)(1 - F_{i(n)}(p)). \end{aligned} \tag{2.51}$$

Introducing now for $i = 1, ..., m$ the value

$$\kappa_i = \lambda_i \sum_{n=0}^{N-1} 1_{\{\tau_n \in [a_i, a_{i+1})\}} \Delta\tau_n \tag{2.52}$$

we obtain after some checking and using relation (2.51) that

$$\sum_{n=0}^{N-1} \upsilon(Q_n) = \sum_{i=1}^{m} \kappa_i \sup_{c \le p \le p_{\max}} (p-\theta)(1 - F_i(p)). \tag{2.53}$$

Hence, in the first step of the algorithm we need to solve for every $i = 1, ..., m$ the problems (recall we use a discretization $\{p_1, ..., p_J\}$ instead of the set $[c, p_{\max}]$)

$$\upsilon(Q_i^{(d)}) = \sup_{p \in \{p_1, ..., p_J\}} (p-\theta)(1 - F_i(p)). \tag{2.54}$$

In the next section, we will give some theoretical results regarding the optimal stopping sets and a numerical counter example showing that the function $U_0(x) - cx$ is in general not discrete concave. We will first start with the numerical counter example.

2.5 The Behavior of the Function $U_0(x) - cx$

In this section, we show by means of a numerical counter example that in general the function U_0 or equivalently for linear procurement costs the function $x \to U_0(x) - cx$ is not discrete concave. To verify this, we use the parameter settings as discussed in Chapter 4. For convenience, we again report the base scenario problem parameters in the table below. For the dynamic model with a possibility to exit the market, we draw in Figure 2.1 the graph of the function $x \to U_0(x) - cx$. The uppermost plotting of Figure 2.1 shows the graph of the function $U_0(x) - cx$ for order quantity x in the range 0 until x_U. We obtain the middle plotting by zooming in and restricting x to the range 0 and 1000. Looking at these two graphs, it seems that the function is discrete concave; but, zooming in more further and restricting x from 340 to 390, the non-concavity of the function $U_0(x) - cx$ can be seen clearly.

In Figure 2.2, we draw the graph of the same function for the no stopping model (NSM). Again from this figure we clearly see that the function $x \to U_0(x) - cx$ is not concave. In [29], it is claimed that the function is concave for the discrete time continuous demand model with no exiting allowed; but unfortunately, the used proof is incorrect. Due to an incorrect application of a result under the condition that 2×2 Hessian matrix is negative definite, the authors claim that the function $(x, p) \to xd(p)$ is concave in (x, p) but in general it is not.

Even for no stopping model, it seems that the one period dynamic programming operator does not preserve discrete concavity and so it is unclear that the optimal pricing policy has a nice structure. Also we observe that Figure 2.1 and Figure 2.2 show almost the same behavior since in the base scenario, the possibility of exiting the market is very low (see Chapter 4 for more information). In the next section, we will show for a particular case of the arrival rate function that the optimal stopping sets have a nice structure.

T	c	$p_{\max}$	h	θ	ϵ	a_i	τ_n	λ_i	μ_i
18	60	350	25	50	10	0, 6, 12	0, 6, 12	400, 200, 100	$\frac{1}{150}, \frac{1}{90}, \frac{1}{55}$

Table 2.1: Base scenario problem parameters for $i = 1, 2, 3$ and $n = 0, 1, 2$.

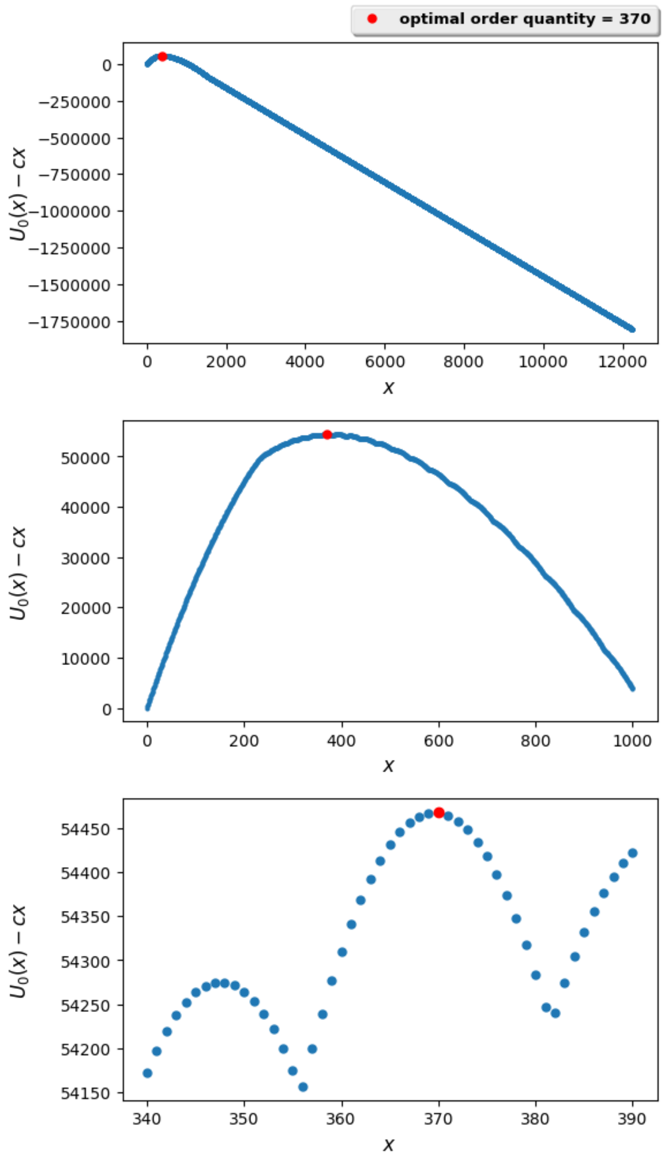


Figure 2.1: Graph of $x \mapsto U_0(x) - cx$ for the base scenario of the dynamic model with exiting allowed.

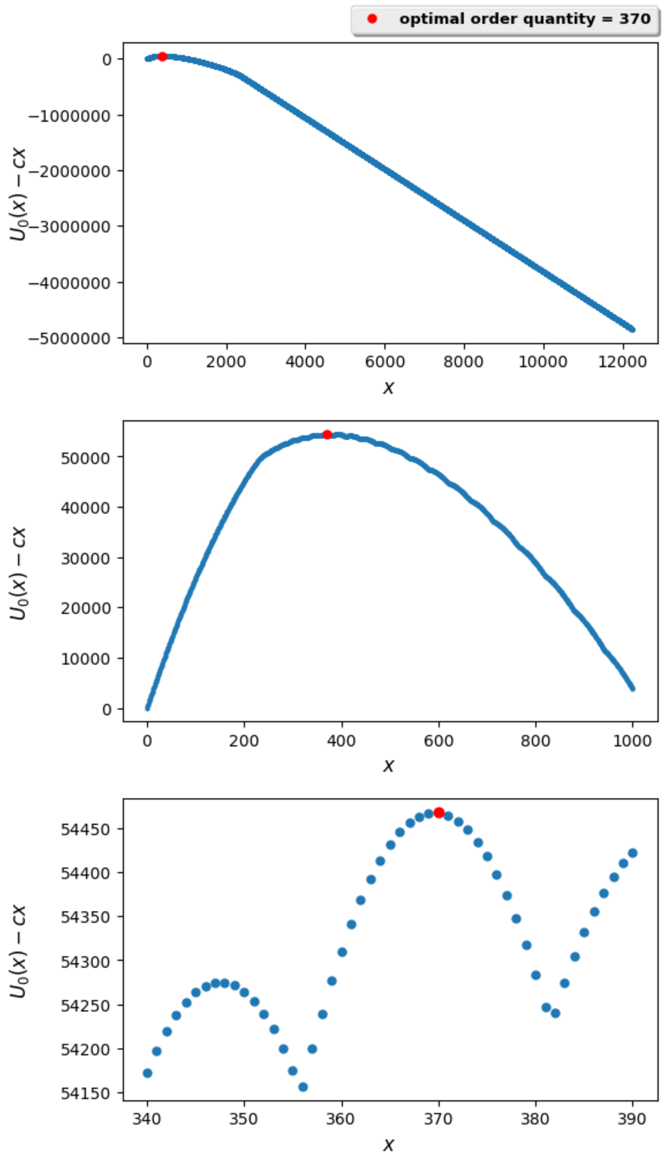


Figure 2.2: Graph of $x \mapsto U_0(x) - cx$ for the base scenario of the dynamic model with no exiting allowed.

2.6 The Structure of the Optimal Stopping Sets

In this section, we will derive some properties of the optimal stopping sets in case the arrival rate function only depends on the offered price. Firstly, observe that the optimal stopping sets $S_n \subseteq \mathbb{Z}_+$ at time $n = \{1, ..., N-1\}$ in our considered problem are given by

$$S_n = \{x \in \mathbb{Z}_+ : \theta x \geq U_n(x)\} = \{x \in \mathbb{Z}_+ : V_n(x) = \theta x\}.$$

Introducing now $\Delta\tau_n = \tau_{n+1} - \tau_n, n = 0, ..., N-1$, we show the following result for the DP equation.

Lemma 6 *If $\Delta\tau_n = \Delta > 0$ for every $n = 0, ..., N-1$ and the intensity rate function only depends on the price p, then*

$$\theta x = V_N(x) \leq V_{N-1}(x) \leq .... \leq V_1(x) \tag{2.55}$$

for every $x \in \mathbb{Z}_+$.

Proof. By relation (2.4) and (2.14) it follows that

$$V_N(x) = \theta x \leq \max\{\theta x\, , U_{N-1}(x)\} = V_{N-1}(x).$$

Assume now by induction that $V_{n+1}(x) \leq V_n(x)$ for a given $2 \leq n \leq N-1$. Since for every given price p the function $\lambda(t, p) = \lambda(p)$ only depends on p and implying $\Delta\tau_n = \Delta$ as $\tau_n = n\Delta$, it follows for a given price p and inventory level x at both times $n\Delta$ and $(n-1)\Delta$ that the random variable

$$N_p((n-1)\Delta + (\Delta \wedge \sigma_x^{(n-1)}) - N_p((n-1)\Delta)$$

has the same CDF as

$$N_p(n\Delta + (\Delta \wedge \sigma_x^n) - N_p(n\Delta).$$

By the same argument, we also obtain that the random variable $N_p((n-1)\Delta + s) - N_p((n-1)\Delta)$ has the same CDF as $N_p(n\Delta + s) - N_p(n\Delta)$ for every $0 \leq s \leq \Delta$. This shows by the definition of the one period revenue in interval $[n\Delta, (n+1)\Delta]$ in relation (2.8) that the random variable $r_{n-1}(x, p)$ has the same CDF as $r_n(x, p)$. This implies by relation (2.5) and using our induction hypothesis $V_{n+1}(x) \leq V_n(x)$ for every x that

$$\begin{aligned} U_n(x) &= \sup_{\overline{p}\in\mathcal{P}}\{\mathbb{E}[r_n(x,\overline{p}) + V_{n+1}((x-(N_{\overline{p}}(\tau_{n+1}) - N_{\overline{p}}(\tau_n))^+)]\} \\ &\leq \sup_{\overline{p}\in\mathcal{P}}\{\mathbb{E}[r_{n-1}(x,\overline{p}) + V_n((x-(N_{\overline{p}}(\tau_{n+1}) - N_{\overline{p}}(\tau_n))^+)]\} \\ &= \sup_{\overline{p}\in\mathcal{P}}\{\mathbb{E}[r_{n-1}(x,\overline{p}) + V_n(x-(N_{\overline{p}}(\tau_n) - N_{\overline{p}}(\tau_{n-1}))^+)]\} \\ &= U_{n-1}(x). \end{aligned} \tag{2.56}$$

This shows by relation (2.4) that

$$V_n(x) = \max\{\theta x, U_n(x)\} \geq \max\{\theta x, U_{n-1}(x)\} = V_{n-1}(x)$$

and we have verified $V_1(x) \geq V_2(x) \geq .... \geq V_N(x)$. □

Using the above result one can show the following structure of the optimal policy.

Lemma 7 *If the conditions of Lemma 6 are satisfied, then* $S_n \subseteq S_{n+1}$ *for every* $n = 1, ..., N-1$.

Proof. If $x \in S_n$, then by definition $V_n(x) = \theta x$. By Lemma 6, it follows that $V_n(x) \leq V_{n+1}(x)$ and this shows by relation (2.4)

$$\theta x = V_n(x) \leq V_{n+1}(x) = \max\{\theta x, U_{n+1}(x)\} \leq \theta x$$

and so $V_{n+1}(x) = \theta x$. This shows the result. □

In the next chapter, we will discuss the static model in detail.

Chapter 3

Solving The Static Model Using NLP

In this chapter, we analyze in more detail the static version of the continuous time demand pricing model. In this case, the time horizon is given by $[0, T]$ and at time 0 we need to select a price p and an order quantity x. To analyze the random revenue in this T period model, we assume that the arrival process $N = \{N(t) : t \geq 0\}$ is a non homogeneous Poisson process with arrival intensity $\lambda(t, p), 0 \leq t \leq T, c \leq p \leq p_{\max}$. The random revenue in this T period model with inventory level x at time 0 and selected price p is clearly given by

$$r_0(x, p) = pN_p(T \wedge \sigma_x) - h \int_0^T (x - N_p(u))^+ du + \theta(x - N_p(T))^+ \tag{3.1}$$

with $T \wedge \sigma_x := \min\{T, \sigma_x\}$. The parameter $h > 0$ denotes the inventory holding cost per item per unit of time, p the price of the item and θ the salvage value at the end of the horizon at time T. Since by definition $\sigma_0 = 0$, it follows that $r_0(x, p) = 0$ and so $\mathbb{E}(r_0(0, p)) = 0$. Before discussing the next lemma, we first introduce the definition.

Definition 8 *A function $f : \mathbb{Z}_+ \to \mathbb{R}$ is called discrete concave if the first order difference $\Delta f(x) := f(x + 1) - f(x)$ is decreasing in $x \in \mathbb{Z}_+$.*

It is now easy to show the following result. Observe this result holds for any nonexplosive simple point process (cf.[4]) with $\sigma_n, n \in \mathbb{N}$ denoting the time of arrival of the nth customer. The proof only uses that the sample paths of any arrival process are increasing and $N(\sigma_x) = x$.

Lemma 9 *For any $p \geq \theta$ the function $x \mapsto \mathbb{E}(r_0(x,p))$ is discrete concave.*

Proof. Since for any given p the non homogeneous Poisson process N_p has increasing sample paths and $N_p(\sigma_x) = x$, it follows that

$$\begin{aligned} pN_p(T \wedge \sigma_x) + \theta(x - N_p(T))^+ &= pN_p(T \wedge \sigma_x) + \theta(x - N_p(T \wedge \sigma_x)) \\ &= (p-\theta)N_p(T \wedge \sigma_x) + \theta x \qquad (3.2) \\ &= (p-\theta)(N_p(T) \wedge x) + \theta x. \end{aligned}$$

Hence we obtain

$$\mathbb{E}(pN_p(T \wedge \sigma_x) + \theta(x - N_p(T))^+) = (p-\theta)\mathbb{E}(N_p(T) \wedge x) + \theta x.$$

Since $x \mapsto N_p(T) \wedge x$ is clearly discrete concave, we obtain for any $p \geq \theta$ that the function

$$x \mapsto \mathbb{E}(pN_p(T \wedge \sigma_x) + \theta(x - N_p(T))^+)$$

is discrete concave. Using $x \mapsto \mathbb{E}\left(\int_0^T (x - N_p(u))^+ du\right)$ is discrete convex the result follows by relation (3.1). □

An application of Lemma 9 is given by the following. Since $c > 0$ is the cost of each ordered item, we need to solve for the static problem with $t = T$ the optimization problem

$$v(S) = \sup_{x \in \mathbb{Z}_+, p_0 \in \mathcal{P}} \{\mathbb{E}(r_0(x,p_0)) - cx\}$$

and $\mathcal{P}$ the set of feasible prices. Using a bi-level approach optimizing first for a given $p \in \mathcal{P}$ over $x \in \mathbb{Z}_+$, it is obvious that

$$v(S) = \sup_{p_0 \in \mathcal{P}} \{\Phi(p_0)\} \qquad (S)$$

with

$$\Phi(p_0) := \sup_{x \in \mathbb{Z}_+} \{\mathbb{E}(r_0(x,p_0)) - cx\}. \qquad (P_\Phi(p))$$

For every $p \in \mathcal{P}$, we know by Lemma 9 that the function

$$x \mapsto \mathbb{E}(r_0(x,p)) - cx$$

is discrete concave. This implies that an optimal solution $x(p) \in \mathbb{Z}_+$ of optimization problem $(P_\Phi(p))$ is given by

$$x(p) = \inf\{x \in \mathbb{Z}_+ : \Delta_x \mathbb{E}(r_0(x,p)) - c \leq 0\} \tag{3.3}$$

with $\Delta_x \mathbb{E}(r_0(x,p))$ denoting the first difference operator defined by

$$\Delta_x \mathbb{E}(r_0(x,p)) = \mathbb{E}(r_0(x+1,p)) - \mathbb{E}(r_0(x,p)), x \in \mathbb{Z}_+. \tag{3.4}$$

It is also obvious that

$$\mathbb{E}(r_0(x,p)) = \sum_{k=0}^{x-1} \Delta_x(\mathbb{E}(r_0(k,p)). \tag{3.5}$$

To compute the optimal order quantity $x(p)$ for a given feasible price p, we need to calculate $\Delta_x \mathbb{E}(r_0(x,p))$ for $k \in \mathbb{Z}_+$ until it satisfies the first order conditions given in relation (3.3) and at the same time using (3.5), we obtain an expression for $\mathbb{E}(r_0(x(p),p))$. The optimal objective value is then given by

$$\Phi(p) = \mathbb{E}(r_0(x(p),p)) - cx(p).$$

It is now easy to verify the next result. It is a straightforward generalization of the result in Lemma 1 including the salvage value costs.

Lemma 10 *It follows for every $k \in \mathbb{Z}_+$ and p given that*

$$\begin{aligned} \Delta_x \mathbb{E}(r_0(k,p)) &= (p-\theta)\mathbb{P}(N_p(T) \geq k+1) + h\int_0^T \mathbb{P}(N_p(s) \geq k+1)ds + \theta - hT \\ &= p + (\theta - p)\sum_{j=0}^k \mathbb{P}(N_p(T) = j) - h\sum_{j=0}^k \int_0^T \mathbb{P}(N_p(s) = j)ds. \end{aligned} \tag{3.6}$$

Proof. It is easy to check for every $s \geq 0$ and $k \in \mathbb{Z}_+$ that

$$N_p(s \wedge \sigma_k) = \sum_{n=1}^k 1_{\{\sigma_n \leq s\}}.$$

This shows

$$(k - N_p(s))^+ = k - N_p(s \wedge \sigma_k) = k - \sum_{n=1}^k 1_{\{\sigma_n \leq s\}}$$

and we obtain by relation (3.1)

$$\begin{aligned} r_0(k,p) &= (p-\theta)\sum_{n=1}^{k} 1_{\{\sigma_n \leq T\}} + h\sum_{n=1}^{k}\int_0^t 1_{\{\sigma_n \leq s\}}ds + k(\theta - hT) \\ &= (p-\theta)\sum_{n=1}^{k} 1_{\{N_p(T)\geq n\}} + h\sum_{n=1}^{k}\int_0^t 1_{\{N_p(s)\geq n\}}ds + k(\theta - hT). \end{aligned} \tag{3.7}$$

This shows

$$r_0(k+1,p) - r_0(k,p) = (p-\theta)1_{\{N_p(T)\geq k+1\}} + h\int_0^t 1_{\{N_p(s)\geq k+1\}}ds + \theta - hT$$

and we have verified the first equality. The second equality follows using

$$\mathbb{P}(N_p(s) \geq k+1) = 1 - \mathbb{P}(N_p(s) \leq k) = 1 - \sum_{j=0}^{k}\mathbb{P}(N_p(s) = j)$$

for every $s \leq t$ and the first equality. $\square$

Since the arrival process is a non homogeneous Poisson process with intensity function $(t,p) \mapsto \lambda(t,p)$, it is well known (cf.[26]) that

$$\mathbb{P}(N_p(s) = k) = e^{-\Lambda(s,p)}\frac{\Lambda(s,p)^k}{k!}, k \in \mathbb{Z}_+ \tag{3.8}$$

with the so-called mean value function given by

$$\Lambda(s,p) := \int_0^s \lambda(u,p)du. \tag{3.9}$$

To solve optimization problem ($P_\Phi(p)$), we need to apply the following algorithm for any given p.

Algorithm to solve optimization problem $P_\Phi(p)$ for a selected p

- **Step 1.** $\mathbb{E}(r_0(0,p)) = 0$
- **Step 2.** Evaluate (see Lemma 10)

$$\Delta_x(\mathbb{E}(r_0(0,p))) = p + (\theta - p)\mathbb{P}(N_p(T) = 0) - h\int_0^T \mathbb{P}(N_p(s) = 0)ds. \tag{3.10}$$

- **Step 3.** For $k = 1$ up to the first order conditions in relation (3.3) do the following:

Compute (see Lemma 10)

$$\mathbb{E}(r_0(k,p)) = \Delta_x(\mathbb{E}(r_0(k-1,p)) + \mathbb{E}(r_0(k-1,p))$$

and

$$\alpha(k,p) := (\theta - p)\mathbb{P}(N_p(T) = k) - h\int_0^T \mathbb{P}(N_p(s) = k)ds \tag{3.11}$$

and

$$\Delta_x(\mathbb{E}(r_0(k,p))) = \Delta_x\left(\mathbb{E}(r_0(k-1,p))\right) + \alpha(k,p).$$

- **Step 4.** Output optimal solution $x(p)$ and objective value

$$\Phi(p) = \mathbb{E}(R(T, x(p), p)) - cx(p).$$

This shows that the above algorithm is a black box to calculate $\Phi(p)$ for $p \in \mathcal{P}_0$. To solve optimization problem (S) approximately, we need the following result.

Lemma 11 *It follows for every $\overline{p} > p > \theta$ that*

$$\Phi(\overline{p}) - \Phi(p) \leq (\overline{p} - p)\Lambda(T,p).$$

Proof. It follows for every $\overline{p} > p$ using $\lambda(\overline{p}, s) \leq \lambda(p,s)$ for every s that $N_{\overline{p}}(s) \leq N_p(s)$ with probability 1 for every s. This shows for every k and s that

$$\mathbb{P}(N_{\overline{p}}(s) \geq k+1) \leq \mathbb{P}(N_p(s) \geq k+1) \tag{3.12}$$

implying

$$h\int_0^T \mathbb{P}(N_{\overline{p}}(s) \geq k+1)ds - h\int_0^T \mathbb{P}(N_p(s) \geq k+1)ds \leq 0.$$

Hence by Lemma 10, relation (3.12) and $p_1 > \theta$ it follows that

$$\begin{aligned}
\Delta_x\mathbb{E}(r_0(k,\overline{p})) - \Delta_x\mathbb{E}(r_0(k,p)) &\leq (\overline{p} - \theta)\mathbb{P}(N_{\overline{p}}(T) \geq k+1) - (p-\theta)\mathbb{P}(N_p(T) \geq k+1) \\
&\leq (\overline{p} - \theta)\mathbb{P}(N_p(T) \geq k+1) - (\overline{p}-\theta)\mathbb{P}(N_p(T) \geq k+1) \\
&= (\overline{p} - p)\mathbb{P}(N_p(T) \geq k+1).
\end{aligned}$$

This shows for every $x \in \mathbb{Z}_+$ that

$$\begin{aligned} \mathbb{E}((r_0(x,\overline{p})) - \mathbb{E}((r_0(x,p)) &\leq (\overline{p}-p)\textstyle\sum_{k=0}^{x-1}\mathbb{P}(N_p(T) \geq k+1) \\ &\leq (\overline{p}-p)\mathbb{E}(N_p(T)) \\ &= (\overline{p}-p)\textstyle\int_0^T \lambda(s,p)ds \end{aligned}$$

and we have shown the result. □

By the above result, we construct as follows a discretization of the interval $[c, p_{\max}]$ with $p_{\max} < \infty$. Fix the error $\epsilon > 0$ and start with $p_1 = c$. Once we have selected $p_m > p_{m-1} > p_{m-2} > ... > p_1$, we select the next point p_{m+1} as follows

$$p_{m+1} = p_m + \epsilon(\Lambda(T, p_m))^{-1}.$$

Using Lemma 11, it follows for every $p_n \leq p \leq p_{n+1}$ that

$$\Phi(p) - \Phi(p_m) \leq (p - p_m)\Lambda(T, p_m) \leq \epsilon.$$

Clearly, the number of terms in the constructed finite sequence $\mathcal{D} = (p_n)$ is bounded by $M = \epsilon(T\Lambda(T, p_{\max}))^{-1}$ and it follows by Lemma 11 that

$$\max\nolimits_{p\in\mathcal{P}_0}\Phi(p) - \max\nolimits_{p\in\mathcal{D}}\Phi(p) \leq \epsilon.$$

Using the above algorithm and relations (3.8) and (3.9), we need to compute the expressions

$$\mathbb{P}(N(T) = k) = e^{-\Lambda(T,p)}\frac{\Lambda(T,p)^k}{k!}$$

and

$$\int_0^T \mathbb{P}(N(s) = k)ds = \int_0^T e^{-\Lambda(s,p)}\frac{\Lambda(s,p)^k}{k!}ds$$

efficiently. A possible way to do this is given in the next section.

3.1 Solving the Static Model for Piecewise Constant Arrival Intensity Functions

As in [33], we need to specify the arrival intensity function. Contrary to [33], we also include the inventory costs in the objective function. In [33] the following model is adapted. Let $R_n, n \in \mathbb{N}$ denote a sequence of independent distributed random variables with conditional CDF

$$F_t(p) = \mathbb{P}(R_n \leq p \mid T_n = t), n \in \mathbb{N}, 0 \leq t \leq T$$

with R_n the reservation price of customer n. The nth arriving customer buys the product if and only if $R_n > p$ with p denoting the present price of the product. Now we set

$$\lambda(s,p) = \lambda_c(s)(1 - F_s(p)) \tag{3.13}$$

with λ_c denoting the arrival intensity function of the non homogeneous Poisson arrival process of potential customers. This shows that

$$\Lambda(t,p) = \int_0^t \lambda_c(s)(1 - F_s(p))ds.$$

Hence we need to give an elementary formula for

$$\int_0^t e^{-\Lambda(s,p)} \frac{\Lambda(s,p)^k}{k!} ds.$$

In general this should be done by numerical integration and since this takes a lot of computation time, we use the following special case in our calculations: replacing numerical integration by applying elementary formulas. Select a sequence

$$0 = a_1 < a_2 < .... < a_{m+1} = T.$$

If we set $A_i = [a_i, a_{i+1}), i = 1, ..., m-1$ and $A_m = [a_m, a_{m+1}]$, then we consider

$$\lambda(s,p) = \sum\nolimits_{i=1}^{m} \lambda_i (1 - F_i(p)) 1_{A_i}(s) \tag{3.14}$$

with $\lambda_1, ..., \lambda_m$ arbitrary positive numbers. This means that within each time interval A_i, the overall arrival rate of arriving potential customers is constant and within this interval, the CDF of the reservation is the same. A special case is given by $F_i(p)$ same for every

i and so the CDF of the reservation price is the same for all customers. Observe any function $t \mapsto \lambda_c(t)(1 - F_t(p))$ can be approximated by an above sequence of so-called simple functions.

Lemma 12 *It follows for every* $1 \leq i \leq m$ *that*

$$\Lambda(a_{i+1}, p) = \sum\nolimits_{j=1}^{i} \lambda_j (1 - F_j(p))(a_{j+1} - a_j) \tag{3.15}$$

and

$$\Lambda(t, p) = \sum\nolimits_{i=1}^{m} (\Lambda(a_i, p) + \lambda_i(1 - F_i(p))(t - a_i)) 1_{A_i}(t). \tag{3.16}$$

Proof. Since for every p the arrival intensity function is constant within the interval $A_i, i = 1, ..., m$ and given by $\lambda_i(1 - F_i(p))$, it follows that

$$\Lambda(a_{i+1}, p) - \Lambda(a_i, p) = \lambda_i(1 - F_i(p))(a_{i+1} - a_i).$$

This proves relation (3.15). The relation (3.16) is a direct consequence of

$$\Lambda(t, p) = \Lambda(t, p) - \Lambda(a_i, p) + \Lambda(a_i, p)$$

for every $t \in A_i$ and $\lambda_i(1 - F_i(p))$ is the constant intensity function on A_i. $\square$

By the algorithm in Step 3 for the computation of $\alpha(k, p)$, we know from Lemma 12 that

$$\mathbb{P}(N_p(T) = k) = e^{-\Lambda(T,p)} \frac{\Lambda(T, p)^k}{k!}$$

with

$$\Lambda(T, p) = \sum\nolimits_{j=1}^{m} \lambda_j(1 - F_j(p))(a_{j+1} - a_j).$$

In Step 3 of the algorithm for the arrival intensity function given in relation (3.14), we also need to compute the expression

$$\int_0^T \mathbb{P}(N_p(s) = k) ds$$

for any $k \in \mathbb{N}$.

Lemma 13 *If the arrival intensity function is given by*

$$\lambda(s,p) = \sum\nolimits_{i=1}^{m} \lambda_i(1 - F_i(p))1_{A_i}(s)$$

then for every $k \in \mathbb{N}$

$$\int_0^T \mathbb{P}(N_p(s) = k)ds = \sum\nolimits_{i=1}^{m} \lambda_i^{-1}(1 - F_i(p))^{-1}[\mathbb{P}(N_p(a_i) \leq k) - \mathbb{P}(N_p(a_{i+1}) \leq k)].$$

Proof. It follows that

$$\int_0^T \mathbb{P}(N_p(s) = k)ds = \sum\nolimits_{i=1}^{m} \int_{a_i}^{a_{i+1}} \mathbb{P}[N_p(s) = k)ds.$$

By Lemma 12, we obtain substituting $v = \lambda_{i,p}s + \beta_{i,p}$ with $\lambda_{i,p} = \lambda_i(1 - F_i(p))$ and $\beta_{i,p} = \Lambda(a_i, p) - \lambda_i a_i(1 - F_i(p))$ that

$$\begin{aligned} \textstyle\int_{a_i}^{a_{i+1}} \mathbb{P}(N(s) = k)ds &= \textstyle\int_{a_i}^{a_{i+1}} e^{-(\lambda_{i,p}s+\beta_{i,p})} \frac{(\lambda_{i,p}s+\beta_{i,p})^k}{k!} ds \\ &= \lambda_i^{-1}(1 - F_i(p))^{-1} \textstyle\int_{\Lambda(a_i,p)}^{\Lambda(a_{i+1},p)} e^{-v} \frac{v^k}{k!} dv. \end{aligned}$$

It is well known that

$$\int_w^\infty e^{-v} \frac{v^k}{k!} dv = e^{-w} \sum\nolimits_{j=0}^{k} \frac{w^j}{j!}.$$

This shows that

$$\begin{aligned} & \lambda_i(1 - F_i(p)) \textstyle\int_{a_i}^{a_{i+1}} \mathbb{P}(N(s) = k)ds \\ = \; & e^{-\Lambda(a_i,p)} \textstyle\sum_{j=0}^{k} \frac{\Lambda(a_i,p)^j}{j!} - e^{-\Lambda(a_{i+1},p)} \textstyle\sum_{j=0}^{k} \frac{\Lambda(a_{i+1},p)^j}{j!} \\ = \; & \mathbb{P}(N_p(a_i) \leq k) - \mathbb{P}(N_p(a_{i+1}) \leq k) \end{aligned}$$

and we obtain the desired result. □

An immediate consequence of the above lemma is given by the following result.

Lemma 14 *Introducing*

$$\beta(k,p) = \int_0^T \mathbb{P}(N_p(s) = k)ds, k \in \mathbb{Z}_+$$

it follows for the arrival intensity function given by

$$\lambda(s,p) = \sum\nolimits_{i=1}^m \lambda_i(1 - F_i(p))1_{A_i}(s)$$

that for every $k \in \mathbb{N}$

$$\beta(k,p) = \beta(k-1,p) + \sum\nolimits_{i=1}^m \lambda_i^{-1}(1 - F_i(p))^{-1}[\mathbb{P}(N_p(a_i) = k) - \mathbb{P}(N_p(a_{i+1}) = k)].$$

Proof. No proof is required since the result is obvious. □

In the algorithm to solve $P_\Phi(p)$, we start with $k = 0$. For the piecewise constant arrival intensity case given by $\lambda(s,p) = \sum_{i=1}^m \lambda_i(1 - F_i(p))1_{A_i}(s)$, we need to evaluate the expression

$$\begin{aligned} & \Delta_x(\mathbb{E}(r_0(0,p))) \\ = \quad & p + (\theta - p)\mathbb{P}(N_p(T) = 0) - h \textstyle\int_0^T \mathbb{P}(N_p(s) = 0)ds \\ = \quad & p + (\theta - p)e^{-\Lambda(T,p)} - h \textstyle\sum_{i=1}^m \lambda_i^{-1}(1 - F_i(p))^{-1}[\mathbb{P}(N_p(a_i) = 0) - \mathbb{P}(N(a_{i+1}) = 0)] \\ = \quad & p + (\theta - p)e^{-\Lambda(T,p)} - h \textstyle\sum_{i=1}^m \lambda_i^{-1}(1 - F_i(p))^{-1}[e^{-\Lambda(a_i,p)} - e^{-\Lambda(a_{i+1},p)}]. \end{aligned}$$

To compute $\alpha(k,p)$ for this case, we observe the following. We know from relation (3.11) and Lemma 13 that

$$\begin{aligned} & \alpha(k,p) \\ = \quad & (\theta - p)\mathbb{P}(N_p(T) = k) - h \textstyle\int_0^T \mathbb{P}(N_p(s) = k)ds \\ = \quad & (\theta - p)\mathbb{P}(N_p(T) = k) - \textstyle\sum_{i=1}^m \lambda_i^{-1}(1 - F_i(p))^{-1}[\mathbb{P}(N(a_i) \le k) - \mathbb{P}(N(a_{i+1}) \le k)]. \end{aligned}$$

In the next section, we will apply an alternative approach to solve the static optimization problem.

3.2 Global Properties of the Objective Function

Another way of solving the static problem is given by the following bi-level approach.

$$\upsilon(S) = \sup_{x\in\mathbb{Z}_+}\{\Psi(x) - cx\} \tag{S}$$

with

$$\Psi(x) := \sup_{p_0\in\mathcal{P}_0}\{\mathbb{E}(r_0(x,p_0))\}. \tag{$P_\Psi(x)$}$$

Observe in this section we assume that the set of feasible prices is convex. Since the optimization problem $(P_\Psi(x))$ is a continuous one-dimensional optimization problem, we need to check under which conditions the function $p \mapsto \mathbb{E}(r_0(x,p))$ has nice concavity type properties. Hence we are now interested in the properties of the function $p \mapsto \mathbb{E}(r_0(x,p))$. By relation (3.1) we know

$$\mathbb{E}(r_0(x,p)) = \mathbb{E}\left(pN_p(T\wedge\sigma_x) + \theta(x - N_p(T))^+ - h\int_0^T (x - N_p(u))^+ du\right). \tag{3.17}$$

Since

$$pN_p(T\wedge\sigma_x) + \theta(x - N_p(T))^+ = (p-\theta)N_p(T\wedge\sigma_x) + \theta x$$

and

$$\int_0^T (x - N_p(u))^+ du = xT - \int_0^t N_p(u\wedge\sigma_x)du \tag{3.18}$$

we obtain

$$\mathbb{E}(r_0(x,p)) = (p-\theta)\mathbb{E}(N_p(T\wedge\sigma_x)) + h\int_0^T \mathbb{E}(N_p(u\wedge\sigma_x))du + x(\theta - hT). \tag{3.19}$$

This means we have to solve the optimization problem

$$\upsilon(S) = \sup_{x\in\mathbb{Z}_+}\{\overline{\Psi}(x) + x(\theta - c - hT)\}$$

with

$$\overline{\Psi}(x) := \sup_{p_0 \in \mathcal{P}} \left\{ (p_0 - \theta)\mathbb{E}(N_{p_0}(T \wedge \sigma_x)) + h \int_0^T \mathbb{E}(N_{p_0}(u \wedge \sigma_x))du \right\}. \quad (P_{\overline{\Psi}}(x))$$

Before discussing under which conditions the objective function of optimization problem $(P_{\overline{\Psi}}(x))$ is concave, we first give an upper bound on the optimal order quantity x. Observe that we already showed the next result in Chapter 2 for the dynamic model (see Lemma 5). Take in that result $N = 1$. For completeness, we give a simplified proof for the static case.

Lemma 15 *If $\theta < c$ and $\upsilon(Q) = \sup_{p_0 \in \mathcal{P}_0}\{(p_0 - \theta)\Lambda(T, p_0)\} < \infty$, then any optimal x_{opt} of optimization problem (S) satisfies $x_{opt} \leq x_U = \left\lceil \frac{\upsilon(Q)}{\theta - c} \right\rceil$.*

Proof. Since $\mathbb{E}(N_{p_0}(T \wedge \sigma_x)) \leq \mathbb{E}N_{p_0}(T) = \Lambda(T, p_0)$ and $\mathbb{E}(N_{p_0}(u \wedge \sigma_x)) \leq x$ for every $u \leq T$ it follows by relation (3.19) that

$$\mathbb{E}(r_0(x, p)) - cx \leq (p - \theta)\Lambda_p(T, p) + x(\theta - c).$$

This shows for every $p \in \mathcal{P}_0$ and $x \geq \left\lceil \frac{\upsilon(Q)}{\theta - c} \right\rceil + 1$ that

$$\mathbb{E}(r_0(x, p)) - cx \leq \upsilon(Q) + x(\theta - c) < 0. \tag{3.20}$$

Since for $x = 0$ the objective function in optimization problem (S) equals zero, it follows that $\upsilon(S) \geq 0$ and by relation (3.20) we obtain the desired result. □

By Lemma 15 it follows that

$$\upsilon(S) = \sup_{x \in \mathbb{Z}_+, x \leq x_U}\{\overline{\Psi}(x) + x(\theta - c - hT)\}. \tag{3.21}$$

In the remainder of this section, we will analyze the properties of the function

$$p \mapsto (p - \theta)\mathbb{E}(N_p(T \wedge \sigma_x)) + h \int_0^T \mathbb{E}(N_p(u \wedge \sigma_x))du. \tag{3.22}$$

Remember we always assume that the arrival intensity function λ is positive and bounded, and for every $0 \leq t \leq T$ the function $p \mapsto \lambda(t, p)$ is decreasing. This implies for every $p \in \mathcal{P}$ that the function $t \mapsto \Lambda(t, p)$ with $\Lambda(t, p)$ given in relation (3.9) is strictly increasing and for every $t \leq T$ the function $p \mapsto \Lambda(t,p)$ is decreasing. Applying now Doob's optimal stopping theorem ($t \mapsto N(t) - \Lambda(t,p)$ is a right continuous martingale), we obtain for every $u \leq T$

$$\begin{aligned} \mathbb{E}(N_p(u \wedge \sigma_x)) &= \mathbb{E}\left(\int_0^{u\wedge\sigma_x} \lambda(s,p)ds\right) \\ &= \mathbb{E}(\min\{\Lambda(u,p), \Lambda(\sigma_x,p)\}. \end{aligned} \tag{3.23}$$

Since the arrival intensity function λ is positive, it follows that the random variable $\Lambda(\sigma_x, p)$ has the same CDF as the random variable

$$\rho_x = \inf\{t \geq 0 : \overline{N}(t) \geq x\}$$

with $\overline{N} = \{\overline{N}(t) : t \geq 0\}$ a homogeneous Poisson process with arrival rate 1. Hence by relation (3.23), we obtain for every $u \leq T$

$$\mathbb{E}(N_p(u \wedge \sigma_x)) = \mathbb{E}\left(\min\{\Lambda(u,p), \rho_x\}\right). \tag{3.24}$$

Also by relation (3.18), we obtain

$$\begin{aligned} \mathbb{E}\left(\int_0^T (x - N_p(u)^+ du\right) &= \int_0^T \mathbb{E}((x - N_p(u \wedge \sigma_x))du \\ &= xT - \int_0^T \mathbb{E}\left(\min\{\Lambda(u,p), \rho_x\}\right) du. \end{aligned} \tag{3.25}$$

Introducing the function $\overline{g} : \mathbb{R}_+ \times \mathbb{Z}_+$ given by

$$\overline{g}(u,x) := \mathbb{E}(\min\{u, \rho_x\}) = \int_0^u \mathbb{P}(\rho_x > v)dv = \int_0^u \mathbb{P}(\overline{N}(v) \leq x-1)dv \tag{3.26}$$

it follows by relations (3.24) and (3.25) that

$$v(S) = \sup_{x\in\mathbb{Z}_+}\{\overline{\Psi}(x) + x(\theta - c - hT)\}$$

with

$$\overline{\Psi}(x) = \sup_{p\in\mathcal{P}_0}\left\{(p-\theta)\overline{g}(\Lambda(T,p),x) + h\int_0^T \overline{g}(\Lambda(u,p),x)du\right\}. \tag{$P_{\overline{\Psi}}(x)$}$$

In the next result, we show under which conditions the objective function of optimization problem ($P_{\overline{\Psi}}(x)$) is concave on $\mathcal{P}_0$.

Lemma 16 *If for every $0 \leq u \leq T$ the function $p \mapsto \Lambda(u,p)$ is concave and differentiable on the convex interval $\mathcal{P} \subseteq (c,\infty)$, then for every $x \in \mathbb{Z}_+$ the function*

$$p \mapsto (p-\theta)\overline{g}(\Lambda(T,p),x) + h\int_0^T \overline{g}(\Lambda(u,p),x)du$$

is concave on $\mathcal{P}$.

Proof. By relation (3.26) it follows that

$$(p-\theta)\overline{g}(\Lambda(T,p),x) = \mathbb{E}(\min\{(p-\theta)\Lambda(T,p),(p-\theta)\rho_x\}). \qquad (3.27)$$

Since by assumption the function $p \mapsto \lambda(T,p)$ is a decreasing positive function, the function $p \mapsto \Lambda(T,p)$ is also decreasing and positive. To show the concavity of the function $p \mapsto (p-\theta)\Lambda(T,p)$ on $\mathcal{P}_0$ we now proceed as follows. We first observe that the derivative of the function $\gamma(p) = (p-\theta)\Lambda(T,p)$ is given by

$$\gamma'(p) = \Lambda(T,p) + (p-\theta)\frac{\partial\Lambda}{\partial p}(T,p).$$

Since the decreasing function $p \mapsto \Lambda(T,p)$ is positive and concave, this shows that the function $p \mapsto \frac{\partial\Lambda}{\partial p}(T,p)$ is decreasing and non-positive. Hence it follows using $\theta \leq c$ that the function $p \mapsto \gamma'(p)$ is decreasing on $\mathcal{P} \subseteq [c,\infty)$ and so the function γ is concave on $\mathcal{P}$. Using the fact that $\min$ operator preserves concavity and this implies by relation (3.27) that the function $p \mapsto (p-\theta)\overline{g}(\Lambda(T,p),x)$ is concave. By the same reasoning, the function

$$p \mapsto \overline{g}(\Lambda(u,p),x) = \mathbb{E}(\min\{\Lambda(u,p),\rho_x\})$$

is also concave on $\mathcal{P}$ for every $u \leq T$ and this shows that the function $p \mapsto h\int_0^T \overline{g}(\Lambda(u,p),x)du$ is concave on $\mathcal{P}$. Since adding two concave functions preserves concavity, the desired result follows. □

It can also be shown by a similar analysis that the objective function is concave without assuming differentiability of the function $p \mapsto \Lambda(u,p)$. We only impose this condition since the proof of concavity of the function $p \mapsto (p-\theta)\overline{g}(\Lambda(T,p),x)$ can then be done using the derivative. As in relation (3.13) we assume that

$$\lambda(t,p) = (1 - F_t(p))\lambda_c(t).$$

In case $\mathcal{P} = [c, p_{\max}]$ with $p_{\max} < \infty$ and for every $t \leq T$ the function $p \mapsto F_t(p)$ is convex on $\mathcal{P}$, it follows that the conditions of Lemma 16 are satisfied.

If $p \mapsto \Lambda(u,p)$ is concave and decreasing for every $u \leq T$, then an alternative algorithm to solve the static problem has the following structure.

Alternative algorithm to solve the static problem

- **Step 1.** Solve the concave maximization problem

$$v(Q) = \sup_{p\in\mathcal{P}}\{(p-\theta)\Lambda(T,p)\}.$$

- **Step 2.** For $x = 0$ up to $\left\lceil \frac{v(Q)}{\theta - c} \right\rceil$ solve the concave optimization problem

$$\overline{\Psi}(x) = \sup_{p\in\mathcal{P}}\left\{(p-\theta)\overline{g}(\Lambda(T,p),x) + h\int_0^T \overline{g}(\Lambda(u,p),x)du\right\}$$

  and compute its optimal solution $p(x)$.

- **Step 3.** Determine

$$x_{opt} = \arg\max\left\{x \in \mathbb{Z}_+, x \leq \left\lceil \frac{v(Q)}{\theta - c} \right\rceil : \overline{\Psi}(x) + x(\theta - c - hT)\right\}$$

  and $p(x_{opt})$.

In case $\lambda(s,p) = \lambda_c(s)(1 - F_s(p))$ and for every s, the CDF F_s has a density f_s and it follows that the derivative of function

$$p \mapsto (p-\theta)\Lambda(T,p) = (p-\theta)\int_0^T \lambda_c(s)(1-F_s(p))ds_c$$

is given by

$$p \mapsto \int_0^T \lambda_c(s)[1 - F_s(p) - f_s(p)(p-\theta)]ds.$$

As before, we can now use in the computational section the special case considered in relation (3.14). The above derivative becomes then easy. A one dimensional concave maximization problem over a convex interval can be easily solved by using the derivative.

Chapter 4

Numerical Results

4.1 Computational Results for the Base Scenario

In our numerical experiments, we consider for a given base scenario the static and dynamic model, and compare the performances of both models. The algorithm derived in this thesis is coded in Python and executed on a laptop with a 2.40 GHz processor. Applying different variations of the base scenario to the dynamic model take on average 6 minutes and applying to the static model the same variations take on average 20 seconds.

In the base scenario, we set the length of the season to 18 weeks and so the parameter T is given by 18. The ordering cost c of an item is given by 60 and 350 is the maximum selected price p_{max}. Also the discrete set of possible prices the supplier can select from at the decision moments is given by $\{60, 70, 80, ..., 350\}$. This corresponds to a discretization parameter $\epsilon = 10$ on the interval $[c, p_{max}]$. The value h denoting the inventory holding costs per item per unit of time is set to 25.

During the season of 18 weeks, potential customers are arriving. In both models, we divide the season into three equal parts each lasting 6 weeks. This means we consider three equal time intervals $[a_1, a_2)$,$[a_2, a_3)$ and $[a_3, a_4)$ with $a_1 = 0$, $a_2 = 6$, $a_3 = 12$ and $a_4 = 18$. At the beginning of the season the product is highly popular, so we face on average the highest number of potential customers during that period. Due to the decreasing popularity of the product, the average number of potential customers decreases during the season. Consequently, on average we face a high interest in the product during the first 6 weeks, a medium interest from week 6 until week 12 and a low interest during the last 6 weeks of the season. To capture this behavior, we take $\lambda_{i+1} = \alpha\lambda_i$, $i = 1, 2$ with λ_i denoting the average number of potential customers per unit of time during period i,

$i = 1, 2, 3$ of the season, and $0 < \alpha < 1$ is some chosen parameter. In our base scenario, we set $\alpha = 0.5$ and $\lambda_1 = 400$.

Next to being interested in buying the offered product, we also need to specify the willingness to pay of a customer for a given price. This is known as the reservation price of each customer. Since the reservation price is a random variable, we assume in our computational setup that its cumulative distribution function is given by an exponential distribution with parameter μ. Assuming that the popularity of the product will decrease over time, the willingness to pay also decreases over time. This behavior is captured by a different choice of the parameter μ in different periods of the season. Since an exponential cumulative distribution with parameter μ has first moment $\frac{1}{\mu}$ and so, $\frac{1}{\mu}$ is the expected price a customer is willing to pay, we choose increasing μ_i values in our base scenario. In this base scenario, we set the parameters of the exponential cumulative distribution to $\mu_1 = \frac{1}{150}$, $\mu_2 = \frac{1}{90}$ and $\mu_3 = \frac{1}{55}$. This corresponds to an expected price a customer is willing to pay of 150 in the first period, 90 in the second period, and 55 in the last period of the season. Also in our computations, we consider the arrival rate function

$$\lambda(t, p) = \sum_{i=1}^{3} \lambda_i (e^{-\mu_i p}) 1_{[a_i, a_{i+1})}(t).$$

In the static model, we decide only on the price of the product at the beginning of the season and keep the same price during the whole season. Also we will not decide within the season to stop selling the product and leave the market. In the dynamic model, it is decided at the beginning of the season at some selected times τ_n, $1 \leq n \leq N-1$ to reconsider the price of the product or stop selling the product and leaving the market. When leaving the market within the season or at the end of the season, we obtain for each leftover item a salvage value θ. In our base scenario θ equals 50. Also in our base scenario, we will only take either adjusting price or leaving the market decisions at the moments where the average demand of potential customers is changing. Hence in our base scenario $\tau_0 = 0$, $\tau_1 = 6$ and $\tau_2 = 12$. Since we have more options in the dynamic model, it is clear that the expected profit given by the dynamic model is always greater than or equal to the expected profit given by the static model. For completeness, we summarize the values of all parameters of our base scenario in Table 4.1.

T	c	$p_{\max}$	h	θ	ϵ	a_i	τ_n	λ_i	μ_i
18	60	350	25	50	10	0, 6, 12	0, 6, 12	400, 200, 100	$\frac{1}{150}, \frac{1}{90}, \frac{1}{55}$

Table 4.1: Base scenario problem parameters for $i = 1, 2, 3$ and $n = 0, 1, 2$.

In Table 4.2, we report the results of the base scenario for the static and dynamic version of the model. We observe both for the dynamic and static model that the optimal initial price is given by 290. In the static model this price will stay the same during the whole season; while depending on the realized demand, this price will change for the dynamic model. For the dynamic model, the optimal order quantity equals 370 and this value is 5 units higher than the optimal order quantity for the static model. Since for the static model we select the optimal price of 290, the expected number of sold items equals 398.11. This means for the selected optimal price and optimal order quantity, one can easily compute the probability of no left-over items at the end of the season for the static model. This probability is given by

$$
\begin{aligned}
\mathbb{P}(N_p(T) > 365) &= 1 - \mathbb{P}(N_p(T) \leq 365) \\
&= 1 - \sum_{j=0}^{365} \mathbb{P}(N_p(T) = j) \\
&= 1 - \sum_{j=0}^{365} e^{-\Lambda(T,p^*)} \frac{\Lambda(T,p^*)^j}{j!} \\
&= 1 - 0.05 \\
&= 0.95
\end{aligned}
$$

To compare the difference in maximum expected profits for the static and dynamic model, we compute the percentage

$$PR = 100 \times \frac{\upsilon(P) - \upsilon(S)}{\upsilon(S)}.$$

In this ratio, $\upsilon(P)$ and $\upsilon(S)$ denote the maximum expected profits of dynamic and static model respectively. For the base scenario, the relative difference is found to be $\%1.18$ which means the maximum expected profit of the dynamic model is $\%1.18$ higher than the maximum expected profit of the static model. Also in Figure 4.1, we list for the static model the expected profit for every chosen price within the interval $[c, p_{max}]$. The red point in this figure denotes the optimal objective profit for the static model.

Dynamic Model

$\upsilon(P)$	x^*	p_0^*
54468.14	370	290

Static Model

$\upsilon(S)$	$\Lambda(T, p^*)$	x^*	p^*
53833.86	398.11	365	290

Table 4.2: Dynamic and static model outcomes for the base scenario parameters.

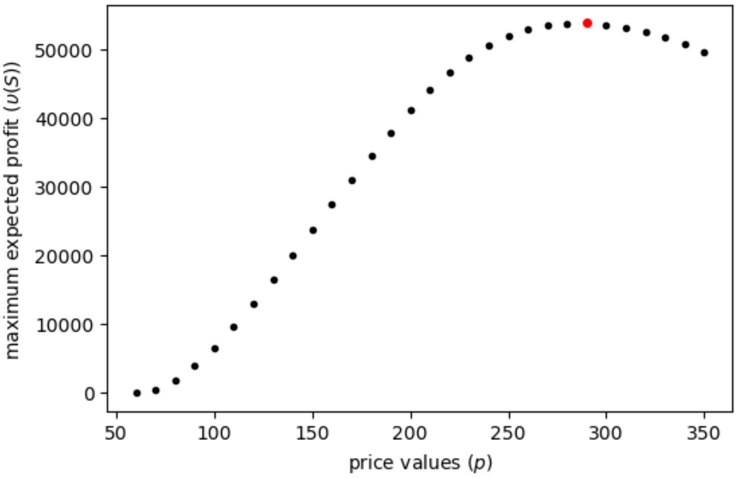


Figure 4.1: Outcome of the static model.

Applying the dynamic programming algorithm, we start with the boundary condition (2.47) at $\tau_3 = 18$ and using backward iterations, we compute the maximum expected incremental revenues $V_n(x)$ at τ_n, $n = 1, 2$ and the optimal actions depending on the current inventory level at each decision moment. At τ_0, we compute $U_0(x)$ to decide on the optimal order quantity (2.50). Since the decision moments are scheduled at the beginning of the season, the objective function of optimization problem (P) is determined using only three iterations. In case it is profitable to enter the market, we then determine the optimal order quantity x^* and the optimal initial price p_0^* in the first period. The optimal actions are reported in the optimal policy table. Hence looking at the remaining inventory level at $\tau_0 = 0$, $\tau_1 = 6$ and $\tau_2 = 12$, the optimal expected incremental profit $V_n(x)$, the optimal price to select at that time and the expected number of customers buying the product for that selected price until the next decision moment can be found in Table 4.3. If it is optimal to leave the market at time τ_n, the optimal price and the expected number of customers are set to zero and only the total salvage value is listed. Since in the base scenario it is optimal to order 370 items in the dynamic model, the optimal policy table consists of 371 rows and 4 columns and in each of those columns it is listed what to do at the decision moments 0, 6 and 12. According to our results, the supplier should order 370 units and his maximum expected incremental revenue until the end of the season with the optimal initial price 290 is given by 76668.14. Also the expected number of sales in the first 6 weeks at the price 290 equals 347.2. The cells at which it is optimal to leave the market can be found under the $\tau_1 = 6$ column between rows $x = 370$ and $x = 297$ and under the $\tau_2 = 12$ column between rows $x = 370$ and $x = 65$. At the end of the season at $\tau_3 = 18$, we certainly leave the market and obtain a salvage value of 50 per unit for the remaining inventory. The last row, $x = 0$ contains all zeros since no inventory means no sales and hence no profits.

One should give the following interpretation of the optimal policy table; after ordering 370 units and setting a price of 290 at $\tau_0 = 0$, we assume for example that 75 items are sold during the first 6 weeks. The remaining inventory at $\tau_1 = 6$ is then 295 and according to the optimal policy table, the supplier should set the new price at 130 during week 6 until week 12 to reach the maximum remaining expected revenue until the end of season which is given by 14929.99. Also in the same table, it is listed that the expected number of sales at the given price 130 during week 6 until 12 is given by 283.05. Consequently, the policy listed in Table 4.3 should be applied at every decision moment based on the remaining inventory level. Examining Table 4.3 more closely, we also observe that after having sold 368 items during the first 6 weeks and left with 2 items at the beginning of the second period of the season the items are more popular than expected), we will set the new price at the maximum level. This is higher than the optimal initial price of 290 and this means, we do not always reduce the price and give a discount.

	$\tau_0 = 0$	$\tau_1 = 6$	$\tau_2 = 12$	$\tau_3 = 18$
$x = 370$	76668.14, 290, 347.2	18500, 0, 0	18500, 0, 0	18500
$x = 369$	76607.73, 290, 347.2	18450, 0, 0	18450, 0, 0	18450
$x = 368$	76543.85, 290, 347.2	18400, 0, 0	18400, 0, 0	18400
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x = 297$	70933.89, 320, 284.26	14850, 0, 0	14850, 0, 0	14850
$x = 296$	70830.17, 320, 284.26	14871.56, 130, 283.05	14800, 0, 0	14800
$x = 295$	70722.62, 320, 284.26	14929.99, 130, 283.05	14750, 0, 0	14750
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x = 140$	42638.63, 350, 232.73	16308.44, 190, 145.32	7000, 0, 0	7000
$x = 139$	42378.86, 350, 232.73	16278.79, 190, 145.32	6950, 0, 0	6950
$x = 138$	42118.45, 350, 232.73	16246.13, 190, 145.32	6900, 0, 0	6900
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x = 64$	21059.41, 350, 232.73	11789.18, 250, 74.61	3200, 0, 0	3200
$x = 63$	20750.66, 350, 232.73	11702.8, 260, 66.77	3202.94, 110, 81.2	3150
$x = 62$	20441.26, 350, 232.73	11613.62, 260, 66.77	3210.18, 110, 81.2	3100
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$x = 2$	698.07, 350, 232.73	681.68, 350, 24.56	428.84, 260, 5.31	100
$x = 1$	349.36, 350, 232.73	343.89, 350, 24.56	234.64, 280, 3.69	50
$x = 0$	0	0	0	0

Table 4.3: Optimal policy table for the base scenario.

To visualize the optimal stopping regions, we also show this in a separate plot by a two dimensional vector consisting of the time and the corresponding inventory levels. For the dynamic model, Figure 4.2 shows the set of points at which it is optimal to leave the market and sell the remaining leftover inventory at the salvage value 50 per left over item. As also shown in Table 4.3, we will only exit the market at $\tau_1 = 6$ if the remaining inventory is between 370 and 297, and at $\tau_2 = 12$, if the remaining inventory is between 370 and 64. For any order size $x_{ord} \geq 370$, we can also compute the probability of leaving the market before the end of the season. First, we compute the probability of leaving the market after 6 weeks, so $\Delta\tau_0 = \tau_1 - \tau_0 = 6$ and by the shape of the optimal stopping set, this is given by

$$
\begin{aligned}
&\mathbb{P}(\text{exit market at } \tau_1 = 6) \\
&= \mathbb{P}(N_p(\tau_1) \leq x_{ord} - 297) \qquad (4.1) \\
&= e^{-\lambda_1(e^{-\mu_1 p^*})6} \sum_{j=0}^{x_{ord}-297} \frac{(\lambda_1(e^{-\mu_1 p^*})6)^j}{j!}
\end{aligned}
$$

Also it is easy to compute the probability of leaving the market after 12 weeks and again by the shape of the optimal stopping set, we obtain

$$
\begin{aligned}
&\mathbb{P}(\text{exit market at } \tau_2 = 12) \\
&= \mathbb{P}(\text{not exit market at } \tau_1 = 6, \text{exit market at } \tau_2 = 12) \\
&= \mathbb{P}(N_p(\tau_2) \leq x_{ord} - 64, N_p(\tau_1) > x_{ord} - 297) \\
&= \sum_{j=x_{ord}-296}^{x_{ord}-64} \mathbb{P}(N_p(\tau_2) \leq x_{ord} - 64, N_p(\tau_1) = j) \qquad (4.2) \\
&= \sum_{j=x_{ord}-296}^{x_{ord}-64} \mathbb{P}(N_p(\tau_2) \leq x_{ord} - 64 \mid N_p(\tau_1) = j)\mathbb{P}(N_p(\tau_1) = j) \\
&= \sum_{j=x_{ord}-296}^{x_{ord}-64} \mathbb{P}(N_p^{(1)}(\Delta\tau_1) \leq x_{ord} - 64 - j \mid N_p(\tau_1) = j)\mathbb{P}(N_p(\tau_1) = j)
\end{aligned}
$$

For the base scenario, probability of exiting the market is very low; therefore, we will consider the case where we order 1025 units, $x_{ord} = 1025$ which is much more than the optimal order quantity. From Table 4.5, we observe that $U_0(1025) = 72174.47$ and the expected profit can be calculated again with $U_0(1025) - c(1025)$ which equals 10674.47. Hence, it is still profitable to enter the market. With the optimal initial price $p_0^* = 170$, the expected number of buying customers in the first 6 weeks is $\lambda_1(e^{-\mu_1 p_0^*})6 = 772.7$. As Figure 4.3 shows, it is optimal to leave the market at two different decision moments, so we need to sum the probability of exiting at τ_1 and the probability of exiting at τ_2 to find $\mathbb{P}$(exiting the market until T). Consequently, this scenario happens with probability 0.5878.

We know that $N_p(t)$ gives the total demand up to time t, $N_p^{(n)}(t) = N_p(t + \tau_n) - N_p(\tau_n)$ with price p. Also, we would like to note that $N_p(\tau_1) = j$ and $N_p^{(1)}(\Delta\tau_1) \leq x_{ord} - 64 - j$ are correlated. We can explain this situation with an example where $x_{ord} = 370$, which is the optimal order quantity in the base scenario. If we assume that demand is 230 in the first 6 weeks, then the remaining inventory will be $x = 370 - 230 = 140$ at $\tau_1 = 6$. Hence, we should look to the cell in the optimal policy table under $\tau_1 = 6$ column and the row where $x = 140$. From Table 4.3, we find $V_1(140) = 16308.44$, optimal price until $\tau_2 = 12$ is 190 and expected number of buying customers with price 190 until $\tau_2 = 12$ is 145.32. And to calculate $P(N_p^{(1)}(\Delta\tau_1) \leq x_{ord} - 64 - j) = P(N_p^{(1)}(\Delta\tau_1) \leq 76)$, we should use those pieces of information in the corresponding cell. We would like to note that demand during week 6 and week 12 should be no more than 76 in order to exit the market at $\tau_2 = 12$. Consequently, $N_p(\tau_1) = j$ specifies the remaining inventory at $\tau_1 = 6$ which gives the $x_{ord} - j$, and this will lead to a different optimal price to adjust and different expected number of buying customers until the next decision moment.

In Section 2.1, we mentioned about the no stopping model and the different part of its algorithm from the dynamic model (see 2.11 & 2.12). When we apply the no stopping model (NSM) algorithm with the base scenario parameters, we obtain that optimal order quantity and maximum expected profit values are the same with the dynamic model. As it can be seen from Table 4.3 and Table 4.4, $U_0(370)$ is the same for the dynamic and no stopping models. This is because the probability of exiting the market probability is almost zero for the base scenario. However, we also observe some differences in Table 4.4. For instance, if we can not sell any items during the first period, by considering optimal policy table for the no stopping model, we should adjust our price to 110 at $\tau_1 = 6$ and the maximum expected incremental revenue that we can gain from then on wards is 11400.61. In the dynamic model, if we can not sell any items during the first period, it is optimal to exit the market at $\tau_1 = 6$. Since in the no stopping model we can not exit the market until the end of the season, the model tries to set the price in such a way that we can still earn income. Obviously, this income value in the no stopping model is smaller than the revenue that we get when we exit the market in dynamic model, i.e. revenue equals to θx. Moreover, we also observe negative values in Table 4.4. For instance, if we are left with 295 items at $\tau_2 = 12$, we should set the price to 60 and still the expected number of customers are less than the inventory on hand, also we have to pay holding cost per item per unit of time; therefore, maximum expected incremental revenue becomes negative. In the dynamic model with the same inventory level at $\tau_2 = 12$, we choose to exit the market since it is more profitable.

As discussed above, the probability of leaving the market in the dynamic model is 0.5878 when $x_{ord} = 1025$. Since this probability is much higher than the one in the base scenario, we compare the expected profits of the dynamic and no stopping model when we order 1025 items. Optimal policy for both models, for only inventory level equals 1025 can be found in Table 4.5. DM refers to the dynamic model and NSM refers to the no stopping model. When we look closer, we see that in DM, $U_0(1025)$ equals 72174.47 with initial price 170 and the expected profit 10674.47. However in NSM, $U_0(1025)$ equals 61902.97 with initial price 140 and the expected profit 403. In this case, we can see how big the difference of the profits for both models. Also in DM, it is optimal to leave the market at $\tau_1 = 6$ and $\tau_2 = 12$; but in NSM, we observe negative maximum expected incremental revenues at both decision moments. The reason is that when the possibility of leaving the market is high, the option of leaving the market becomes a major factor which creates a more profitable situation.

	$\tau_0 = 0$	$\tau_1 = 6$	$\tau_2 = 12$	$\tau_3 = 18$
$x = 370$	76668.14, 290, 347.2	11400.61, 110, 353.49	-19868.54, 60, 201.55	18500
$x = 369$	76607.73, 290, 347.2	11472.67, 110, 353.49	-19768.54, 60, 201.55	18450
$x = 368$	76543.85, 290, 347.2	11542.65, 110, 353.49	-19668.54, 60, 201.55	18400
⋮	⋮	⋮	⋮	⋮
$x = 297$	70933.89, 320, 284.26	14810.33, 130, 283.05	-12568.54, 60, 201.55	14850
$x = 296$	70830.17, 320, 284.26	14871.35, 130, 283.05	-12468.54, 60, 201.55	14800
$x = 295$	70722.62, 320, 284.26	14929.82, 130, 283.05	-12368.54, 60, 201.55	14750
⋮	⋮	⋮	⋮	⋮
$x = 140$	42638.63, 350, 232.73	16308.44, 190, 145.32	1054.3, 60, 201.55	7000
$x = 139$	42378.86, 350, 232.73	16278.79, 190, 145.32	1098.5, 60, 201.55	6950
$x = 138$	42118.45, 350, 232.73	16246.13, 190, 145.32	1141.95, 60, 201.55	6900
⋮	⋮	⋮	⋮	⋮
$x = 64$	21059.41, 350, 232.73	11789.18, 250, 74.61	3196.45, 100, 97.39	3200
$x = 63$	20750.66, 350, 232.73	11702.8, 260, 66.77	3202.94, 110, 81.2	3150
$x = 62$	20441.26, 350, 232.73	11613.62, 260, 66.77	3210.18, 110, 81.2	3100
⋮	⋮	⋮	⋮	⋮
$x = 2$	698.07, 350, 232.73	681.68, 350, 24.56	428.84, 260, 5.31	100
$x = 1$	349.36, 350, 232.73	343.89, 350, 24.56	234.64, 280, 3.69	50
$x = 0$	0	0	0	0

Table 4.4: Optimal policy table for no stopping model.

		$\tau_0 = 0$	$\tau_1 = 6$	$\tau_2 = 12$	$\tau_3 = 18$
DM	$x = 1025$	72174.47, 170, 772.7	51250, 0, 0	51250, 0, 0	51250
NSM	$x = 1025$	61902.97, 140, 943.78	-94334.91, 60, 616.1	-85368.54, 60, 201.55	51250

Table 4.5: Comparing dynamic model and no stopping model when 1025 items are ordered.

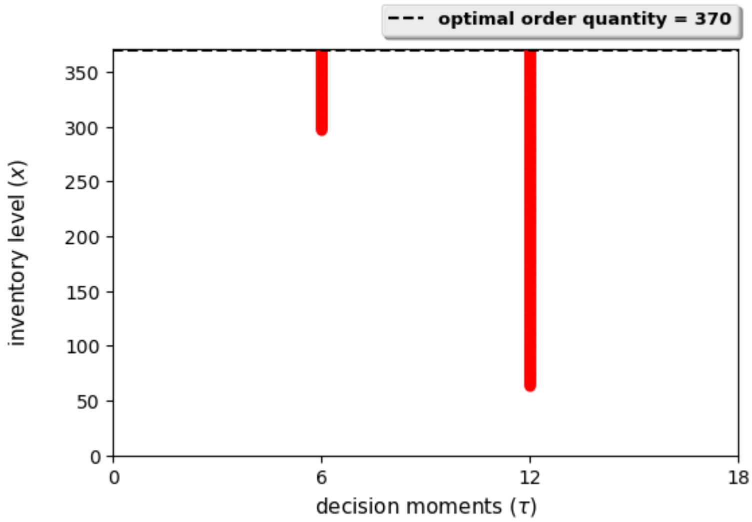


Figure 4.2: Stopping set for the base scenario.

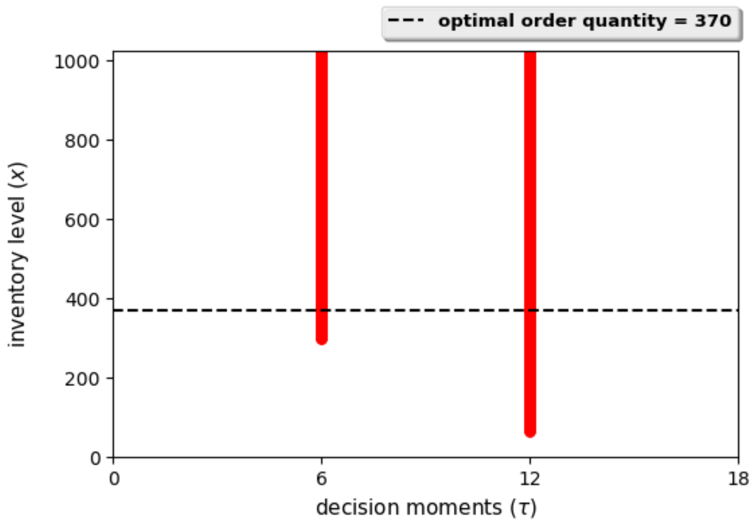


Figure 4.3: Stopping set when 1025 items are ordered.

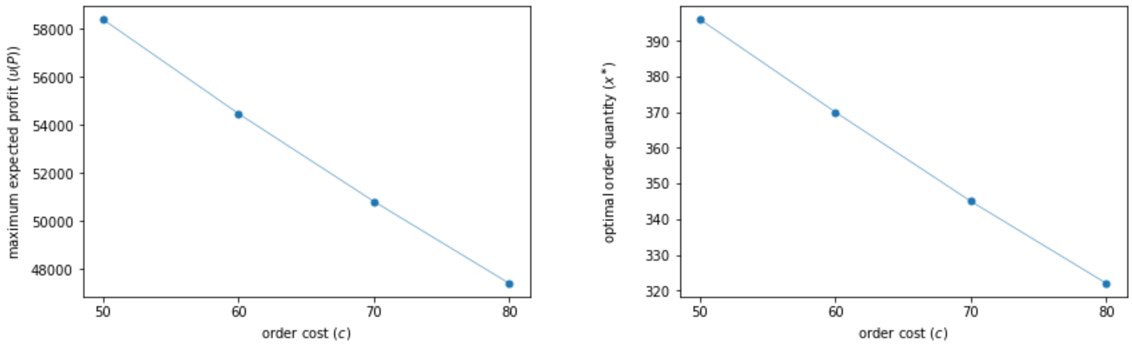
4.2 Sensitivity Analysis on the Different Parameters

In this section, we apply some sensitivity analysis on the problem parameters used in the base scenario and list the results in Table 4.8 for both the static and dynamic models. In this table, for the different scenarios we show the optimal objective values for both the static and dynamic model. We also list again for the different scenarios the optimal order quantity x^* for both models, the optimal initial price p_0^* for the dynamic model and the optimal fixed price p^* and the expected sales $\Lambda(T, p^*)$ during the season of 18 weeks for the static model. Finally, in the last column we compute the percentage difference in the optimal objective values between the static and dynamic model.

Each of the 30 different scenario is obtained by changing the value of only one parameter within the base scenario, while the other parameters remained the same. The results for the base scenario can be seen in this table in the starred rows. Observe the parameters ϵ and τ are selected in such a way that the prices and decision moments chosen in the base scenario will be a subset of all the other scenarios. As an example, we mention that in the base scenario the price discretization parameter $\epsilon = 10$. This means given $p_{max} = 350$ and $c = 60$ that the different prices are selected from the set $\{60, 70, 80, ..., 350\}$. In another scenario with $\epsilon = 5$ and the other parameters as in the base scenario, the different prices are selected from the set $\{60, 65, 70, 75, ..., 350\}$. The same discretization approach is also applied to the decision moments and the different sets of decision moments are listed in Table 4.8.

For all the scenarios represented in Table 4.8, it is obvious that the optimal objective value of the dynamic model is larger than the optimal objective value of the static model. Since under the same scenario price adjustments or exiting the market are allowed in the dynamic model, this result is to be expected.

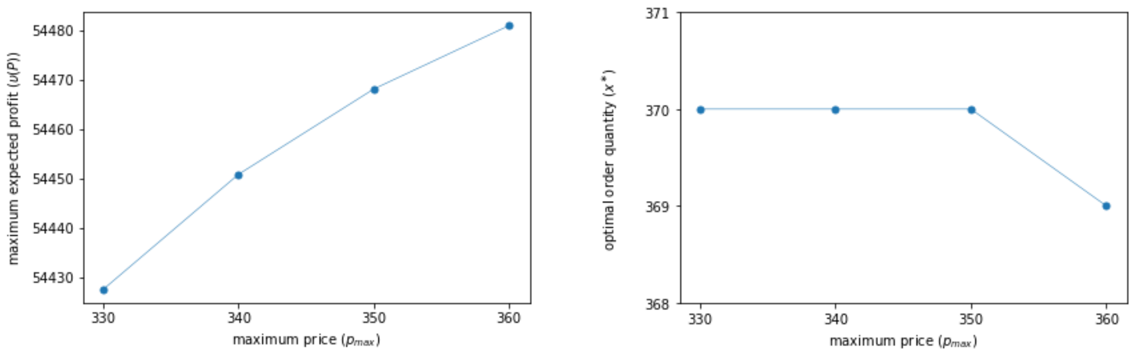
In Table 4.8, we observe that increasing the order cost yields decrease in the objective values and the optimal order quantities for the both models. Before giving an explanation for this result, we first mention that for a given price discretization parameter, the set of possible prices does not change when order cost value changes. The above result is to be expected since increasing the order cost per item leads to order less items which increases the optimal initial price. As a consequence, the expected demand drops and the profit decreases. According to our assumptions, order cost value should be always bigger than the salvage value; therefore, it is not possible to set c smaller than 50. Figure 4.4 shows the sensitivity results of the optimal objective value and the order quantity for the dynamic model due to changing order cost values.



(a) **(b)**

Figure 4.4: Sensitivity analysis plots for order cost.

Different from the order cost, if we increase/decrease the maximum price values, we can select from a bigger/smaller set of possible prices and hence the optimal objective value of the dynamic model increases/decreases. For example if we set $p_{\max} = 330$ and $\epsilon = 10$, the prices are selected from the set $\{60, 70, 80, ..., 320, 330\}$ and so in this case it is possible to raise the price until 330. This reduces the flexibility of the dynamic model and since optimal order quantity does not change with lower $p_{\max}$ values, the maximum expected profit decreases. Figure 4.5 shows the sensitivity of both the optimal order quantity and the optimal expected profit for the dynamic model due to changing maximum prices. Also in the static model we do not see these effects. This is related to the fact that the optimal price is not equal to the maximum price (hence the maximum price is not a restriction) and so increasing the maximum price will not change the optimal price and expected profit.



(a) **(b)**

Figure 4.5: Sensitivity analysis plots for maximum price.

Compared to c and $p_{\max}$, we observe that the optimal profit is more sensitive to changing holding cost. As h increases from 0 to 5, the optimal objective value decreases rapidly in both models. Additionally, we also observe two obvious facts in case the inventory holding cost is zero. Firstly, if we have no inventory costs, we will never exit the market in the dynamic model since keeping inventory until the end of the season does not cost anything and we have a longer time to sell the products. Secondly, the optimal order quantity is more than the expected sales in the static model since keeping more inventory does not cost anything and we might still sell all of the products. However, for large inventory holding costs per item per unit of time, the optimal order quantity decreases to avoid high inventory costs in both models. By the same reasoning, the optimal initial price and the optimal fixed price increase due to sufficient demand which will cover partially for the high inventory cost of leftover items. However, these higher prices will not compensate for all of this and so the profit decreases. Figure 4.6 shows the maximum expected profit and the optimal order quantity for the dynamic model in case the holding cost is increasing by 0.5 from 0 to 35 for two different cases. In panel a and panel b, the price discretization parameter ϵ equals 10. Due to the linear relation of the holding costs in the revenue function, the convex decreasing behavior of the maximum expected profit is expected in the parameter h in panel a. We also notice the jumps in the optimal order quantity as h increases in panel b. For instance, when $h = 14.5$ the optimal order quantity is 512; but when $h = 15$ the optimal order quantity drops to 480. Detailed information can be found in Table 4.6. These big differences are caused by the restrictions imposed on the possible prices. A small change in the inventory cost will result in a big difference in the selected price due to the size of the price discretization and this causes a big change in the optimal order size. Observe that for $\epsilon = 10$ the prices are selected from the set $\{60, 70, 80, ..., 350\}$, while for $\epsilon = 1.25$ the prices are selected from the set $\{60, 61.25, 62.5, ..., 350\}$. To show this behavior we also plotted the same increase in the holding costs for $\epsilon = 1.25$. In this case the big jumps in the optimal order sizes disappear. Results for the same h values when $\epsilon = 1.25$ is given in Table 4.7. For this case, effect of the holding cost increment on optimal order quantity can be seen more clearly in panels c and d of Figure 4.6. Jumps still occur, but since the sizes of the jumps in the optimal selected price are smaller due to the smaller price discretization, we have smaller jumps in the optimal order quantities.

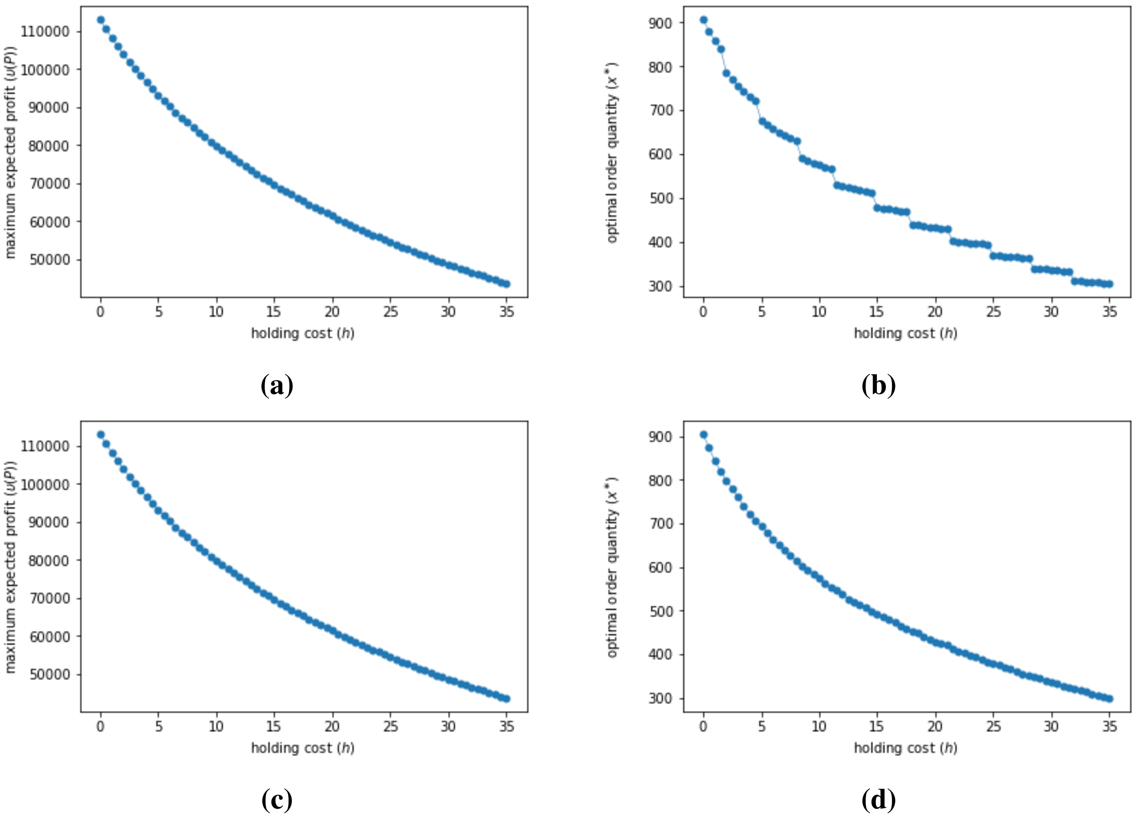


Figure 4.6: Sensitivity analysis plots for holding cost.

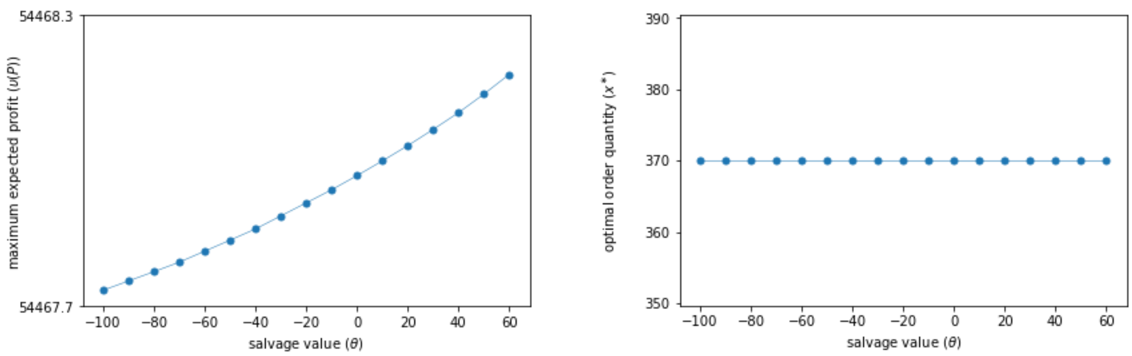
h	$\upsilon(P)$	x^*	p_0^*	$\upsilon(S)$	$\Lambda(T,p^*)$	x^*	p^*	PR
14.5	70478.28	512	250	69389.9	534.28	509	250	1.57
15	69567.92	480	260	68473.94	534.28	507	250	1.6

Table 4.6: A jump example for holding cost when epsilon is 10 (base scenario).

h	$\upsilon(P)$	x^*	p_0^*	$\upsilon(S)$	$\Lambda(T,p^*)$	x^*	p^*	PR
14.5	70519.93	497	255	69411.95	519.63	495	253.75	1.6
15	69603.65	491	256.25	68517.74	514.84	490	255	1.58

Table 4.7: Same holding cost values when epsilon is 1.25. No jump occurs.

Salvage value per item can be either negative or positive. If the salvage value is negative, it will reduce expected profit since disposing every remaining item from stock at the end of the season is a cost. It is still a cost compared to the order cost in case of a positive salvage value but the loss is less. Looking at the selected salvage values in Table 4.8, we observe that salvage value has almost no effect on the optimal order quantity and optimal price in both models. Also, the expected sales in the static model are not influenced by the salvage value variation because it depends on the optimal fixed price (Lemma 12). In both models, the optimal order sizes and prices are selected in such a way that is very likely that at the end of the season no item is left over and so we incur no loss due to salvage. Figure 4.7 shows the effect of salvage value on maximum expected profit and optimal order quantity in the dynamic model. We increased the salvage value by 10 in the range -100 and 60. According to our assumptions, order cost value should be always bigger than the salvage value; therefore, it is not possible to set θ bigger than 60.

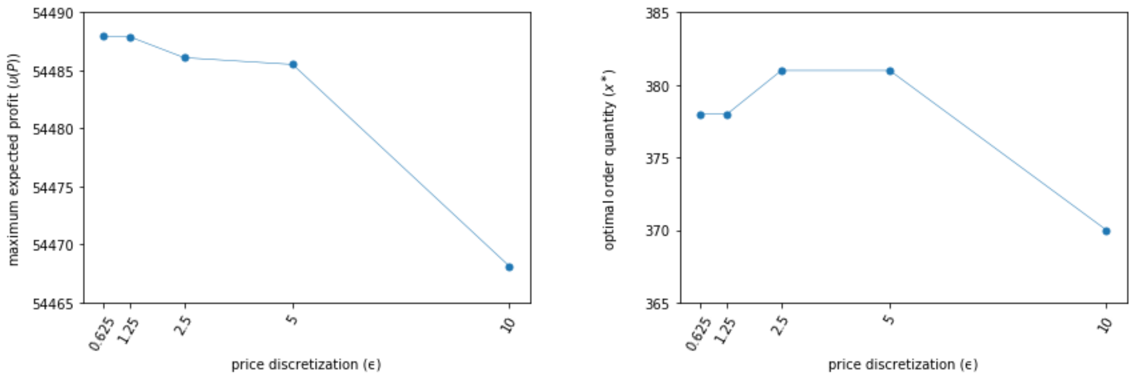


(a) **(b)**

Figure 4.7: Sensitivity analysis plots for salvage value.

The parameter ϵ denotes the price discretization between c and $p_{\max}$. This means that smaller epsilon values will lead to a larger set $\mathcal{P}$ of different prices. We observe that changing ϵ from 10 to 5 creates the biggest alteration in both models compared to other scenarios for different ϵ values. Lowering ϵ to 5 yields more flexibility in selecting optimal prices in both models, hence the optimal expected profits increase. However, lowering ϵ more than 5 leads almost no change in $v(S)$ and a small increment in $v(P)$. This is caused by the fact that the prices which can be selected in the set $\mathcal{P}$ are much closer to each other when ϵ is very low. It is interesting to see that when ϵ decreases from 10 to 5, optimal order quantities of both models increase and optimal prices decrease; however, when ϵ

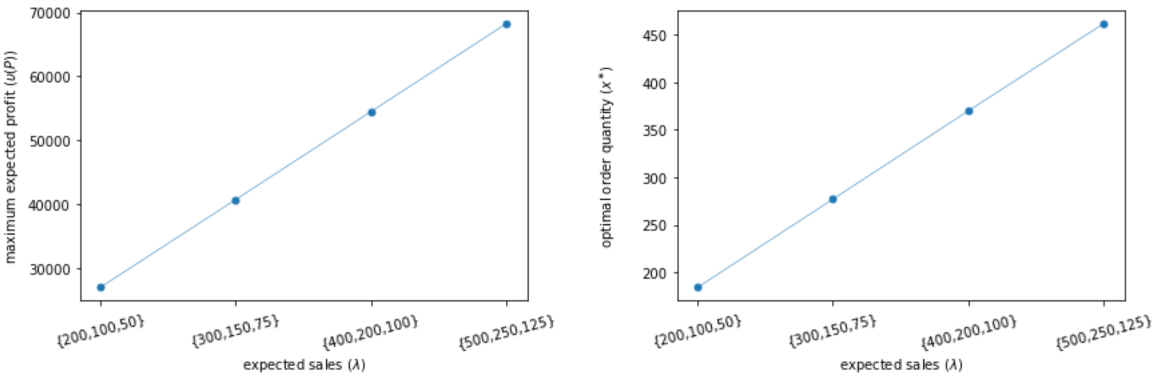
decreases to 1.25 from 5, optimal order quantities of both models decrease and optimal prices increase. This may be the reason for this situation; when epsilon is 1.25 or 0.625, optimal prices of both models are selected as 286.25 which is the price that can not be selected when epsilon is 2.5 or more. And costs are reduced by ordering less products but at the same time it is possible to get the same or more profit in both models. Figure 4.8 visualizes the results of sensitivity analysis for the dynamic model based on ϵ values in Table 4.8.



(a) **(b)**

Figure 4.8: Sensitivity analysis plots for price discretization parameter.

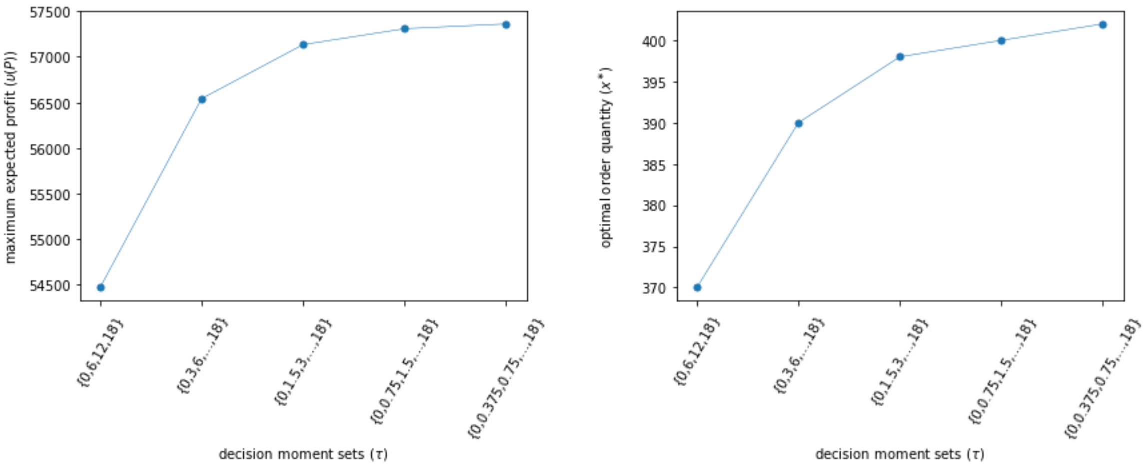
The expected number of buying customers also have a great effect on the optimal expected profit for both models. In Table 4.8, it is possible to see that when we double the λ values in each period and get the set $\{400, 200, 100\}$ from $\{200, 100, 50\}$, the optimal expected profits and the optimal order quantities for both models become almost twice as those in the base scenario. Only optimal initial price does not change; but optimal fixed price increases. Moreover, we observe that when λ values increase too much, the difference between the dynamic and the static model is diminishing. For example when λ equals to 500 in the first period and optimal order quantity is 462, more or less all products can be sold before entering the second period which will reduce the value of the flexibility of the dynamic model to reset the prices. Also we observe that the optimal order quantity of the static model is more than the optimal order quantity in the dynamic model for the last two scenarios. A possible explanation might be that in the static model we do not leave the market during the season. In combination with a large number of expected sales in the third period we might still sell a lot of products even if the popularity decreases. The behavior of the maximum expected profit and the optimal order size for the dynamic model as the λ set changes can be found in Figure 4.9. Also a linear increase is observed in Figure 4.9 since λ sets increase linearly.



(a) **(b)**

Figure 4.9: Sensitivity analysis plots for expected number of buying customers.

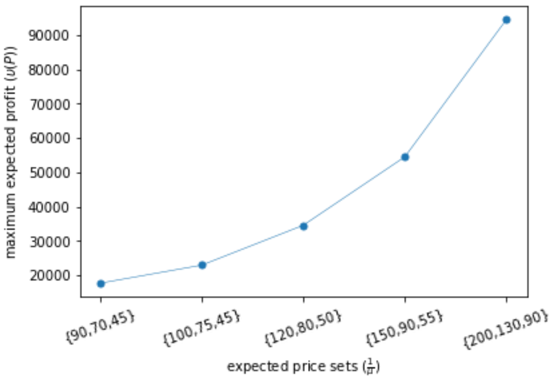
The static model is not influenced by changing the number of decision moments since no decision can be made during the season. Hence we will only investigate the effects of enlarging the set of decision moments for the dynamic model. To nest the decision moments, we double the number of decision moments by including in each new scenario the middle of two decision moments of the previous scenario. We start with base scenario in which we take every 6 weeks a decision about leaving the market or resetting the prices. We observe that as the total number of decision moments increase, the optimal expected profit also increases, but the size of the increase decreases. For example, the increase of $v(P)$ is more when changing decision moments from $\{0, 6, 12, 18\}$ to $\{0, 3, 6, ..., 18\}$ than changing decision moments from $\{0, 0.75.1.5, ..., 18\}$ to $\{0, 0.375, 0.75, ..., 18\}$. This is because in the base scenario, a decision can be made every 6 weeks which corresponds to the time points at which the demand changes; however, making decision at every 3 weeks includes different moments than the demand changing time points which allows more price adjustments or quitting market options. Nevertheless, making decision at every $\frac{3}{8}$ weeks instead of every $\frac{3}{4}$ weeks does not cause a major difference since the moments of decisions are very close. Also, we observe that the optimal initial price value decreases and optimal order quantity increases with the increase in the number of decision moments since the supplier will have more flexibility to adjust the price. The results of increasing the number of decision moments for the dynamic model can be seen more clearly in Figure 4.10.



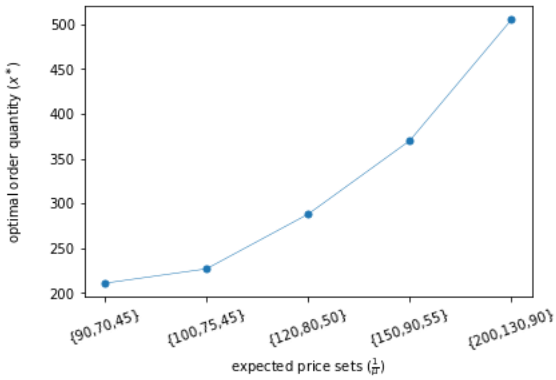
(a) **(b)**

Figure 4.10: Sensitivity analysis plots for decision moments.

The highest optimal objective values for both models in the sensitivity analysis results can be reached with different $\frac{1}{\mu}$ values. Likewise for high values of the expected number of buying customers, we observe similar optimal order quantities and optimal prices for both models when expected price a customer is willing to pay for each period is taken as $\{200, 130, 90\}$. Also with this set of values, we obtain the lowest relative difference (PR) in the sensitivity analysis results. This can be explained as follows; since in the first period there are enough customers who are willing to pay a high price, more or less all the items can be sold in the first period. Hence the flexibility of the dynamic model in the remaining periods is of little use. Also for the same scenario, optimal order quantities and optimal objective values for both models show significant increments, optimal initial price and optimal fixed price become 340 which is the second highest price in the price set $\mathcal{P}$. However, a small increase in expected prices still causes a big difference. For instance, taking $\frac{1}{\mu}$ as $\{100, 75, 45\}$ instead of $\{90, 70, 45\}$ causes $\upsilon(P)$ to increase as 5250.69 and $\upsilon(S)$ as 5169.8. Therefore, it would be fair to say that the expected prices that customers are willing to pay have a great contribution on maximum expected profits of both models. The importance of expected prices can be seen in Figure 4.11 by looking at the increasing behavior of maximum expected profit and optimal order quantity for the dynamic model.



(a)



(b)

Figure 4.11: Sensitivity analysis plots for expected reservation price.

c	$v(P)$	x^*	p_0^*	$v(S)$	$\Lambda(T,p^*)$	x^*	p^*	PR	
50	58385.15	396	280	57711.71	428.29	393	280	1.17	
60	54468.14	370	290	53833.86	398.11	365	290	1.18	*
70	50813.64	345	300	50207.48	370.18	339	300	1.21	
80	47403.27	322	310	46832.57	370.18	337	300	1.22	
$p_{\max}$									
330	54427.59	370	290	53833.86	398.11	365	290	1.1	
340	54450.87	370	290	53833.86	398.11	365	290	1.15	
350	54468.14	370	290	53833.86	398.11	365	290	1.18	*
360	54480.97	369	290	53833.86	398.11	365	290	1.2	
h									
0	112958.33	906	210	108580.78	840.53	883	190	4.03	
5	93100.62	676	230	91018.38	668.78	668	220	2.29	
10	79753.22	575	240	78375.73	575.57	560	240	1.76	
15	69567.92	480	260	68473.94	534.28	507	250	1.6	
25	54468.14	370	290	53833.86	398.11	365	290	1.18	*
35	43659.53	306	310	43288.21	344.3	304	310	0.86	
θ									
-90	54467.75	370	290	53783.39	398.11	364	290	1.27	
-30	54467.89	370	290	53805.83	398.11	364	290	1.23	
20	54468.03	370	290	53821.15	398.11	365	290	1.2	
50	54468.14	370	290	53833.86	398.11	365	290	1.18	*
60	54468.18	370	290	53838.1	398.11	365	290	1.17	
ϵ									
10	54468.14	370	290	53833.86	398.11	365	290	1.18	*
5	54485.52	381	285	53857.41	412.9	377	285	1.17	
2.5	54486.1	381	285	53857.41	412.9	377	285	1.17	
1.25	54487.89	378	286.25	53857.74	409.15	374	286.25	1.17	
0.625	54487.93	378	286.25	53857.74	409.15	374	286.25	1.17	
λ									
500,250,125	68270.65	462	290	67564.87	497.64	458	290	1.04	
400,200,100	54468.14	370	290	53833.86	398.11	365	290	1.18	*
300,150,75	40681.83	277	290	40117.56	321.21	291	280	1.41	
200,100,50	26921.54	184	290	26441.63	214.14	191	280	1.81	
τ									
0,6,12,18	54468.14	370	290	53833.86	398.11	365	290	1.18	*
0,3,6,...,18	56541	390	250	53833.86	398.11	365	290	5.03	
0,1.5,3,...,18	57133.98	398	230	53833.86	398.11	365	290	6.13	
0,0.75,1.5,...,18	57308.6	400	220	53833.86	398.11	365	290	6.45	
0,0.375,0.75,...,18	57361.6	402	210	53833.86	398.11	365	290	6.55	
$1/\mu$									
200,130,90	94427.82	505	340	93730.34	539.93	501	340	0.74	
150,90,55	54468.14	370	290	53833.86	398.11	365	290	1.18	*
120,80,50	34548.89	288	260	34084.92	324.78	286	260	1.36	
100,75,45	22938.98	227	240	22522.72	269.53	224	240	1.85	
90,70,45	17688.29	211	220	17352.92	264.57	206	220	1.93	

Table 4.8: Sensitivity analysis results. Starred rows are for the base scenario.

Chapter 5

Conclusion and Future Research

In this thesis, we deal with the problem of selling a seasonal product in a retail store over a finite horizon. A particular example of our model is the sales of fashion clothes during a season. Taking pricing decisions over time in order to increase sales and maximize profit is an important issue in revenue management. However, most of the literature with random sales process assume that the cost components are independent of the used pricing policy which is not practical. Since inventory holding costs are not included in those papers, exiting the market is not an option due to no additional costs of staying in the market. Therefore, we aim to develop the existing models by considering the inventory holding cost per item per unit of time and the possibility to exit the market before the end of season.

We model the general sales process by considering continuous and discrete time demand models. In the continuous time demand model, we assume that potential customers arrive according to a non homogeneous Poisson process over time. We assume that accumulated sales in discrete time demand model are independent which enables us to solve it by stochastic dynamic programming. The main model that we intend to improve is the continuous time demand model and customers are assumed to be myopic. During the season, potential customers use a so-called reservation price to decide whether to buy the product or not. This reservation price is a random variable with an arbitrary cumulative distribution function. Under these conditions, the supplier has to make a decision at predetermined moments in time during the sales season. At these moments, the supplier should either decide to stay in the market and select the optimal price from a given price set, or exit the market and sell the remaining inventory at a certain salvage value. During the season, inventory replenishment or backlogging is not allowed. The initial order quantity is also a decision variable in our model; therefore, we consider ordering cost per item at the beginning of the season. The supplier will certainly exit the market at the end of the season and sell the possibly remaining products at the same salvage value.

In the first chapter, we make a review on the most important dynamic pricing models which appeared in the literature and continue with the more detailed introduction of our considered model. In the second chapter, we formulate the mathematical model for both continuous and discrete time demand models with dynamic programming techniques in order to determine the optimal dynamic pricing policy. We study the static version of the continuous time demand model in Chapter 3. In static model, we determine the optimal order quantity and the demanded price at the beginning of the season, and exiting the market is not allowed until the end. This model is solved by nonlinear programming techniques. In Chapter 4, we conduct an extensive numerical study for the continuous time demand model in order to evaluate the optimal pricing policy for a given base scenario. For computational convenience, we assume that the reservation price distribution is exponential with one parameter. We provide the optimal policy table for the base scenario and give an example how to apply the optimal pricing policy. Also we show the optimal stopping regions with figures for the optimal order quantity obtained by the base scenario and for the case of ordering more than the optimal order quantity. Moreover, we provide an example of no stopping model where price can be updated but exiting the market is not allowed at predetermined decision moments. Finally, we perform a sensitivity analysis on the optimal policy by changing the values of the various problem parameters and compare the results of the dynamic and static models.

To conclude, in this thesis we add inventory holding cost and the action to exit the market to the dynamic pricing models with stochastic demand discussed in the literature. By this way, we extend the paper of Frenk et al. (2017). With numerical results, we verify that maximum expected profit given by the dynamic model is always above or equal to the maximum expected profit given by the static model. We also show that if staying in the market is costly, exiting the market option brings along a higher expected profit. Since in our model we assume that the parameters are given, directions for future research include the statistical estimation of the used functions and parameters based on a given data set. Another potential future research would be to explore the possibility of re-ordering at predetermined decision moments. If high demand is observed, this may be a viable option. In this model, a relatively short lead time can be included to make the problem more realistic.

Bibliography

[1] Y. Aviv and A. Pazgal. A partially observed markov decision process for dynamic pricing. *Management Science*, 51(9):1400–1416, 2005.

[2] G. Bitran, R. Caldentey, and S. Mondschein. Coordinating clearance markdown sales of seasonal products in retail chains. *Operations Research*, 46(5):609–624, 1998.

[3] G. R. Bitran and S. Mondschein. Periodic pricing of seasonal products in retailing. *Management Science*, 43(1):64–79, 1997.

[4] P. Bremaud. *Point Processes and Queues: Martingale Dynamics*. Springer-Verlag, New York, 1981.

[5] E. Çınlar. *Probability and Stochastics*. Springer-Verlag, New York, 2011.

[6] R. Chatwin. Optimal dynamic pricing of perishable products with stochastic demand and a finite set of prices. *European Journal of Operational Research*, 125(1):149–174, 2000.

[7] N. Chen and G. Gallego. Welfare analysis of dynamic pricing. *Management Science*, 2018. doi: 10.1287/mnsc.2017.2943.

[8] X. Chen, P. Hu, and Z. Hu. Efficient algorithms for the dynamic pricing problem with reference price effect. *Management Science*, 63(12):4389–4408, 2017.

[9] A. Şen. A comparison of fixed and dynamic pricing policies in revenue management. *Omega*, 41(3):586–597, 2013.

[10] P. Du and Q. Chen. Skimming or penetration: Optimal pricing of new fashion products in the presence of strategic consumers. *Annals of Operations Research*, 257(1):275–295, 2017.

[11] W. Elmaghraby and P. Keskinocak. Dynamic pricing in the presence of inventory considerations: Research overview, current practices, and future directions. *Management Science*, 49(10):1287–1309, 2003.

[12] Y. Feng and G. Gallego. Optimal starting times for end-of-season sales and optimal stopping times for promotional fares. *Management Science*, 41(8):1371–1391, 1995.

[13] Y. Feng and G. Gallego. Perishable asset revenue management with markovian time dependent demand intensities. *Management Science*, 46(7):941–956, 2000.

[14] Y. Feng and B. Xiao. Optimal policies of yield management with multiple predetermined prices. *Operations Research*, 48(2):332–343, 2000.

[15] J. B. G. Frenk, S. O. Sezer, and C. Pehlivan. Order and exit decisions under non-increasing price curves for products with short life cycles. *Submitted to MMOR*, 2017.

[16] J. B. G. Frenk, S. Javadi, and S. O. Sezer. An optimal stopping approach for the end-of-life inventory problem. *Submitted to MMOR*, 2018.

[17] J. B. G. Frenk, S. Javadi, S. O. Sezer, and M. Pourakbar. An exact static solution approach for the service parts end-of-life inventory problem. *European Journal of Operational Research*, 2018. doi: 10.1016/j.ejor.2018.06.041.

[18] J. Huang, M. Leng, and M. Parlar. Demand functions in decision modeling: A comprehensive survey and research directions. *Decision Sciences*, 44(3):15–24, 2013.

[19] U. K. Khedlekar and D. Shukla. Dynamic pricing model with logarithmic demand. *OPSEARCH*, 50(1):1–13, 2013.

[20] K. Y. Lin. A sequential dynamic pricing model and its applications. *Naval Research Logistics*, 51:501–521, 2004.

[21] W. Liu, W. Tang, L. Feng, and J. Zhang. Dynamic pricing under temperature control for perishable foods. *Journal of Systems Science and Systems Engineering*, 23(3): 252–265, 2014.

[22] S. Netessine. Dynamic pricing of inventory/capacity with infrequent price changes. *European Journal of Operational Research*, 174(1):553–580, 2006.

[23] R. Phillips, A. S. Şimşek, and G. Van Ryzin. The effectiveness of field price discretion: Empirical evidence from auto lending. *Management Science*, 61(8): 1741–1759, 2015.

[24] D. F. Pyke, J. M. Hall, and M. Fleischmann. A dynamic pricing model for coordinated sales and operations. *ERIM Report Series Research in Management*, 2005.

[25] S. M. Ross. *Stochastic Processes*. Wiley Series in Probability and Statistics. Wiley, New York, 1983.

[26] S. M. Ross. *Introduction to Probability Models*. Academic Press, 6 edition, 1997.

[27] D. Simchi-Levi, D. S. Wu, and Z. Shen. *Handbook of Quantitative Supply Chain Analysis: Modeling in the E-Business Era*. International Series in Operations Research & Management Science. Springer-Verlag, Berlin, 2004.

[28] D. Simchi-Levi, X. Chen, and J. Bramel. *The Logic of Logistics (second edition)*. Springer Series in Operations Research. Springer-Verlag, New York, 2005.

[29] K. T. Talluri and G. Van Ryzin. *The Theory And Practice Of Revenue Management*. International Series in Operations Research & Management Science. Springer, New York, 2004.

[30] O. Tang, S. N. Musa, and J. Li. Dynamic pricing in the news-vendor problem with yield risks. *International Journal of Production Economics*, 139(1):127–134, 2012.

[31] G. Van Ryzin and G. Gallego. Optimal dynamic pricing of inventories with stochastic demand over finite horizons. *Management Science*, 40(8):999–1020, 1994.

[32] D. Zhang and L. Weatherford. Dynamic pricing for network revenue management: A new approach and application in the hotel industry. *INFORMS Journal on Computing*, 29(1):18–35, 2017.

[33] W. Zhao and Y. Zheng. Optimal dynamic pricing for perishable assets with nonhomogeneous demand. *Management Science*, 46(3):375–388, 2000.