Strategy-Proof Size Improvement: Is it Possible?*

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Abstract

In unit-demand and multi-copy object allocation problems, we say that a mechanism size-wise dominates another mechanism if the latter never allocates more objects than the former does, while the converse is true for some problem. Our main result shows that no individually rational and strategy-proof mechanism size-wise dominates a non-wasteful, truncation-invariant, and extension-responding mechanism. As a corollary of this, the well-known deferred-acceptance, serial dictatorship, and Boston mechanisms are not size-wise dominated by an individually rational and strategy-proof mechanism. We also show that whenever the number of agents does not exceed the total number of object copies, no group strategy-proof and efficient mechanism, such as top trading cycles mechanism, is size-wise dominated by an individually rational, weakly population-monotonic, and strategy-proof mechanism.

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1 Introduction

Strategy-proofness, which requires right incentives for agents to reveal their true preferences, is a major desideratum in the design of object allocation mechanisms. In terms of positive economics, a lack of strategy-proofness undermines the allocative performance of a mechanism. In normative ground, on the other hand, non strategy-proof mechanisms may favor strategic agents at the expense of sincere ones (see Pathak and Sönmez (2008)); hence fairness would be a concern. Market practitioners pay attention to strategy-proofness, and indeed, in some matching markets, the previously used manipulable mechanisms have been replaced with strategy-proof ones. Some of such examples are Boston, Chicago, and Wake County school-choice programs (see Abdulkadiroğlu et al. (2005b), Pathak and Sönmez (2013), Dur et al. (2017)).

Individual rationality is another fundamental property. Agents are often free to opt out of an object assignment and go with their outside option. Individual rationality guarantees agents to be at least as better off as they would be with their outside option.[1] In other words, it rationalizes agents participation in an object assignment.

In the design of object allocation mechanisms, another natural desideratum may be *maximizing the number of assigned agents*. This objective becomes most apparent in organ exchanges as it means maximizing the number of transplants (see Roth et al. (2005) and Ergin et al. (2017)). It also significantly matters in a matching of asylum seekers to countries (see Andersson and Ehlers (2016)) as well as dorm assignments in colleges. Maximality is important for school choice programs as well. As described in Abdulkadiroğlu et al. (2005a), one of the main problems of the old New York City high school match system was that a large proportion of students were remaining unassigned under the normal process. These students then, through an administrative process, were placed at schools, which were not necessarily stated in their preferences. Maximality is indeed the primary goal in some school-choice programs, as indicated by the following quote from the Frankfurt and North Rhine-West

[1] More formally, a matching is individually rational if agents never would rather receive no object.

secondary school districts in Germany:

> *"The organization of the "Frankfurt School Mechanism" is shared between State, city and school. Its primary goal is to give as many applicants as possible one of their preferred schools. Each school decides for itself which students to admit..."* (Basteck et al., 2015).

In this paper, we focus on these three desiderata and study whether we can improve a given benchmark mechanism in terms of the number of assigned agents through an individually rational and strategy-proof mechanism. More formally, we say that a mechanism *size-wise dominates* another mechanism if the latter never assigns more agents than the former does, while the converse holds at some problem.

Our main result shows that for any given benchmark mechanism that satisfies certain arguably mild properties,[2] there is no individually rational and strategy-proof mechanism that size-wise dominates the former. As the well-known deferred-acceptance (DA), serial dictatorship (SD), and Boston (BM) mechanisms satisfy all these properties, neither of them is size-wise dominated by an individually rational and strategy-proof mechanism. This, along with the strategy-proofness of SD and DA, spells out that if a market designer's three goals are individual rationality, strategy-proofness, and maximizing the number of assigned agents, then SD and DA are unbeatable. We also show that the impossibility result is tight in the sense that it does not hold without either individual rationality or strategy-proofness.

Above result does not say anything about the other well-known top trading cycles (TTC) mechanism as TTC does not satisfy all the benchmark axioms. However, we obtain that whenever the number of agents does not exceed the total number of object copies, no group strategy-proof and efficient mechanism,[3] such as TTC, is size-wise dominated by an individually rational and strategy-proof mechanism that also satisfies a weak solidarity condition.

[2]These properties are non-wastefulness, extension-responsiveness, and truncation invariance. They are all defined in Section 2.

[3]A mechanism is group strategy-proof if no group of agents benefit from misreporting their preferences. A mechanism is efficient if its outcome is never unanimously less preferred to another matching.

To our knowledge, this is the first paper that provides a general axiomatic analysis for a mechanism comparison in terms of the number of assigned agents. In a recent study, Afacan et al. (2017) demonstrate that there is no size-wise domination relation between any pair of TTC, SD, BM, and stable mechanisms, which is partly implied by our main result. Kesten and Kurino (2016) show that no strategy-proof mechanism Pareto dominates DA.[4,5] This result is then generalized by Alva and Manjunath (2017) and Hirata and Kasuya (2017). The former shows that there exists at most one strategy-proof mechanism that improves the agents' welfare relative to any individually rational and participation-maximal mechanism. In a more general matching with contracts setting, Hirata and Kasuya (2017) obtain that no individually rational and strategy-proof mechanism Pareto dominates a stable and strategy-proof rule.

2 Model

Let I and O be the finite sets of agents and objects, respectively. Each agent has a strict preference relation over the objects and being unassigned option, which is denoted by $\emptyset$. Let $P = (P_i)_{i \in I}$ be the preference profile of the agents where P_i is the preference relation of agent i. Let R_i be the weak preference relation associated to P_i such that for any $c, c' \in O$, cR_ic' if and only if cP_ic' or $c = c'$. An object c is **acceptable** to agent i if $cP_i\emptyset$, and **unacceptable** otherwise. For any $I' \subset I$, let $P_{I'} = (P_i)_{i \in I'}$ and $P_{-I'} = (P_i)_{i \in I \setminus I'}$. Each object $c \in O$ can come with multiple copies, and q_c is the number of object c copy. Let $q = (q_c)_{c \in O}$.

An object allocation problem is a 4-tuple (I, O, P, q). In the rest of the paper, we fix I, O, q, and denote the problem with P. A **matching** $\mu : I \to O \cup \{\emptyset\}$ is a function such that for each object $c \in O$, $|\mu^{-1}(c)| \leq q_c$. For any $k \in I \cup O$, we write μ_k to denote the assignment of k. Let $|\mu| = |\{i \in I : \mu_i \neq \emptyset\}|$, that is, the number of agents assigned to an object under

[4] A mechanism Pareto dominates another one if the latter's outcome is never strictly preferred by some agent to the former's, while the converse holds at some problem.

[5] This result generalizes both results of Kesten (2010), which shows that no efficient and strategy-proof mechanism Pareto dominates DA, and of Abdulkadiroğlu et al. (2009), which, under the presence of an outside option, obtains the same conclusion with Kesten and Kurino (2016)'s result.

μ. We call it "*size*" of matching. A matching μ is **individually rational** if for any agent $i \in I$, $\mu_i R_i \emptyset$. A matching μ is **non-wasteful** if there does not exist an agent-object pair (i, c) such that $c P_i \mu_i$ and $|\mu_c| < q_c$. A matching μ is **efficient** if there is no matching μ' such that for any agent $i \in I$, $\mu'_i R_i \mu_i$, with this holding strictly for some agent.

A **mechanism** ψ is a procedure that selects a matching in any problem P. The matching selected by mechanism ψ in problem P is denoted by $\psi(P)$. Mechanism ψ is individually rational (efficient) <non-wasteful> if, for any problem P, $\psi(P)$ is individually rational (efficient) <non-wasteful>. Mechanism ψ is **strategy-proof** if there exist no problem P, agent i, and P'_i such that $\psi_i(P'_i, P_{-i}) P_i \psi_i(P)$. Mechanism ψ is **group strategy-proof** if there exist no problem P, group of agents $I' \subseteq I$, and false preference profile $P'_{I'}$ such that for each agent $i \in I'$, $\psi_i(P'_{I'}, P_{-I'}) R_i \psi_i(P)$, with this holding strictly for some agent $j \in I'$. Here is our size comparison criterion. A mechanism ψ **size-wise dominates** mechanism ϕ if, for every problem P, $|\psi(P)| \geq |\phi(P)|$, and $|\psi(P')| > |\phi(P')|$ for some problem P'.

We next introduce the properties for a benchmark mechanism. A preference relation P'_i is the **truncation** of P_i from object c if, for any pair of objects c', c'', $c' P_i c''$ if and only if $c' P'_i c''$, and for any c' such that $c P_i c'$, $\emptyset P'_i c'$. Mechanism ψ is **truncation-invariant** if, for any problem P and agent i such that $\psi_i(P) \neq \emptyset$, $\psi(P) = \psi(P'_i, P_{-i})$ where P'_i is the truncation of P_i from $\psi_i(P)$. Truncation-invariance is a well-studied property requiring the assignment to remain the same after an agent truncates his preferences below his assignment. DA is truncation-invariant (Ehlers and Klaus, 2016). It is immediate to see from their definitions that SD, TTC, and BM are all truncation-invariant as well.[6]

For an object c, a preference relation P'_i is the **object c extension of** P_i if, for any $c', c'' \in O \setminus \{c\}$, $c' P_i c''$ if and only if $c' P'_i c''$, $\emptyset P_i c$, and object c is the least preferred acceptable object under P'_i. Mechanism ψ is **extension-responding** if, for any problem P, agent i, and object c such that $\psi_i(P) = \emptyset$ and $|\psi_c(P)| < q_c$, $\psi_i(P'_i, P_{-i}) = c$ and $\psi_j(P'_i, P_{-i}) = \psi_j(P)$ for every $j \in I \setminus \{i\}$ where P'_i is the object c extension of P_i. To our knowledge, this is a

[6]The definitions of these mechanisms are provided in Appendix A.

new property. It basically requires an originally unassigned agent to receive a leftover object after he starts demanding it while the others remain unaffected. It is easy to verify that DA, BM, and SD are all extension-responding.

3 The Results

Now, we are ready to present our main result, which indicates that any non-wasteful, truncation-invariant, and extension-responding mechanism cannot be size-wise dominated by an individually rational and strategy-proof mechanism.

Theorem 1. *Let ψ be a non-wasteful, truncation-invariant, and extension-responding mechanism. Then, there is no individually rational and strategy-proof mechanism that size-wise dominates ψ.*

Proof. See Appendix B. □

More explicitly, Theorem 1 implies that for any non-wasteful, truncation-invariant, and extension-responding mechanism ψ, and an individually rational and strategy-proof mechanism ϕ, either $|\psi(P)| = |\phi(P)|$ for every P or $|\psi(P')| > |\phi(P')|$ for some P'.

As aforementioned, DA, SD, and BM are all truncation-invariant and extension-responding. They are all non-wasteful as well. Hence, as a corollary of Theorem 1, we obtain that neither of them is size-wise dominated by an individually rational and strategy-proof mechanism. Moreover, this coupled with the well-known rural hospital theorem (Roth, 1984) generalizes it to the set of all stable mechanisms.[7]

Corollary 1. *There is no individually rational and strategy-proof mechanism that size-wise dominates any of BM, SD, and a stable mechanism.*

It is well-known that TTC, SD, and DA are all individually rational and strategy-proof. Hence, we also have the following result, which is independently obtained by Afacan et al. (2017) as well.

[7]Rural hospital theorem says that the number of assigned agents is the same under any stable matching.

Corollary 2. *Neither of BM, SD, DA is size-wise dominated by any of TTC, SD, and DA.*

Theorem 1 is tight in the sense that the impossibility does not hold without either individual rationality or strategy-proofness. For instance, let us consider a mechanism such that in every problem, it assigns each object to the same group of agents in a way that either all the agents receive an object or no object is leftover. It is easy to see that this mechanism is strategy-proof and that size-wise dominates all DA, SD, and BM. Yet, it is not individually rational. On the other hand, consider a mechanism that always assigns as many agents as possible subject to individual rationality. Such a mechanism is individually rational and size-wise dominates all DA, SD, and BM. Yet, it is not strategy-proof.[8]

Remark 1. BM and stable mechanisms take an object priority profile over agents as an input of the problem. However, our setting and results are priority-free. In other words, all of our results hold for any priority ordering. Therefore, Corollary 1 holds for BM and any stable mechanism with any priority ordering. Likewise, SD uses an agent ordering. We do not specify any such ordering as well. Therefore, the result above holds for SD with any agent ordering. By the same token, Corollary 2 holds for any priority and agent ordering.[9]

Theorem 1 does not say anything about TTC as it is not extension-responding.[10] While we do not know whether TTC is size-wise dominated by an individually rational and strategy-proof mechanism, we obtain a similar result below by adding another axiom.

[8] To see this, let us consider $I = \{i, j\}$ and $O = \{a, b\}$, each with unit quota. The preferences are such that $P_i = P_j : \ a, b, \emptyset$. Let ψ be a mechanism that assigns as many agents as possible subject to individual rationality. In this problem, without loss of generality, suppose $\psi_i(P) = a$ and $\psi_j(P) = b$. Then, under ψ, agent j can obtain object a by reporting false preferences P'_j under which only object a is acceptable.

[9] More explicitly, Corollary 2 shows that none of BM and DA under any priority ordering and SD under any agent ordering is size-wise dominated by any of TTC and DA under any priority ordering, and SD under any agent ordering.

[10] In order to see this, let us consider $I = \{i, j, k\}$ and $O = \{a, b\}$, each with unit quota. The preference profile P is as follows: $P_i = P_j : \ a, \emptyset$, and $P_k : \emptyset$. Let us denote the objects' strict priority orders by $\succ = (\succ_c)_{c \in O}$. They are as follows: $\succ_a$: k, i, j and $\succ_b$: j, k, i. The TTC outcome at P is such that $TTC_i(P) = a$ and $TTC_j(P) = TTC_k(P) = \emptyset$. Let us now consider $P'_k : b, \emptyset$ and write $P' = (P'_k, P_{-k})$. The TTC outcome at P' is such that $TTC_i(P') = \emptyset$, $TTC_j(P') = a$, and $TTC_k(P') = b$. Hence, TTC is not extension-responding.

A mechanism ψ is **weakly population-monotonic** if, for any problem P and agent i such that $\psi_i(P) = \emptyset$, we have $\psi_j(P'_i, P_{-i}) R_j \psi_j(P)$ for each $j \in I \setminus \{i\}$ where P'_i is such that $\emptyset P'_i c$ for any $c \in O$. In words, after an originally unassigned agent stops demanding an object, weak population-monotonicity requires the other agents be at least as better off as before. This is a weaker version of the well-studied population monotonicity condition.[11] DA is population monotonic (Kojima and Manea, 2010), hence weakly population-monotonic. It is easy to verify that BM, SD, and TTC are all weakly population-monotonic.

Theorem 2. *Let $\Sigma_{c \in O} q_c \geq |I|$. Let ψ be an efficient and group strategy-proof mechanism. Then, there is no individually rational, weakly population-monotonic, and strategy-proof mechanism that size-wise dominates ψ.*[12]

Proof. See Appendix B. □

As TTC is efficient and group strategy-proof, we have the following corollary.[13]

Corollary 3. *Let $\Sigma_{c \in O} q_c \geq |I|$. Then, TTC is not size-wise dominated by an individually rational, weakly population-monotonic, and strategy-proof mechanism, such as SD and DA.*

When each object has only one copy, the class of trading cycles of Pycia and Ünver (2017), including the hierarchial exchange rules of Pápai (2000), are efficient and group strategy-proof. Hence, we also have the following result.

Corollary 4. *Let $q_c = 1$ for each object $c \in O$ and $|O| \geq |I|$. Then, no trading cycles of Pycia and Ünver (2017), hence, in particular, no hierarchial exchange rule of Pápai (2000), is*

[11] A mechanism is population monotonic if, after an agent stops demanding an object, no one else becomes worse off.

[12] By following the steps of the proof of Theorem 2, we can also straightforwardly show that no efficient and group strategy-proof mechanism is size-wise dominated by BM.

[13] Although Pápai (2000) obtains the group strategy-proofness of TTC in a unit-copy object allocation setting, her results easily imply it holding in the multi-copy case as follows. A mechanism ψ is non-bossy if, for any problem P, agent i, and P'_i, $\psi_i(P'_i, P_{-i}) = \psi_i(P)$, then $\psi(P'_i, P_{-i}) = \psi(P)$. Pápai (2000) shows that TTC is non-bossy. It is immediate to see that TTC is non-bossy in the multi-copy object case as well. She also obtains that non-bossiness and strategy-proofness is equivalent to group strategy-proofness (the proof of this result does not rely on the unit-copy assumption). Hence, these, along with the strategy-proofness of TTC, shows that TTC is group strategy-proof in the current multi-copy object assignment model.

size-wise dominated by an individually rational, weakly population-monotonic, and strategy-proof mechanism.

Any mechanism under which each agent always receives the same assignment and no agent is left unassigned unless all the objects are exhausted is strategy-proof, weakly population-monotonic, and that size-wise dominates TTC. Yet, it is not individually rational. Let us consider a mechanism such that it always assigns as many agents as possible subject to individual rationality and no unassigned agent can alter the outcome by not demanding any object. This mechanism is individually rational, weakly population-monotonic, and size-wise dominates TTC; yet it is not strategy-proof (see Footnote 8). We do not know whether Theorem 2 holds without weak population-monotonicity, hence it remains to be an open question.

Appendices

Appendix A

Deferred Acceptance Mechanism (DA)

Let us fix a priority ordering for the objects. Then, DA runs as follows.

Step 1. Each agent applies to his best acceptable object. Each object tentatively accepts the highest priority applicants one by one up to its capacity and rejects the rest.

In general,

Step k. Each rejected agent applies to his next best acceptable object. Each object tentatively accepts the highest priority agents among the tentatively accepted and currently applying agents one by one up to its capacity and rejects the rest.

The algorithm terminates whenever every agent is either tentatively accepted or has gotten rejection from all of his acceptable objects. The assignments in the terminal round realize as the final DA outcome.

Boston Mechanism (BM)

For a fixed priority ordering of the objects, BM works as follows.

Step 1. Each agent applies to his best acceptable object. Each object permanently accepts the highest priority applicants one by one up to its capacity and rejects the rest.

In general,

Step k. Each rejected agent applies to his next best acceptable object. Each object permanently accepts the highest priority applicants up to its remaining capacity and rejects the rest.

The algorithm terminates whenever each agent is either permanently accepted or has gotten rejection from all of his acceptable objects. The assignments in the terminal round realize as the final BM outcome.

Top Trading Cycles Mechanism (TTC)

For a fixed priority profile of the objects, TTC runs as follows.

Step 1. Each agent points to his best acceptable object. Each object points to the highest priority agent. As everything is finite, there exists a cycle. Assign the agents in these cycles to the objects they are pointing to. The assigned agents leave the problem, and each assigned object's capacity is decreased by one.

In general,

Step k. Each agent points to his best acceptable object with a remaining capacity. Each object points to the highest priority remaining agent. As everything is finite, there exists a cycle. Assign the agents in these cycles to the objects they are pointing to. The assigned agents leave the problem, and each assigned object's capacity is decreased by one.

The algorithm terminates whenever each agent receives an object or all of his acceptable objects are exhausted. The assignments by the end of the terminal step realize as the final TTC outcome.

Serial Dictatorship (SD)

For a fixed agent ordering, SD runs as follows.

Step 1. The first agent in the ordering chooses his best acceptable object. The chosen object's capacity is decreased by one.

In general,

Step k. The k^{th} agent in the ordering chooses his remaining best acceptable object. The chosen object's capacity is decreased by one.

The algorithm terminates after the choice-turn of the last agent in the ordering. The chosen objects are the final SD assignment of the agents.

Appendix B

Lemma 1. *Let ψ be a non-wasteful mechanism. If, for any problem P and individually rational mechanism ϕ, $|\psi(P)| < |\phi(P)|$, then there exists an agent i such that $\psi_i(P)P_i\phi_i(P)P_i\emptyset$.*

Proof. Let ψ and ϕ be a non-wasteful and individually rational mechanisms, respectively. Suppose $|\psi(P)| < |\phi(P)|$ for a problem P. Then, there exists an object $c \in O$ such that $|\psi_c(P)| < |\phi_c(P)| \leq q_c$. Let $i \in \phi_c(P) \setminus \psi_c(P)$. As ϕ is individually rational, $cP_i\emptyset$. On the other hand, the non-wastefulness of ψ and $|\psi_c(P)| < q_c$ imply that $\psi_i(P)P_ic$. Therefore, we have $\psi_i(P)P_icP_i\emptyset$, where $\phi_i(P) = c$, which finishes the proof. □

Lemma 2. *Let ψ be a non-wasteful mechanism. For a problem P and an individually rational mechanism ϕ, if $|\psi(P)| = |\phi(P)|$ and there exists no agent i such that $\psi_i(P)P_i\phi_i(P)P_i\emptyset$, then $|\psi_c(P)| = |\phi_c(P)|$ for every object c.*

Proof. Let ψ and ϕ be a non-wasteful and individually rational mechanisms, respectively. In a problem P, suppose that $|\psi(P)| = |\phi(P)|$ and there exists no agent i such that $\psi_i(P)P_i\phi_i(P)P_i\emptyset$.

If there exists an object c such that $|\psi_c(P)| > |\phi_c(P)|$, then, as $|\psi(P)| = |\phi(P)|$, there exists another object d such that $|\psi_d(P)| < |\phi_d(P)|$. Hence, without loss of generality, assume for a contradiction that $|\psi_d(P)| < |\phi_d(P)|$ for some object d. This implies that there exists an agent $i \in \phi_d(P) \setminus \psi_d(P)$. From the individual rationality of ϕ, $dP_i\emptyset$. Because ψ is non-wasteful and $|\psi_d(P)| < q_d$, it implies that $\psi_i(P)P_id$. Hence, these show that $\psi_i(P)P_idP_i\emptyset$, where $d = \phi_i(P)$, contradicting our starting supposition. □

Proof of Theorem 1. Let ψ be a non-wasteful, truncation-invariant, and extension-responding mechanism. Assume for a contradiction that an individually rational and strategy-proof mechanism ϕ size-wise dominates ψ. Let P be a problem such that $|\psi(P)| < |\phi(P)|$. We prove the result in two steps.

Step 1. In this step, we construct a preference profile in which there exists no agent who is assigned under both mechanisms' outcomes while preferring his assignment under ψ to that under ϕ.

As $|\psi(P)| < |\phi(P)|$, from Lemma 1, there exists an agent i such that $\psi_i(P)P_i\phi_i(P)P_i\emptyset$. Let P'_i be the truncation of P_i from $\psi_i(P)$. Let us write $P' = (P'_i, P_{-i})$. From the truncation-invariance of ψ, we have $\psi(P) = \psi(P')$. On the other hand, due to the strategy-proofness of ϕ, $\phi_i(P') = \emptyset$.

Let us now consider problem P'. If there exists no agent j such that $\psi_j(P')P'_j\phi_j(P')P'_j\emptyset$, then move to Step 2. Otherwise, pick such an agent j. Because $\phi_i(P') = \emptyset$, $j \neq i$. By following the same arguments above, we let agent j truncate his preferences from $\psi_j(P')$ and write P'_j for this truncated preferences. If we write $P'' = (P'_i, P'_j, P_{-\{i,j\}})$, then by the truncation-invariance of ψ, we have $\psi(P'') = \psi(P') = \psi(P)$. On the other hand, because of the strategy-proofness of ϕ, $\phi_j(P'') = \emptyset$

We now consider P''. If there exists no agent k such that $\psi_k(P'')P''_k\phi_k(P'')P''_k\emptyset$, then move to Step 2. Otherwise, we repeat the same arguments to such an agent k. Because $\psi(P'') = \psi(P') = \psi(P)$, under $\psi(P'')$, both i and j are assigned to their least preferred acceptable objects with respect to P''_i and P''_j, respectively. Therefore, $k \in I \setminus \{i, j\}$. But then, as there are finitely many agents, this case cannot hold all the time. Therefore, we eventually come up with a preference profile $\tilde{P}$ such that there exists no agent i such that $\psi_i(\tilde{P})\tilde{P}_i\phi_i(\tilde{P})\tilde{P}_i\emptyset$ and move to Step 2.

Step 2. From Lemma 1, we have either $|\psi(\tilde{P})| > |\phi(\tilde{P})|$ or $|\psi(\tilde{P})| = |\phi(\tilde{P})|$. If the former is the case, then we reach a contradiction, which finishes the proof. Let us consider the other case of $|\psi(\tilde{P})| = |\phi(\tilde{P})|$.

Let k be the last agent whose preferences is truncated above in obtaining $\tilde{P}$. Then, we have $\phi_k(\tilde{P}) = \emptyset$ and $\psi_k(\tilde{P}) \neq \emptyset$. This, along with $|\psi(\tilde{P})| = |\phi(\tilde{P})|$, implies that there exists an agent i such that $\psi_i(\tilde{P}) = \emptyset$ and $\phi_i(\tilde{P}) \neq \emptyset$. Moreover, $|\psi(\tilde{P})| = |\psi(P)| < |\phi(P)|$ implies that there exists an object c such that $|\psi_c(\tilde{P})| < q_c$. By the non-wastefulness of ψ, $\emptyset\tilde{P}_i c$.

Let us now consider the object c extension of $\tilde{P}_i$ and write $\hat{P}_i$ for it. Let $\hat{P} = (\hat{P}_i, \tilde{P}_{-i})$. As ψ is extension-responding, we have $\psi_i(\hat{P}) = c$ and $\psi_j(\hat{P}) = \psi_j(\tilde{P})$ for any other agent $j \in I \setminus \{i\}$. Hence, $|\psi(\hat{P})| = |\psi(\tilde{P})| + 1$. On the other hand, from the individual rationality of ϕ, $\phi_i(\tilde{P})\tilde{P}_i\emptyset\tilde{P}_i c$. Therefore, due to the strategy-proofness of ϕ, $\phi_i(\hat{P}) = \phi_i(\tilde{P})$.

As $|\psi(\hat{P})| = |\psi(\tilde{P})| + 1$ and $|\psi(\tilde{P})| = |\phi(\tilde{P})|$, if $|\phi(\hat{P})| \leq |\phi(\tilde{P})|$, then $|\psi(\hat{P})| > |\phi(\hat{P})|$, which yields a contradiction; hence finishing the proof. Suppose $|\phi(\hat{P})| > |\phi(\tilde{P})|$. Without loss of generality, we assume that there is no agent j such that $\psi_j(\hat{P})\hat{P}_j\phi_j(\hat{P})\hat{P}_j\emptyset$. This supposition is legitimate as, otherwise, we can invoke Step 1 for $\hat{P}$. As, in each preference truncation iteration within Step 1, we are to truncate a different agent's preferences and there are finitely many agents, this case cannot hold all the time.

As there is no agent j such that $\psi_j(\hat{P})\hat{P}_j\phi_j(\hat{P})\hat{P}_j\emptyset$, from Lemma 1, we have $|\psi(\hat{P})| \geq |\phi(\hat{P})|$. If it is strict, then we reach a contradiction, finishing the proof. Hence, suppose $|\psi(\hat{P})| = |\phi(\hat{P})|$.

As $|\phi(\hat{P})| > |\phi(\tilde{P})|$, there exists an object c such that $|\phi_c(\hat{P})| > |\phi_c(\tilde{P})|$. This in turn implies that for some agent j, $\phi_j(\hat{P}) = c$ and $\phi_j(\tilde{P}) \neq c$. Note that as $\phi_i(\hat{P}) = \phi_i(\tilde{P})$, agent j is different than agent i. Moreover, from Lemma 2, we have $|\psi_{c'}(\tilde{P})| = |\phi_{c'}(\tilde{P})|$ for every object $c' \in O$. This, along with $|\phi_c(\hat{P})| > |\phi_c(\tilde{P})|$, implies that $|\psi_c(\tilde{P})| < q_c$. Hence, by the non-wastefulness of ψ, we have $\psi_j(\tilde{P})\tilde{R}_j c$. Suppose it is strict. Then, as $j \neq i$, we have $\tilde{P}_j = \hat{P}_j$. As $\psi_j(\hat{P}) = \psi_j(\tilde{P})$, we have $\psi_j(\hat{P})\hat{P}_j\phi_j(\hat{P})\hat{P}_j\emptyset$, contradicting our supposition in Step 2. Otherwise, $\psi_j(\tilde{P}) = c$. By invoking Lemma 2, $|\psi_c(\tilde{P})| = |\phi_c(\tilde{P})|$. Moreover, $\phi_j(\tilde{P}) \neq c$. These imply that there exists an agent h such that $\phi_h(\tilde{P}) = c$ and $\psi_h(\tilde{P}) \neq c$. As $|\psi_c(\tilde{P})| < q_c$ and $c\tilde{P}_h\emptyset$ (by the individual rationality of ϕ), by the non-wastefulness of ψ, $\psi_h(\tilde{P})\tilde{P}_h c\tilde{P}_h\emptyset$, where $\phi_h(\tilde{P}) = c$, contradicting our finding in Step 1, finishing the proof. $\square$

Proof of Theorem 2. Assume for a contradiction that ψ is an individually rational, weakly population-monotonic, and strategy-proof mechanism that size-wise dominates an efficient and group strategy-proof mechanism ϕ. Let P be a problem such that $|\psi(P)| > |\phi(P)|$.

For a matching μ, let $U(\mu) = \{i \in I : \mu_i = \emptyset\}$, that is, the set of unassigned agents under matching μ. As efficiency implies non-wastefulness, for any problem P and agent-object pair (i, c) such that $i \in U(\phi(P))$ and $|\phi_c(P)| < q_c$, we have $\emptyset P_i c$. We do the proof in the following two steps.

Step 1. As $|\psi(P)| > |\phi(P)|$, there exists an agent $i \in U(\phi(P)) \setminus U(\psi(P))$. From the non-wastefulness of ϕ and our supposition that $\sum_{c\in O} q_c \geq |I|$, there exists an object c such that $|\phi_c(P)| < q_c$ and $\emptyset P_i c$.

Let P'_i be the object c extension of P_i, and we write $P' = (P'_i, P_{-i})$. We now claim that $\phi_i(P') = c$. If it is not, then by the strategy-proofness and the efficiency of ϕ, $\phi_i(P') = \phi_i(P) = \emptyset$. But then, by invoking group strategy-proofness of ϕ, $\phi(P) = \phi(P')$. Hence, $|\phi_c(P')| < q_c$ and $cP'_i\emptyset$, contradicting the non-wastefulness of ϕ. Therefore, $\phi_i(P') = c$. On the other hand, by the individual rationality of ψ, $\psi_i(P)P_i\emptyset P_i c$ (recall that $i \in U(\phi(P)) \setminus U(\psi(P))$). As ψ is strategy-proof, we have $\psi_i(P') = \psi_i(P)$. Therefore, $\psi_i(P')P'_i\phi_i(P')P'_i\emptyset$.

Let us now consider P'. If $U(\phi(P')) \setminus U(\psi(P')) = \emptyset$, then we move to Step 2. Otherwise, We invoke Step 1 for P'. That is, let $k \in U(\phi(P')) \setminus U(\psi(P'))$. Note that as $\phi_i(P') \neq \emptyset$, $k \neq i$. As $\sum_{c\in O} q_c \geq |I|$, there exists an object c' such that $|\phi_{c'}(P')| < q_{c'}$ and $\emptyset P'_k c'$ (note that $P'_k = P_k$). Let P''_k be the object c' extension of P'_k, and we write $P'' = (P'_i, P''_k, P_{-\{i,k\}})$. Then, by the same arguments above, we have $\phi_k(P'') = c'$ and $\psi_k(P'')P''_k\phi_k(P'')P''_k\emptyset$.

We next consider P''. If $U(\phi(P''))\backslash U(\psi(P''))=\emptyset$, then we move to Step 2. Otherwise, we repeat Step 1 for P''. In each iteration of Step 1, we pick someone who is unassigned under ϕ yet assigned under ψ and extend his preferences by adding an object to his acceptable set. However, as both the agents and objects are finite, this case cannot hold all the time. Therefore, we eventually come up with a preference profile $\tilde{P}$ such that $U(\phi(\tilde{P}))\backslash U(\psi(\tilde{P}))=\emptyset$. Moreover, for the last agent whose preferences is extended in obtaining $\tilde{P}$, say agent ℓ, we have $\psi_\ell(\tilde{P})\tilde{P}_\ell\phi_\ell(\tilde{P})\tilde{P}_\ell\emptyset$.

Step 2. As $U(\phi(\tilde{P}))\setminus U(\psi(\tilde{P}))=\emptyset$ and $|\psi(\tilde{P})|\geq|\phi(\tilde{P})|$, we have $U(\phi(\tilde{P}))=U(\psi(\tilde{P}))$. Let us now construct a new problem $\hat{P}$ as follows. For any agent $i\notin U(\phi(\tilde{P}))$, $\hat{P}_i=\tilde{P}_i$; and for any $j\in U(\phi(\tilde{P}))$, $\emptyset\hat{P}_jc$ for every $c\in O$. By the efficiency and the group strategy-proofness of ϕ, $\phi(\hat{P})=\phi(\tilde{P})$. On the other hand, by the weak population-monotonicity of ψ, $\psi_i(\hat{P})\tilde{R}_i\psi_i(\tilde{P})\tilde{P}_i\emptyset$ for any $i\notin U(\psi(\tilde{P}))$, and by the individual rationality of ψ, $\psi_j(\hat{P})=\emptyset$ for any $j\in U(\psi(\tilde{P}))$.

In Step 1, we find that $\psi_\ell(\tilde{P})\tilde{P}_\ell\phi_\ell(\tilde{P})\tilde{P}_\ell\emptyset$. From above, $\psi_\ell(\hat{P})\tilde{R}_\ell\psi_\ell(\tilde{P})$ and $\phi_\ell(\hat{P})=\phi_\ell(\tilde{P})$. Hence, $\psi_\ell(\hat{P})\hat{P}_\ell\phi_\ell(\hat{P})$ (note that by construction, $\tilde{P}_\ell=\hat{P}_\ell$). As ϕ is efficient, it implies that there exists an agent i such that $\phi_i(\hat{P})\hat{P}_i\psi_i(\hat{P})$. Moreover, because $U(\phi(\hat{P}))=U(\psi(\hat{P}))$ and ψ is individually rational, $\psi_i(\hat{P})\hat{P}_i\emptyset$.

Let $\bar{P}_i$ be the truncation of $\hat{P}_i$ from $\phi_i(\hat{P})$. Let $\bar{P}=(\bar{P}_i,\hat{P}_{-i})$. By the group strategy-proofness of ϕ, we have $\phi(\bar{P})=\phi(\hat{P})$. By the strategy-proofness of ψ, $\psi_i(\bar{P})=\emptyset$. This, along with the individual rationality of ψ, implies that $|\psi(\bar{P})|<|\psi(\hat{P})|=|\phi(\hat{P})|=|\phi(\bar{P})|$, contradicting our starting supposition that ψ size-wise dominates ϕ. □

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