NEW CAPACITY ALLOCATION POLICIES IN REVENUE MANAGEMENT

by

NURŞEN AYDIN

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by NURŞEN AYDIN

APPROVED BY:

Prof. Dr. Ş. İlker Birbil ....
(Thesis Advisor)

Prof. Dr. Fikri Karaesmen ....

Assoc. Prof. Dr. Kemal Kılıç ....

Assoc. Prof. Dr. Koray D. Şimşek ....

Assoc. Prof. Dr. Hüseyin Topaloğlu ....

DATE OF APPROVAL: 06.08.2014

to my mother Ayşe

NEW CAPACITY ALLOCATION POLICIES IN REVENUE MANAGEMENT

Nurşen Aydın

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Thesis Advisor: Prof. Dr. Ş. İlker Birbil

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In this dissertation, we study three emerging problems in revenue management. First problem is about optimal capacity allocation in single-leg airline revenue management with overbooking. We propose new static and dynamic models. The static problems are difficult to solve optimally. Therefore, we introduce approximate models, which provide upper and lower bounds on the optimal expected revenues. In the dynamic case, we propose a model based on two streams of events; the arrivals of booking requests and cancellations. Following the characterization of the optimal policy, we also present the nested structure of the optimal allocations.

Second problem is about optimal capacity allocation in the presence of a contingent commitment option. This option has been recently offered by airline systems to provide purchase flexibility to the customers. The problem becomes finding the revenue maximizing policy for contingent commitments and advance bookings. We first propose a dynamic programming model. Since it is computationally intractable, we develop an alternate dynamic model based on geometric approximation. In our numerical study,

we investigate the effect of the commitment option on various test instances.

In the third problem, we investigate optimal room allocation policies in hotel revenue management. Long-term stays are very common in hotel industry. Therefore, it is crucial to consider allocation of multiple-day capacities when responding to a request. This requirement leads to solving large-scale network problems, which are computationally challenging. Therefore, we work on various decomposition methods to find reservation policies for walk-in and stay-over customers. We also devise solution algorithms to solve large problems efficiently.

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Bu tezde, gelir yönetimi alanındaki üç güncel problem çalışılmıştır. İlk problemde amaç, kapasite üstü rezervasyona izin verilen bir uçuşun toplam kapasitesinin, geliri enbüyükleyecek şekilde yolcu sınıflarına ayrılmasıdır. Bu problem için yeni statik ve dinamik modeller önerilmiştir. Statik problemlerin karmaşık yapılarından dolayı, en iyi beklenen gelir için alt ve üst sınırları veren yeni modeller sunulmuştur. Dinamik problemde ise, rezervasyon ve iptaller için iki akış temelli bir dinamik programlama modeli önerilmiştir. Ayrıca elde edilen en iyi politikanın yapısı incelenerek, en uygun kapasite dağıtımının içiçe bir yapıda olduğu gösterilmiştir.

İkinci problemde geçici rezervasyon seçeneğini içeren kapasite dağıtım problemi incelenmiştir. Bu seçenek, müşterilere alım esnekliği sağlaması amacıyla havayolları rezervasyon sistemleri tarafından yakın zamanda sunulmaya başlanmıştır. Bu doğrultuda ele aldığımız problemin amacı, geçici ve kesin rezervasyonlar için karar politikasının belirlenmesidir. İlk önce bir dinamik programlama modeli önerilmiştir. Ancak bu modelin çözülmesi çok güç olduğundan, geometrik yakınsamaya dayalı alternatif bir dinamik programlama modeli geliştirilmiştir. Geçici rezervasyon seçeneğinin etkileri sayısal örnekler üzerinde test edilmiştir.

Üçüncü problemde, otel gelir yönetiminde oda dağıtımı incelenmiştir. Müşterilerin uzun süreli konaklaması otel endüstrisinde çok yaygındır. Bu nedenle, müşteri taleplerine cevap verirken kapasitesi kullanılan bütün günleri göz önünde bulundurmak çok önemlidir. Dolayısıyla karşımıza büyük ölçekli ve çözülmeleri oldukça zor ağ problemleri çıkmaktadır. Bu çalışmada, rezervasyonsuz gelen ve kalış süresini uzatmak isteyen müşteriler için rezervasyon politikalarını belirleyen ayrıştırma yöntemleri incelenmiştir. Ayrıca büyük ölçekli problemleri daha etkili çözmeyi sağlayacak çözüm algoritmaları geliştirilmiştir.

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Chapter 1

INTRODUCTION

Revenue management's (RM) focus upon the techniques and strategies in product availability and pricing makes RM one of the most important operations research practices. Historically, airline industry plays the steering role in revenue management. This prominence can be attributed to the quick responses of the airline executives, who have realized the importance of controlling the reservation process to increase their gains throughout a fiscal year. The major development in revenue management began with the 1978 deregulation of the U.S. airline industry. With this act, airline companies also began to manage their own schedules and prices. Low-cost carriers entered the market that increased the competition between airlines. Major airlines began implementing revenue management practices to compete with the low-cost carriers. The main problem, then and now, in revenue management has been to determine how to reserve the seats for the requests coming from the passengers. The first studies on airline revenue management (ARM) focused on single-leg flight problems. After airline companies began to use hub-and-spoke networks to manage their operations, network airline revenue management became an active area of research. Hub-and-spoke structure allow many origin-destination pairs to be served with different flights. It is well known today that many airline companies are interested in managing their revenues over a network of flights. However, network problem is difficult to solve because it includes multiple legs and leg capacities are shared among different flights. The state space of the network problem basically becomes the Cartesian product of flight capacities in the network. Due to this complex structure, network problems are treated by using various approximations. Achieving a good balance between the quality and the efficiency of the approximation method becomes the primary challenge. For a historical account of the role of airline industry in revenue management, we refer to [80, Section 1.2] .

Capacity allocation, overbooking and pricing are the main strategies used by airline revenue management specialists. While capacity allocation deals with reserving seats for different fare classes, overbooking is concerned with the number of additional booking requests to be accepted above the physical capacity. It is quite common that a certain percentage of the accepted requests cancel before the departure time (cancellations) or do not show-up at the departure time (no-shows). Consequently, the capacity becomes available for boarding overbooked passengers. Thus, overbooking is used by the airline companies to protect themselves against vacant seats due to no-shows and late cancellations. On the other hand, some of the reservations may be denied boarding due to the lack of capacity at the departure time. In such a case, the airline faces penalties such as monetary compensations, and even worse, suffers from bad public relations. Even though the overbooking decision involves uncertainties regarding the no-shows and cancellations, accepting more booking requests than available capacity is still a commonly-used, profitable strategy because the revenue collected by overbooking usually exceeds the penalties for denied boardings [73]. In capacity allocation and overbooking models, it is assumed that prices are fixed and fare classes are controlled by opening or closing decisions as demand evolved. Pricing strategy deals with the problem of determining the set of prices of the fare classes that will maximize the total expected revenue. After reservation systems started using online sale channels, pricing has thus become an important control mechanism for the airlines [14]. Bungart [67] presents a comprehensive review of the pricing and capacity control strategies and he states that pricing can be considered as a special case of capacity control when prices are used as control variables.

The control of the flight capacities plays an important role in most of the revenue management strategies. Recent studies focus on mitigating the effects of demand uncertainty in the market. Revenue management practices use various types of options as new ways to differentiate products and effectively manage the demand uncertainty [42]. Many customers have uncertain valuations of the product in advance of its delivery. For instance, the travel time of a customer can be changed due to the unknown future constraints or sport fans would not want to attend the tournament if their favored team was eliminated. By offering various options together with the specific products, service providers aim to attract those customers who otherwise would not consider to buy. Upgrades, flexible products, refundable fares and opaque selling can be given as examples to these options. While options offer purchase flexibility to customers, they also provide additional revenue to the service providers [40]. Despite the fact that the airline industry pioneered the use of revenue management techniques, with these new strategies, many other service industries now benefit from revenue management applications. RM techniques can be applicable to any industry with volatile demand and selling fixed, perishable capacity [49]. Today, a wide range of industries, such as; hotels (e.g. Bitran and Mondschein [15]), car rental agencies (e.g. Carol and Grimes [21]), cargo industries (e.g. Popescu [70]), retailers (e.g. Bitran and Mondschein [16]) and Internet providers (e.g. Nair and Bapna [68]) have adopted revenue management practices. Chiang et al. [26] provide a comprehensive review of the revenue management studies in different industries. However, RM applications in those sectors are not common as they are in the airline industry. Ivanov and Zhechev [47] outline that there is a comparable gap between hotel and airline RM literature. Although hotel industry is one of the main application areas of revenue management, the techniques developed for ARM problems have generally been used in the hotel reservation system after simplifying the problem.

In this thesis, we work on capacity control problems in single-leg and network revenue management. Considering new developments in airline and hotel industry, we address the gaps in the literature and concentrate our efforts on new models that are important for the applications of revenue management. In this context, we first work on overbooking and option problems in single-leg revenue management and then focus on network capacity allocation problems in hotel RM. As these problems are difficult to solve, approximation methods have been employed in the literature to simplify. Our focus is to develop more realistic models compared to the proposed models in the literature. Moreover, we introduce new concepts which are widely used in airline RM.

1.1 Motivation and Contributions

Airline revenue management is an active area of research. The high interest of airline companies led to an acceleration in studies in this area after 1990s. Many solution approaches have been designed and theoretical results obtained for capacity allocation problem [80]. Nonetheless, joint capacity allocation and overbooking problem has not been thoroughly studied. Models with overbooking are difficult to handle as the state-space in dynamic formulations increases significantly. Hence, in almost all cases, an approximation to the problem is solved. The first studies in this area focus on finding the overbooking limit by ignoring the capacity allocation. The following studies concentrate both on the capacity control and overbooking decisions. However, most of these studies have considered overbooking limit as an input parameter. A common practice is first setting the virtual capacity and then doing the allocations (c.f. Belobaba [11]). This heuristic approach, in fact, undermines the effects of these two decisions on each other. Therefore, it is natural to study the joint capacity allocation and overbooking problem which is, in general, difficult to solve largely because of the uncertainty in demand, no-shows and cancellations.

This is what we provide in the first part of the thesis. The approach we propose aims to provide joint capacity allocation and overbooking policy. We study the properties of the model and propose new mathematical programming models for static and dynamic single-leg problems that involve no-shows, cancellations, and hence, overbooking. Our first static model focuses on finding the total overbooking limit for multiple classes under the assumption that the fare class requests are accepted as long as the total number of reservations is below the total booking limit. This model allows for class-dependent cancellations and no-shows. To the best of our knowledge, our model is a first in the literature in determining an optimal total booking limit under this broad setting. As a by-product of our approach, we also discover that a well-known heuristic from the literature finds an optimal overbooking limit whenever the particular parameters dictated by our analysis are used. In the second static model, which also considers the class-dependent no-shows and cancellations, we determine simultaneously the total booking limit and the partitioned allocation of the virtual capacity to each fare class. Arriving at a computationally difficult model, we propose upper and lower bounding problems to obtain approximate solutions, which have demonstrated promising performance in our computational study. We also derive bounds on the error introduced by solving the upper bounding problem instead of the corresponding original static model. Our last model involves a dynamic setting based on two independent streams of events; arrivals of booking requests and cancellations. Contrary to the static case, the dynamic setting deals with the class-independent show-ups and cancellations. The proposed model, therefore, can be used as a heuristic in practice for the actual model with class-dependent processes.

The second theme we address in this thesis is a relatively new application, namely contingent commitment options, in the airline reservation systems. Revenue management systems focus on designing services to manage demand risk, improve capacity utilization and increase revenues. Recently, the airline reservation systems offer the contingent commitment option to attract customers who are price sensitive and have uncertain travel time. This option allows passengers to reserve a seat for a certain duration of time within the reservation period before making a buy or a leave decision. Commitment option has been widely adopted by many airline companies that even dedicated web based services, such as OptionsAway, have been launched [91].

From an airline perspective, every committed seat provides an additional revenue from the non-refundable fee. However, offering aforementioned options may cannibalize demand by blocking the expensive fare class customers, if the capacity management is poor. In addition, this option also creates another source of uncertainty causing probable revenue loss due to empty seats. In this thesis, we introduce the commitment concept to the revenue management literature. We develop single-leg revenue management models that consider such contingent commitment decisions. We start with a dynamic programming model of this problem. This model is computationally intractable as it requires storing a multi-dimensional state space due to book-keeping of the committed seats. To alleviate this difficulty, we propose an alternate dynamic programming formulation. We also present a deterministic linear programming model that gives an upper bound on the optimal expected revenue from the intractable dynamic programming model. We study the properties of the model and examine how does offering commitment options to customers affect overall revenue.

The third theme we address in this thesis is related to capacity allocation problem in hotel revenue management. Hotel RM problem has a linear network structure and hence it can be defined as a special case of airline network revenue management problem. However, due to the problem structure, the techniques developed for network problem may not be directly applicable. First, multi-night stay in hotels is quite common. While a flight itinerary generally includes at most three legs, number of nights in a hotel itinerary can be as high as twelve [96]. Second, demand structure is different. Hotel customers can change their length of accommodation even while staying in the hotel. However, it is not possible for an airline customer to alter her reservation once she is on board in a flight [49]. Moreover, while airline customers generally make advance purchases, a good portion of the hotel customers are constituted by walk-ins and even the early reservations in the booking period can cancel for free.

The literature on capacity control problem for hotel industry is not as mature as the one for airline industry. Due to the complexity of the problem, the proposed studies generally focus on the deterministic problem or single day stay only. In this study, we work on the room allocation problem with walk-in and stay-over customers for multi-day stay and formulate the problem as a dynamic programming model. Since the resulting model is a large-scale network problem, we concentrate on decomposition methods to attack the dynamic model. We first work on single-day decomposition and propose new modeling approach by analyzing the structure of the problem. However, day-based decomposition causes a loss of information on the number of customers in each booking type. Therefore, performance of these methods can be poor for stay-over customers. To include these customers, we work on the product-based decomposition. By exploiting the analytical properties of the model, we devise a fast solution algorithm.

1.2 Overview of the Proposed Thesis

All problems in this thesis deal with capacity allocation in revenue management. In Chapter 2, we study joint capacity allocation and overbooking problem. We first discuss the methods proposed in the overbooking and capacity allocation literature. Then, we present our dynamic and static models for single-leg revenue management. We study the properties of the model and propose solution procedures. In Chapter 3, we introduce the concept of contingent commitment option and examine the options presented in revenue management literature. We analyze the consequences of selling this option along with standard bookings of the products. We derive dynamic and static models for the capacity allocation problem. We discuss the analytical properties of the model and propose an alternate tractable model to determine the optimal capacity allocation policy. We conduct a computational study to evaluate how offering options affects the airline's revenue and test the performance of our approach. In Chapter 4, we study the capacity allocation problem in a hotel network. We introduce new modeling approaches for managing seat inventory using dynamic programming methodology. We focus on the day-based and pair-based decomposition approaches by considering walk-in and stay-over customers. We analyze structural properties of the decomposition approaches and solution algorithms. We provide several computational results to test the performances of our approaches. Finally, in Chapter 5 we discuss our results and contributions. We also provide a discussion about our future research.

Chapter 2

SINGLE-LEG PROBLEM: OVERBOOKING OPTION

In this chapter, we discuss our work on the problem of joint capacity allocation and overbooking [6]. Airline revenue management is concerned with identifying the maximum revenue seat allocation policies. Since a major loss in revenue results from cancellations and no-shows, overbooking has received a significant attention in the literature over the years. We first provide a summary of the single-leg capacity control models.

Studies on seat allocation problem starts with Littlewood's [63] work. Littlewood proposes a solution method for the single-leg problem with two fare classes. The idea behind his model is to equate the marginal revenues in each of the two fare classes. Belobaba [12] extends this idea to a multi-class problem and introduces the method of expected marginal seat revenue (EMSR) for the general approach. However, this method can generate optimal booking limits only for the two fare class problem. Curry [30], Wollmer [94], and Brumelle and McGill [20] work on EMSR method and obtain optimal policies for the multi-class static problem.

Lee and Hersh [60] propose a discrete time dynamic programming model and formulate the problem as a Markov decision process (MDP). In this model, the reservation period is divided into sufficiently small time intervals to allow only one arrival. In each period, a reservation request is accepted if its fare is higher than expected marginal revenue of the seat. The work of Lee and Hersh has elicited interest from various researchers and it is refined by Liang [61] and Lautenbacher and Stidham [59]. While Liang reformulates the model in continuous time, Lautenbacher and Stidham

[59] combine the dynamic and static approaches under a common MDP formulation.

Parallel to seat allocation, research on overbooking problem has accelerated. The early overbooking literature concentrates mainly on static models with one or two fare classes and the objective of finding the overbooking limit. The first scientific work on overbooking is proposed by Beckman [10]. He develops a static single fare class overbooking model, which determines the overbooking limit by considering the trade-off between the lost revenue due to empty seats, the total cost of denied boardings and the revenue generated by the go-show passengers. The go-shows are the passengers who show up without any reservation at the departure time. American Airlines adopted Beckman's approach and implemented a related model in 1976 and then revised it in 1987 [75]. Beckman's work is succeeded by Thompson [82], who considers a practical model ignoring the probability distribution of demand and requiring only data on the number of cancellations among the total number of reservations. Given the capacities for two fare classes, Thompson [82] aims at determining the overbooking amount for each fare class so that the probability of overbooking equals to a specified value. He also supports his arguments by a statistical analysis of the involved distributions. The works of Beckman and Thompson are refined by Taylor [81]. Like Thompson, he focuses on a service measure by constraining the number of denied boardings but considers cancellations, no-shows and group sizes explicitly. This influential work of Taylor has attracted the attention of various airlines. Consequently, the variants of this work are implemented, and then, reported in a sequence of papers. The references and the details of this history are given by Rothstein [73].

Chi [25] studies a static overbooking problem with multiple fare classes and formulates it as a dynamic programming model. However, when cancellations and no-shows are considered, the model suffers from the curse of dimensionality because one needs to keep track of the number of reservations for each class. To overcome this difficulty, Chi proposes an approximate model that can be solved in polynomial time. Coughlan [27] also considers a overbooking problem with multiple fare classes, but he assumes that the go-show passengers are given the empty seats at the same price as in Beckman [10]. Unlike the majority of the studies in the literature, Coughlan does not use a Poisson distribution to model the demand but makes the simplifying assumption that both the demand and the number of bookings for each fare class are independent and normally distributed. Coughlan's discussion also supposes implicitly that the minimum of the demand and the number of bookings is also normally distributed; unfortunately, this supposition does not hold mathematically in general. Overall, the author provides a closed form formula for the revenue function and applies heuristic search methods to find a maximizer.

Several researchers have addressed dynamic overbooking models for single-leg problems. Generally, the dynamic overbooking problem is modeled as a Markov Decision Process (MDP). Rothstein [72] proposes two such models, where only one fare class is considered. In the first model, the objective is to find the optimal expected revenue after deducting the cost of denied boardings. Following the work of Thompson [82], the second model adds a constraint to limit the proportion of denied boardings. Alstrup et al. [2] also use a MDP to solve an overbooking model but this time with two fare classes and the cost of downgrading (a cost that is incurred due to reserving cheaper seats for the passengers requesting more expensive fare classes). Chi [25] also discusses two dynamic models with multiple fare classes. Although the first model incorporates the realistic setting of cancellations occurring in time, it is computationally intractable. To ease the computational burden, Chi then assumes in his second model that the cancellations occur right before the departure time. This assumption allows him to solve the resulting model with an approximation similar to the one he uses in the static case. Chatwin [22] analyzes the optimal solution structure of a discrete time dynamic single fare class overbooking model and discusses the conditions, under which a booking limit policy is optimal. Subramanian et al. [76] study a more general setting than Chatwin, where they analyze the overbooking problem with multiple fare classes. The authors consider the arrival of a cancellation, the arrival of a booking request and no arrival of any type as a combined stream and assume that at most one of these events can occur at any discrete time epoch. Under this setting they present two models. In the first model, the cancellation and no-show probabilities do not depend on the fare classes. They show that the resulting problem can be equivalently modeled as a queuing system discussed in the literature [62]. In their second model, they relax the class independence assumption and model a more general problem with class-dependent cancellations and no-shows. Unfortunately, the resulting dynamic programming formulation cannot be solved efficiently because of the high-dimensional state space. Chatwin [23] examines a continuous-time single fare class overbooking problem, where fares and refunds vary over time according to piecewise constant functions. In his model the arrival process of requests is assumed to be a homogeneous Poisson process, and the probabilities to identify the type of a request are independent of time. He assumes that the reservations cancel independently according to an exponential distribution with a common rate, and the arrival process of requests depends on the number of reservations. Under these assumptions, the author formulates the problem as a homogeneous birth-and-death process and shows that a piecewise constant overbooking limit policy is optimal. A closely related study is given by Feng et al. [36]. They consider a continuous-time model with cancellations and no-shows. They derive a threshold type optimal control policy, which simply states that a request should be admitted only if the corresponding fare is above the expected marginal seat revenue (EMSR). Karaesmen and van Ryzin [48] examine the overbooking problem differently. Their model permits that fare classes can substitute for one another. They formulate the overbooking model as a two-period optimization problem. In the first period the reservations are made by using only the probabilistic information of cancellations. In the second period, after observing the cancellations and no-shows, all the remaining customers are either assigned to a reserved seat or denied by considering the substitution options. They give the structural properties of the overall optimization problem, which turns out to be highly nonlinear. Therefore, they propose to apply a simulation based optimization method using stochastic gradients to solve the problem.

In all of the above models probability distributions are used to model uncertainty in demand and cancellations. Recent studies in revenue management focus on the availability of information. Adaptive methods are used when there exists no or limited information about the demand. Most of these methods assume that there is access only to samples from demand distributions. They mainly compute the booking limits based on the past information but also react to the possible inaccuracies related to the estimates of demand [89, 46]. Kunnumkal and Topaloglu [52] consider a capacity allocation problem with limited demand information and develop a stochastic approximation method to compute the optimal protection levels iteratively. They prove that the sequence of protection levels computed by using their approach converge to the optimal ones. Birbil et al. [28] present a robust version of static and dynamic single leg problems. In their model, they take into account the inaccuracies associated with the estimated probability distributions of the demand for different fare classes. Ball and Queyranne [8] use online algorithms to treat also a robust problem. In this way, they eliminate the need for estimating the demand and present the closed-form optimal booking limits. Lan et al. [58] generalize Ball and Queyranne's model by assuming that the demand for each fare class lies in a given interval. By using relative regret and absolute regret as performance criteria, they provide two capacity allocation models which differ in their objective functions. They show that these two models can be analyzed in a unified manner and both models provide nested booking limits. In a related work, Lan et al. [57] formulate a joint overbooking and seat allocation model, where both the random demand and no-shows are characterized using interval uncertainty. They focus on the seller's regret in not being able to find the optimal policy due to the lack of information. The regret of the seller is quantified by comparing the net revenues associated with the policy obtained before observing the actual demand and the optimal policy obtained under perfect information. The model aims to find a policy which minimizes the maximum relative regret.

In the present study, we work on the problem proposed by Aydin [5]. She discusses the static and dynamic overbooking problems and proposes several solution approaches. However, the proposed static models find the capacity allocation policy for a given overbooking limit. In other words, she computes the overbooking limit without considering the capacity allocation policy of multiple fare classes. In addition, the proposed models are computationally intractable for the large-scale problems. On the other hand, in the dynamic model she assumes that cancellation probability is linearly increasing with the number of reservations.

In this chapter, we discuss two static models which allow class-dependent cancellations and no-shows. The first model can be seen as a generalization of the single fare class model discussed in Phillips [69]. The second static model aims at determining both the total booking limit and the partitioned allocation of the virtual capacity to each fare class. Since the resulting problem is difficult to solve, we introduce computationally tractable approximate models. We also work on the error committed by solving the these approximate models instead of the originally proposed model. We then propose a discrete-time dynamic model based on independent streams of arrivals of booking requests and cancellations. Our modeling approach differs from the one based on a combined stream of events (Subramanian et al. [76]) by allowing the arrival and cancellation processes to be independent. In particular, we assume that requests for different fare classes arrive according to independent nonhomogeneous Poisson processes. Moreover, the number of cancellations in any time period, given that there are n number of accepted requests at the beginning of that time period, is a binomially distributed random variable with n independent trials and a period-dependent cancellation probability. Thus, as desired, the arrival process of the booking requests are independent of the number of reservations whereas the cancellation and no-show probabilities depend on the total number of reservations.

2.1 Static Overbooking Models

In this section, we propose two static risk-based overbooking models and analyze them in-depth to obtain efficient solution methods. The risk-based models try to determine a policy considering the trade-off between the potential revenue from accepting an additional request and the cost of an additional denied service. The objective of our first static model is to find an optimal booking limit maximizing the expected net revenue under the assumption that the greedy policy—that is, a request for any fare class is accepted as long as the total number of reservations is below the overbooking limit— is applied. In this model, the probabilistic information comes from the aggregate demand for all fare classes. However, we assume that each booking request belongs to a fare class with a certain probability. Finding an optimal total booking limit in this way is useful in practice, since the overbooking limit can be used as an input to some well-known allocation methods. This kind of heuristic approach first determines the total booking limit and then uses one of the well-known capacity allocation methods, like the famous EMSR heuristics [12, 13], to calculate the nested protection levels for different fare classes. In our second model, on the other hand, the probabilistic information is related to the demand for each fare class. We try to determine both the total booking limit and the partitioned allocation of the virtual capacity to each fare class in such a way that the expected net revenue is maximized. Since the second static model is quite hard to solve, we introduce two computationally tractable models that give upper and lower bounds on the proposed model's optimal expected net revenue.

In the subsequent discussion, we consider a flight with a known seat capacity of C and do not assume that the booking requests for different fare classes arrive in a certain order. In the first model, the booking requests for m different fare classes are accepted until the total booking limit $b \geq C$ is reached, whereas in the second model the booking decisions are based on the capacity allocated to each fare class. An accepted request becomes a reservation and a reservation may cancel at any time before departure or may not show up without cancelling. Let $\beta_i^s > 0$ denote the probability that an accepted fare class i request shows up at the departure time. For the remaining fare class i reservations, if we denote the probability of a cancellation by δ_i, then a fare class i reservation cancels with probability $\beta_i^c := (1 - \beta_i^s)\delta_i$. We assume that a fare class i cancellation is refunded a percentage α_i of the corresponding ticket price r_i, and no-shows do not receive any refund. If the number of shows exceeds the capacity C, then exactly C shows will be on the flight and the rest will be denied boarding. For each denied service, the airline incurs a denied service cost of $\theta > 0$. We refer the interested reader to Chatwin [23] for a discussion on fare class-independent compensation for a denied boarding. In our study, the total booking limit and the individual booking limits are allowed to be infinite; an infinite value corresponds to accepting all the booking requests. Let $\bar{\mathbb{Z}}_+ = \mathbb{Z}_+ \cup \{\infty\}$ denote the set of extended natural numbers. Aside from this notation, the random variables and the vectors are denoted by uppercase and lowercase boldface letters, respectively. If $\mathbf{X}$ and $\mathbf{Y}$ are random variables, then $\mathbf{X} =^d \mathbf{Y}$

means that the cumulative distribution functions of $\mathbf{X}$ and $\mathbf{Y}$ are identical. To simplify the exposition, we also denote $\max\{x, 0\}$ by $[x]^+$.

2.1.1 Total Booking Limit

In this section, we propose a model to determine a total booking limit $b \geq C$. We consider a model, where the probabilistic information is the random total booking requests, and denote this non-negative integer valued random variable by $\mathbf{D}$. We assume that $\mathbf{D}$ has a finite first moment and each booking request belongs to a certain fare class according to a multinomial selection mechanism with given probabilities. These probabilities can be estimated using historical data about the overall market share of each fare class. In particular, each arriving request is for fare class i with probability p_i, $i = 1, \ldots, m$. Clearly, $p_i \geq 0$ and $\sum_{i=1}^m p_i = 1$. Thus, we assume that the random fare class i demand, denoted by $\mathbf{D}_i$, has a binomial distribution with $\mathbf{D}$ independent trials and the success probability of p_i.

We first define a Bernoulli selection type random variable. If $\mathbf{X}$ denotes the non-negative integer random size of a population, then the random variable $\mathbf{B}(p, \mathbf{X})$ denotes the total number within the population of size $\mathbf{X}$ having a certain property under the condition that each member in the population has this property with probability p independent of each other. Hence, the random variable $\mathbf{B}(p, \mathbf{X})$ is given by

$$\mathbf{B}(p, \mathbf{X}) := \begin{cases} \sum_{k=1}^{\mathbf{X}} \mathbf{1}_{\{\mathbf{U}_k \leq p\}}, & \text{if } \mathbf{X} \geq 1; \\ 0, & \text{if } \mathbf{X} = 0, \end{cases} \tag{2.1}$$

where $\mathbf{U}_k, k \in \mathbb{N}$, is a sequence of independent standard uniformly distributed random variables, and the random variable $\mathbf{X}$ is independent of the sequence $\mathbf{U}_k,\ k \in \mathbb{N}$. By relation (2.1), we obtain

$$\mathbb{E}\left(\mathbf{B}(p, \mathbf{X})\right) = p\mathbb{E}\left(\mathbf{X}\right).$$

Furthermore, it is well-known that the generating function of the random variable $\mathbf{B}(p, \mathbf{X})$ is given by

$$\mathbb{E}\left(z^{\mathbf{B}(p,\mathbf{X})}\right) = \mathbb{E}\left((1 - p + pz)^{\mathbf{X}}\right) \tag{2.2}$$

and

$$\mathbf{B}\left(q, \mathbf{B}(p, \mathbf{X})\right) =^d \mathbf{B}(pq, \mathbf{X})$$

for any $0 \leq p, q \leq 1$ [35].

We consider the greedy policy of accepting a booking request for any fare class as long as the total booking limit b is not reached. Under this policy the random total number of reservations is given by $\mathbf{N}(b) := \min\{b, \mathbf{D}\}$. Let $\mathbf{D}_i^r$ designate the random number of reservations for fare class i. Since our policy accepts any request until the booking limit is reached, it is easy to prove the following lemma, which implies that the joint distribution of the random vector $(\mathbf{D}_1^r, \ldots, \mathbf{D}_m^r)$ follows a multinomial distribution with $\mathbf{N}(b)$ independent trials and the success probabilities $p_i, \ i = 1, \ldots, m$.

Lemma 2.1.1 *Under the greedy policy, it follows that* $\mathbf{D}_i^r =^d \mathbf{B}(p_i, \mathbf{N}(b))$.

Proof. Let $\mathbf{D}_i^r$ denote the random number of fare class i reservations. By the definition of the total booking limit b and the used policy, we obtain for every integer k satisfying $k \leq b - 1$ and $y \leq k$ that

$$\mathbb{P}(\mathbf{D}_i^r = y \mid \mathbf{N}(b) = k) = \mathbb{P}(\mathbf{D}_i^r = y \mid \mathbf{D} = k) = \binom{k}{y} p_i^y (1 - p_i)^{k-y}. \tag{2.3}$$

It also follows for every $y \leq b$ that

$$\begin{aligned} \mathbb{P}\left(\mathbf{D}_i^r = y \mid \mathbf{N}(b) = b\right) &= \mathbb{P}\left(\mathbf{D}_i^r = y \mid \mathbf{D} \geq b\right) \\ &= \frac{\mathbb{P}\left(\mathbf{D}_i^r = y, \mathbf{D} \geq b\right)}{\mathbb{P}\left(\mathbf{D} \geq b\right)} = \frac{\sum_{k=b}^{\infty} \mathbb{P}\left(\mathbf{D}_i^r = y, \mathbf{D} = k\right)}{\mathbb{P}\left(\mathbf{D} \geq b\right)} \\ &= \frac{\sum_{k=b}^{\infty} \mathbb{P}\left(\mathbf{D}_i^r = y \mid \mathbf{D} = k\right) \mathbb{P}\left(\mathbf{D} = k\right)}{\mathbb{P}\left(\mathbf{D} \geq b\right)} \qquad (2.4) \\ &= \frac{\sum_{k=b}^{\infty} \binom{b}{y} p_i^y \left(1 - p_i\right)^{b-y} \mathbb{P}\left(\mathbf{D} = k\right)}{\mathbb{P}\left(\mathbf{D} \geq b\right)} = \binom{b}{y} p_i^y (1 - p_i)^{b-y}. \end{aligned}$$

Applying now relations (2.3) and (2.4) yields the desired result. $\square$

As discussed at the beginning of Section 2.1, we distinguish between a no-show and a cancellation to obtain an explicit expression of the revenue obtained from each reservation. By Lemma 2.1.1 and the properties of the Bernoulli selection mechanism, the random number of fare class i shows and fare class i cancellations are given by $\mathbf{B}(\beta_i^s p_i, \mathbf{N}(b))$ and $\mathbf{B}(\beta_i^c p_i, \mathbf{N}(b))$, respectively, (c.f. [82, 22, 27] for similar uses of the Bernoulli selection scheme). Hence, for a given booking limit b the random total revenue generated by any fare class i reservation is given by

$$r_i \mathbf{B}(p_i, \mathbf{N}(b)) - \alpha_i r_i \mathbf{B}(\beta_i^c p_i, \mathbf{N}(b)),$$

where $\alpha_i r_i$ denotes the refund paid for a fare class i cancellation. Introducing now

$$\tau_i = r_i(1 - \alpha_i \beta_i^c), \qquad i = 1, \ldots, m, \tag{2.5}$$

the expected total revenue over all reservations becomes

$$\sum\nolimits_{i=1}^{m} p_i \tau_i \mathbb{E}(\mathbf{N}(b)). \tag{2.6}$$

To incorporate the penalty cost of overbooking, we first observe adding up all the shows that the total number of denied boardings equals

$$\left[\sum\nolimits_{i=1}^{m} \mathbf{B}(\beta_i^s p_i, \mathbf{N}(b)) - C\right]^+ .$$

Since the binomial random variables $\mathbf{B}(\beta_i^s p_i, \mathbf{N}(b)),\ i = 1, \ldots, m$, arise within a multinomial selection experiment with independent trials from the same population, we obtain

$$\left[\sum\nolimits_{i=1}^{m} \mathbf{B}(\beta_i^s p_i, \mathbf{N}(b)) - C\right]^+ =^d \left[\mathbf{B}\left(\sum\nolimits_{i=1}^{m} \beta_i^s p_i, \mathbf{N}(b)\right) - C\right]^+ . \tag{2.7}$$

Then, using relations (2.6) and (2.7) the expected net revenue is obtained as

$$\psi(b) := \sum\nolimits_{i=1}^{m} p_i \tau_i \mathbb{E}(\mathbf{N}(b)) - \theta \mathbb{E}\left(\left[\mathbf{B}\left(\sum\nolimits_{i=1}^{m} \beta_i^s p_i, \mathbf{N}(b)\right) - C\right]^+\right)$$

and the optimal booking limit is found by solving

$$\max\{\psi(b) : b \geq C,\ b \in \bar{\mathbb{Z}}_+\}. \tag{P_T}$$

To analyze the global properties of the function $b \mapsto \psi(b)$, we first observe that $\psi(b) = \mathbb{E}(f(\mathbf{N}(b)))$ with $f : \mathbb{Z}_+ \to \mathbb{R}$ given by

$$f(x) = \sum_{i=1}^{m} p_i \tau_i x - \theta \mathbb{E}\left(\left[\mathbf{B}\left(\sum_{i=1}^{m} \beta_i^s p_i, x\right) - C\right]^+\right). \tag{2.8}$$

In the next lemma we derive an important property of expectations of discrete concave functions of the random variable $\mathbf{B}(p, n)$.

Lemma 2.1.2 *If the function $g : \mathbb{Z}_+ \mapsto \mathbb{R}$ is discrete concave (convex), then the function $n \mapsto \mathbb{E}\left(g\left(\mathbf{B}(p, n)\right)\right)$ is also discrete concave (convex).*

Proof. We need to show that $n \mapsto \mathbb{E}\left(g\left(\mathbf{B}(p, n+1)\right)\right) - \mathbb{E}\left(g\left(\mathbf{B}(p, n)\right)\right)$ is decreasing (increasing). By the definition of $\mathbf{B}(p, n+1)$ given in relation (2.1) and the conditional expectation formula we obtain that

$$\begin{aligned} \mathbb{E}\left(g\left(\mathbf{B}(p, n+1)\right)\right) - \mathbb{E}\left(g\left(\mathbf{B}(p, n)\right)\right) &= p\mathbb{E}\left(g\left(\mathbf{B}(p, n+1)\right) - g\left(\mathbf{B}(p, n)\right) | \mathbf{U}_{n+1} \leq p\right) \\ &= p\mathbb{E}\left(g\left(1 + \mathbf{B}(p, n)\right) - g\left(\mathbf{B}(p, n)\right) | \mathbf{U}_{n+1} \leq p\right) \\ &= p\mathbb{E}\left(g\left(1 + \mathbf{B}(p, n)\right) - g\left(\mathbf{B}(p, n)\right)\right). \end{aligned} \tag{2.9}$$

Since $\mathbf{B}(p, n+1) \geq \mathbf{B}(p, n)$ and g is discrete concave (convex) we obtain that $n \mapsto g\left(1 + \mathbf{B}(p, n)\right) - g\left(\mathbf{B}(p, n)\right)$ is decreasing (increasing) and by relation (2.9) the result follows. □

For any non-negative random variable $\mathbf{D}$, we define the random variable $\mathbf{N}(n) = \min\{n, \mathbf{D}\}$.

Lemma 2.1.3 *If $f : \mathbb{Z}_+ \mapsto \mathbb{R}$ is a discrete concave function and $f(\infty) := \liminf_{n \uparrow \infty} f(n)$, an optimal solution of the optimization problem $\max\{f(n) : n \geq C,\ n \in \bar{\mathbb{Z}}_+\}$ is also an optimal solution of the problem $\max\left\{\mathbb{E}\left(f\left(\mathbf{N}(n)\right)\right) : n \geq C,\ n \in \bar{\mathbb{Z}}_+\right\}$.*

PROOF. The discrete concavity of f implies its discrete unimodality. If its unimodality point n_{opt} equals ∞, or equivalently, f is increasing, the desired result easily follows. On the other hand, if n_{opt} is finite, we obtain for every $n \geq n_{opt}$ that

$$f(n+1) \leq f(n) \tag{2.10}$$

and for every $n < n_{opt}$

$$f(n+1) \geq f(n). \tag{2.11}$$

By the definition of $\mathbf{N}(n)$ it follows that

$$f(\mathbf{N}(n+1)) - f(\mathbf{N}(n)) \quad = \quad (f(n+1) - f(n))\mathbf{1}_{\{\mathbf{D} \geq n+1\}}.$$

This shows

$$\mathbb{E}\left(f(\mathbf{N}(n+1)) - f(\mathbf{N}(n))\right) = (f(n+1) - f(n))\mathbb{P}(\mathbf{D} \geq n+1) \tag{2.12}$$

and by relations (2.10),(2.11) and (2.12) we obtain for every $n \geq n_{opt}$ that

$$\mathbb{E}\left(f\left(\mathbf{N}(n+1)\right)\right) \leq \mathbb{E}\left(f\left(\mathbf{N}(n)\right)\right)$$

and for every $n < n_{opt}$

$$\mathbb{E}\left(f\left(\mathbf{N}(n+1)\right)\right) \geq \mathbb{E}\left(f\left(\mathbf{N}(n)\right)\right).$$

Hence, n_{opt} is also an optimal solution of $\max\left\{\mathbb{E}\left(f\left(\mathbf{N}(n)\right)\right) : n \geq C,\ n \in \bar{\mathbb{Z}}_+\right\}$. $\square$

By Lemma 2.1.2 it follows that the function $x \mapsto \mathbb{E}([\mathbf{B}(\sum_{i=1}^{m} \beta_i^s p_i, x) - C]^+)$ is discrete convex, and this implies that the function $x \mapsto f(x)$ is discrete concave. Therefore, by Lemma 2.1.3 the optimal solution of

$$\max\{f(b) : b \geq C,\ b \in \bar{\mathbb{Z}}_+\}$$

coincides with the optimal solution of problem (P_T). Then, by using the discrete concavity of the function f, an optimal solution to (P_T) is given by

$$b_{opt} = \inf\{b \geq C : f(b+1) - f(b) < 0\}. \tag{2.13}$$

Here we use the convention that the infimum of the empty set is equal to infinity. Introduce $\beta^s := \sum_{i=1}^{m} \beta_i^s p_i$ and let $\mathbf{U}_k$, $k = 1, \ldots, b+1$, be a sequence of independent standard uniformly distributed random variables. Furthermore, let $\mathbf{1}_A$ be the indicator random variable of the event A, i.e, it takes value 1 if the event A occurs, and 0 otherwise. Then, by relation (2.8) and the representation of a binomial distributed random variable given in (2.1) we obtain for every $b \geq C$ that

$$\begin{aligned} f(b+1) - f(b) &= \sum_{i=1}^{m} p_i \tau_i - \theta \mathbb{E}\left(\mathbf{1}_{\{\mathbf{U}_{b+1} \leq \beta^s\}}\right) \mathbb{E}\left(\mathbf{1}_{\{\sum_{k=1}^{b} \mathbf{1}_{\{\mathbf{U}_k \leq \beta^s\}} \geq C\}}\right) \\ &= \sum_{i=1}^{m} p_i \tau_i - \theta \beta^s \mathbb{P}\left(\sum_{k=1}^{b} \mathbf{1}_{\{\mathbf{U}_k \leq \beta^s\}} \geq C\right) \\ &= \sum_{i=1}^{m} p_i \tau_i - \theta \beta^s \mathbb{P}\left(\mathbf{B}(\beta^s, b) \geq C\right). \end{aligned}$$

This shows using $\theta \beta^s > 0$ that

$$f(b+1) - f(b) < 0 \Leftrightarrow \mathbb{P}\left(\mathbf{B}(\beta^s, b) \geq C\right) > \frac{\mu_0}{\mu_1},$$

where

$$\mu_0 = \sum_{i=1}^{m} p_i \tau_i \text{ and } \mu_1 = \theta \beta^s. \tag{2.14}$$

Therefore, by using (2.13), the optimal solution to our optimization problem becomes

$$b_{opt} = \inf\left\{b \geq C : \mathbb{P}\left(\mathbf{B}(\beta^s, b) \geq C\right) > \frac{\mu_0}{\mu_1}\right\}. \tag{2.15}$$

A surprising consequence of this result is that the optimal total booking limit does not depend on the probability distribution function of the total demand $\mathbf{D}$. It is also easy to see that the optimal solution to our overbooking problem is to set $b = \infty$ when $\mu_0 - \mu_1 \geq 0$. An intuitive interpretation of this result is as follows: Since the expected net revenue per fare class i reservation is at least equal to $\tau_i - \theta\beta_i^s$, the expected net revenue per reservation is given by

$$\sum\nolimits_{i=1}^{m} p_i(\tau_i - \theta\beta_i^s) = \mu_0 - \mu_1.$$

This expression being non-negative shows that for the risk-based objective, it is always profitable to accept all requests despite the overbooking cost. Thus, the total booking limit should be set to infinity. When $\mu_0 - \mu_1 < 0$, there exists a finite optimal solution $b_{opt} \geq C$.

We next provide a computationally efficient iterative method to calculate the optimal total booking limit. To determine b_{opt}, we need to evaluate iteratively for $b \geq C$ the increasing sequence

$$\gamma_b = \mathbb{P}\left(\mathbf{B}(\beta^s, b) \geq C\right).$$

For $b = C$, it is obvious that

$$\gamma_C = \mathbb{P}\left(\mathbf{B}(\beta^s, C) \geq C\right) = (\beta^s)^C.$$

Then, we obtain the recursive relation

$$\gamma_{b+1} = \gamma_b + \beta^s \mathbb{P}\left(\mathbf{B}(\beta^s, b) = C - 1\right). \tag{2.16}$$

Our proposed overbooking model is related to the single fare class model discussed in Section 9.3.2 of Phillips [69]. Actually, the optimal booking limit of our model with multiple fare classes is equal to the booking limit obtained by the risk-based overbooking model with a single fare class, where the price is μ_0/β^s, the overbooking cost is θ and the show-up probability is β^s. In Section 9.4.2 of the same book, a heuristic is proposed to determine the total booking limit for multiple fare classes by reducing the problem to a single fare class model. Basically, this method first estimates the values of the parameters associated with a representative single fare class from the fare class-dependent parameters, and then, solves the resulting single fare class model. As a direct consequence of this estimation, only a heuristic method is obtained. Contrary to Phillips [69], we show that under a multinomial selection scheme linking the overall demand to the demand for each fare class and the policy of accepting all the requests until the total booking limit is reached, our proposed model determines the optimal total booking limit. From a different angle, we can state that our analysis provides the values of the price, show-up probability and overbooking cost parameters for which the heuristic proposed by Phillips is exact. As mentioned before, our model can be used to provide the overbooking limit to the capacity allocation heuristics like EMSR-a and EMSR-b. Since we allow class-dependent show-up probabilities, our model could perform better than those standard static models that determine the total overbooking limit when the show-up probabilities do not depend on the fare classes [69]. We note that the performance of the proposed model depends on the accuracy of the estimation of the model parameters. Among the parameters required to determine the optimal total booking limit (see (2.5),(2.14) and (2.15)), we acknowledge that the parameters p_i are the most challenging to estimate due to the non-availability of proper historical data. As emphasized in Talluri and van Ryzin [79], typically, the data on the arrivals is incomplete and only the purchase transaction data are available. In our case, suppose that the p_i parameters associated with more expensive fare classes, and consequently the parameter μ_0 in relation (2.14), are overestimated. Then, this shows by relation (2.15) that we may end up with a higher total booking value.

We conclude this section with two further remarks: (i) The first static model in the airline revenue management literature was proposed by Beckman [10]. He considers the cost minimization for a single fare class and provides a more complex analysis. He also observes that the overbooking limit decision does not depend on the demand distribution. His model can also be analyzed with our simpler approach. (ii) As it is common in the literature [76, 80], the expected total denied boarding cost may be given by an increasing convex function to represent the need to offer higher levels of compensation or incur higher goodwill costs for each additional denied boarding. Given the total booking limit b, this implies that for our model the denied boarding cost equals

$\mathbb{E}(c(\mathbf{N}(b)))$, where $c : \mathbb{Z}_+ \to \mathbb{R}$ is given by

$$c(x) = \mathbb{E}(g(\mathbf{B}(\sum\nolimits_{i=1}^{m} \beta_i^s p_i, x) - C))$$

and $g : \mathbb{R} \to \mathbb{R}$ is an increasing convex function satisfying $g(z) = 0$ for every $z \leq 0$. Again by Lemma 2.1.2 the function c is discrete convex, and consequently, the function $f : \mathbb{Z}_+ \to \mathbb{R}$ given by

$$f(x) = \sum\nolimits_{i=1}^{n} p_i \tau_i x - c(x)$$

is discrete concave. Therefore, as in the previous model, one can show that the optimal booking limit is in the form of (2.13).

2.1.2 Booking Limits for Individual Fare Classes

In this section we focus on a model, in which the partitioned booking limits as well as the overbooking limit are determined. This modeling approach sets us apart from other methods using capacity allocation heuristics, like EMSR-a and EMSR-b [12, 13], after setting the overbooking limit. However, it is important to note that a policy, which strictly maintains the partitioned booking limits, is rarely applied in practice because in such a dynamic setting it is clearly suboptimal to reject a higher fare class request even if there is available capacity for lower fare classes. Therefore, the partitioned booking limits are used to obtain nested booking limits or nested protection levels. Under a nested policy, higher fare classes are allowed to use all the capacity reserved for lower fare classes. From this perspective, whenever the optimal partitioned limits that are obtained in this section are used in a nested way, the resulting method becomes another heuristic but it does not require a predefined overbooking limit.

We assume that the distribution of the demand for fare class i, denoted by $\mathbf{D}_i$, is known and $\mathbb{E}(\mathbf{D}_i) < \infty$ for all $i = 1, \ldots, m$. If b_i is the partitioned booking limit for fare class i, then the random variable $\mathbf{N}_i(b_i) = \min\{b_i, \mathbf{D}_i\}$ denotes the number of reservations for fare class i. Using our notation in the previous section, the random number of fare class i reservations that show up at the departure time and the random number of fare class i cancellations are given by $\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i))$

and $\mathbf{B}(\beta_i^c, \mathbf{N}_i(b_i))$, respectively. Since the random total number of denied boardings is equal to $[\sum_{i=1}^m \mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - C]^+$, the expected net revenue $\phi(\mathbf{b})$ for a vector $\mathbf{b} = (b_1, \dots, b_m) \in \bar{\mathbb{Z}}_+^m$ is given by

$$\phi(\mathbf{b}) = \sum_{i=1}^m \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) - \theta \mathbb{E}\left(\left[\sum_{i=1}^m \mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - C\right]^+\right). \tag{2.17}$$

Thus, we need to solve the following problem to obtain the optimal partitioned booking limits:

$$\max\{\phi(\mathbf{b}) : \mathbf{b} \in \bar{\mathbb{Z}}_+^m\}. \tag{P_I}$$

Observe that $\sum_{i=1}^m b_i$ defines the overbooking limit and as suggested, the problem (P_I) provides the optimal overbooking limit and the optimal partitioned booking limits simultaneously. Unfortunately, due to the expected total overbooking cost, the expected total net revenue is not separable by the fare classes and this makes it difficult to solve the optimization problem (P_I) in an efficient way. Therefore, we consider lower and upper bounding functions on the expected total overbooking cost proposed by Aydin [5] and develop computationally efficient methods to find approximate solutions to problem (P_I).

To compute a lower bounding function on the total expected overbooking cost, we use Jensen's inequality which leads to

$$\begin{aligned}\mathbb{E}\left(\left[\sum_{i=1}^m \mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - C\right]^+\right) &\geq \left[\mathbb{E}\left(\sum_{i=1}^m \mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - C\right)\right]^+ \\ &= \left[\sum_{i=1}^m \beta_i^s \mathbb{E}\left(\mathbf{N}_i(b_i)\right) - C\right]^+.\end{aligned}$$

This shows by relation (2.17) that for every $\mathbf{b} \in \bar{\mathbb{Z}}_+^m$

$$\phi(\mathbf{b}) \leq \sum_{i=1}^m \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) - \theta\left[\sum_{i=1}^m \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i)) - C\right]^+ := \phi_U(\mathbf{b}).$$

Hence, an upper bound on the optimal objective value of problem (P_I) can be obtained by solving the optimization problem

$$\max\{\phi_U(\mathbf{b}) : \mathbf{b} \in \bar{\mathbb{Z}}_+^m\}. \tag{P_I^{UB}}$$

Although its objective function is not separable, it is still possible to use dynamic programming to solve the problem (P_I^{UB}). The main idea is to partition the set of integers into two sets. Let

$$S_1 = \left\{\mathbf{b} \in \bar{\mathbb{Z}}_+^m : \sum\nolimits_{i=1}^m \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i)) \geq C\right\} \text{ and } S_2 := \left\{\mathbf{b} \in \bar{\mathbb{Z}}_+^m : \sum\nolimits_{i=1}^m \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i)) \leq C\right\}.$$

Clearly, $S_1 \cup S_2 = \bar{\mathbb{Z}}_+^m$. Therefore, we have

$$\max\{\phi_U(\mathbf{b}) : \mathbf{b} \in \bar{\mathbb{Z}}_+^m\} = \max\left\{\max\left\{\phi_U(\mathbf{b}) : \mathbf{b} \in S_1\right\}, \max\left\{\phi_U(\mathbf{b}) : \mathbf{b} \in S_2\right\}\right\}.$$

Thus, to compute $\phi_U(\mathbf{b})$, we need to take the maximum of the objective function values of the following two optimization problems

$$\max\left\{\phi_U(\mathbf{b}) : \mathbf{b} \in S_1\right\} = \theta C + \max\left\{\sum\nolimits_{i=1}^m (\tau_i - \theta\beta_i^s)\mathbb{E}(\mathbf{N}_i(b_i)) : \mathbf{b} \in S_1\right\} \tag{2.18}$$

and

$$\max\left\{\phi_U(\mathbf{b}) : \mathbf{b} \in S_2\right\} = \max\left\{\sum\nolimits_{i=1}^m \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) : \mathbf{b} \in S_2\right\}. \tag{2.19}$$

Note that both problems (2.18) and (2.19) are separable and they can be solved by dynamic programming. However, we note that the implementation for solving problem (2.18) demands a special treatment. This is because of the greater-than-equal-to constraint, since one can check this constraint at each stage only when the bookings for all fare classes are known. To overcome this difficulty, we formulate (2.18) as a constrained shortest path problem and solve it using the well-known K-shortest path algorithm [95]. This algorithm returns successively the first K paths from origin to destination on a graph. We apply the same algorithm to return several paths in decreasing order of $\phi_U(\mathbf{b})$ values until we find the first one that satisfies the constraint in (2.18).

We also note that our upper bounding problem is similar to the approximate model proposed in [25, Section 2.3.4]. However, Chi [25] applies one more approximation to solve the resulting model, whereas we solve it to optimality.

Although we solve the problem with the K-shortest path algorithm, it is not computationally efficient. Next, we present a solution method based on a mixed-integer programming formulation, which is easier to follow and seems to be computationally more efficient as demonstrated by our numerical experiments. To restrict the feasible region of the problem (P_I^{UB}) and formulate it as a mixed-integer linear program, we have to introduce upper bounds on the booking limits.

Our objective is to restrict the feasible region of the upper bounding problem to a box. In other words, we introduce bounding constraints $b_i \leq M_i,\ i = 1, \dots, m$, in such a way that the error we make in calculating the objective function is significantly small. Our proposed approach is based on the next lemma.

Lemma 2.1.4 *Suppose that we consider the optimization problem* $\max\{h(\mathbf{b}) : \mathbf{b} \in \bar{\mathbb{Z}}_+^m\}$ *with*

$$h(\mathbf{b}) = \sum\nolimits_{i=1}^{m} f_i(b_i) - g(\mathbf{b}).$$

If the functions $f_i,\ i = 1, \dots, m$, and g are increasing and bounded, then for every $\epsilon > 0$ there exists a box B such that for every $\mathbf{b} \in \bar{\mathbb{Z}}_+^m$ one can find a vector $\hat{\mathbf{b}} \in B \subseteq \mathbb{Z}_+^m$ satisfying

$$h(\mathbf{b}) - h(\hat{\mathbf{b}}) \leq m\epsilon.$$

Proof. Since $\lim_{b \uparrow \infty} f_i(b_i) = f_i(\infty)$, there exists for every $\epsilon > 0$ some $b_i(\epsilon)$ such that

$$f_i(\infty) \leq f_i(b_i(\epsilon)) + \epsilon \quad \forall i = 1, \dots, m.$$

Consider the box $B = \{\mathbf{b} \in \bar{\mathbb{Z}}_+^m : b_i \leq b_i(\epsilon),\ i = 1, \dots, m\}$ and let $\mathbf{b} \notin B$. This shows that the set $I = \{1 \leq i \leq m \ : \ b_i > b_i(\epsilon)\}$ is nonempty and take $\hat{\mathbf{b}} = \{\hat{b}_1, \dots, \hat{b}_m\}$ with

$$\hat{b}_i = \begin{cases} b_i(\epsilon) & \text{if } i \in I \\ b_i & \text{otherwise} \end{cases}$$

Clearly $\hat{\mathbf{b}}$ belongs to B and $\mathbf{b} \geq \hat{\mathbf{b}}$. Using now the assumption that the functions $f_i,\ i = 1, \ldots, m$, and g are increasing and bounded we obtain

$$\begin{aligned} h(\mathbf{b}) - h(\hat{\mathbf{b}}) &= \textstyle\sum_{i=1}^{m}(f_i(b_i) - f_i(\hat{b}_i)) + g(\hat{\mathbf{b}}) - g(\mathbf{b}) \\ &\leq \textstyle\sum_{i=1}^{m}(f_i(\infty) - f_i(\hat{b}_i)) + g(\hat{\mathbf{b}}) - g(\mathbf{b}) \\ &\leq m\epsilon, \end{aligned}$$

and this shows the desired result. □

Observe that the objective function of the upper bounding problem can be written in the form of the function h given in Lemma 2.1.4:

$$\phi_U(\mathbf{b}) = \sum_{i=1}^{m} f_i(b_i) - g(\mathbf{b})$$

with

$$f_i(b_i) = \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) \text{ and } g(\mathbf{b}) = \theta \left[\sum_{i=1}^{m} \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i)) - C\right]^+. \qquad (2.20)$$

It is easy to see that the functions $f_i,\ i = 1, \ldots, m$, and g given in (2.20) are increasing. Since we assume that $\mathbb{E}(\mathbf{D}_i) < \infty$ for all $i = 1, \ldots, m$, the functions f_i and g are bounded. Thus, for a specified error term ϵ we can easily find some integer $b_i(\epsilon)$ satisfying

$$f_i(\infty) - f_i(b_i(\epsilon)) = \tau_i \mathbb{E}([\mathbf{D}_i - b_i(\epsilon)]^+) \leq \epsilon \quad \forall i = 1, \ldots, m.$$

Then, by Lemma 2.1.4, it is guaranteed that considering the feasible region $B = \{b \in \mathbb{Z}_+^m \ :\ b_i \leq b_i(\epsilon),\ i = 1, \ldots, m\}$ instead of $\{b \in \bar{\mathbb{Z}}_+^m\}$ would result in a deviation of at most $m\epsilon$ from the optimal objective function value, i.e., $\phi_U(\mathbf{b}) - \phi_U(\hat{\mathbf{b}}) \leq m\epsilon$ for any $\hat{\mathbf{b}} \in B$ and $\mathbf{b} \geq \hat{\mathbf{b}}$.

Before presenting the mathematical model, let us introduce the binary variables $x_{ij},\ i = 1 \ldots, m,\ j = 0, \ldots, M_i$, where $x_{ij} = 1$ and $x_{ij} = 0$ imply that $b_i = j$ and $b_i \neq j$, respectively. Then, calculating the input parameters $a_{ij} := \mathbb{E}(\mathbf{N}_i(j))$ for all $i = 1 \ldots, m,\ j = 0, \ldots, M_i$, we obtain an alternate formulation of the problem (P_I^{UB}):

$$
\begin{aligned}
\text{maximize} \quad & \sum\nolimits_{i=1}^{m} \tau_i \sum\nolimits_{j=0}^{M_i} a_{ij} x_{ij} - \theta w && (2.21)\\
\text{subject to} \quad & w \geq \sum\nolimits_{i=1}^{m} \beta_i^s \sum\nolimits_{j=0}^{M_i} a_{ij} x_{ij} - C, && (2.22)\\
& w \geq 0, && (2.23)\\
& \sum\nolimits_{j=0}^{M_i} x_{ij} = 1, \quad i = 1, \dots, m, && (2.24)\\
& x_{ij} \in \{0, 1\}, \quad i = 1, \dots, m, \ j = 0, \dots, M_i. && (2.25)
\end{aligned}
$$

By constraints (2.24)-(2.25) and the definition of parameters a_{ij}, it is guaranteed for each fare class i that exactly one of the binary variables $x_{i0}, x_{i1}, \dots, x_{iM_i}$ takes value 1 and $\sum_{j=0}^{M_i} a_{ij} x_{ij} = \mathbb{E}(\mathbf{N}_i(\sum_{j=1}^{M_i} j x_{ij}))$. Let $(\mathbf{x}^*, w^*)$ be an optimal solution of the problem (2.21)-(2.25). Constraints (2.22) and (2.23), and the structure of the objective function (2.21) ensure that

$$
w^* = [\sum\nolimits_{i=1}^{m} \beta_i^s \sum\nolimits_{j=0}^{M_i} a_{ij} x_{ij}^* - C]^+.
$$

Then, it is easy to show that the booking limits $b_i = \sum_{j=1}^{M_i} j x_{ij}^*$, $i = 1 \dots, m$, provide an optimal solution of the problem (P_I^{UB}) under additional bounding conditions. The number of binary variables is $\sum_{i=1}^{m} M_i \leq m \max\{M_1, \dots, M_m\}$. In practice, the number of fare classes is a reasonably small number for a single leg problem, and therefore, the proposed formulation can be very efficiently solved by a standard mixed integer programming solver such as CPLEX.

We note that restricting the feasible region by introducing sufficiently large bounds is not really a concern in determining the optimal policy. Having $b_i = M_i$ at the optimal solution of the problem (2.21)-(2.25) would imply that, in practice, all of the booking requests for fare class i are accepted, since M_i is in general a large number compared to the number of arriving booking requests. However, forcing $b_i \leq M_i$ leads to an error in calculating the objective function value, since the function $\mathbb{E}(\mathbf{N}_i(\cdot)) : \hat{\mathbb{Z}}_+ \to \mathbb{R}$ is increasing, and so $\mathbb{E}(\mathbf{N}_i(M_i)) < \mathbb{E}(\mathbf{N}_i(\infty))$. To this end, we provide an analysis to determine the upper bound values in such a way that the derivation from the optimal objective function value of the problem (P_I^{UB}) is at most $m\epsilon$ for a specified error tolerance ϵ.

To compare the quality of the revenue obtained with the approximate optimization problem (P_I^{UB}) against that provided by the optimization problem (P_I), we next find a lower bound on the optimal objective function of the problem (P_I). To compute an upper bounding function on the expected total overbooking cost, let $\mathbf{y} = (y_1, \dots, y_m) \in \mathbb{Z}_+^m$ with $\sum_{i=1}^m y_i = C$ be a partitioned allocation of available capacity C to each fare class. By the subadditivity of the function $x \mapsto [x]^+$, we observe that

$$\left[\sum\nolimits_{i=1}^m \mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - C\right]^+ = \left[\sum\nolimits_{i=1}^m \left(\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - y_i\right)\right]^+ \leq \sum_{i=1}^m \left[\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - y_i\right]^+.$$

Thus, for any partitioned allocation $\mathbf{y}$ such that $\sum_{i=1}^m y_i = C$, $y_i \in \mathbb{Z}_+$, we have

$$\mathbb{E}\left(\left[\sum\nolimits_{i=1}^m \mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - C\right]^+\right) \leq \sum\nolimits_{i=1}^m \mathbb{E}\left(\left[\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - y_i\right]^+\right),$$

and we obtain by relation (2.17) that

$$\phi(\mathbf{b}) \geq \sum\nolimits_{i=1}^m \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) - \theta \sum\nolimits_{i=1}^m \mathbb{E}\left(\left[\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - y_i\right]^+\right) := \phi_L(\mathbf{b}, \mathbf{y}). \tag{2.26}$$

Hence, a lower bound on the optimal objective value of the problem (P_I) is found by solving

$$\max\{\phi_L(\mathbf{b}, \mathbf{y}) : \sum\nolimits_{i=1}^m y_i = C,\ \mathbf{b} \in \bar{\mathbb{Z}}_+^m,\ \mathbf{y} \in \mathbb{Z}_+^m\}. \tag{P_I^{LB}}$$

Since the optimization problem (P_I^{LB}) is separable, it can be solved by dynamic programming. We first observe that the problem (P_I^{LB}) is equivalent to the optimization problem

$$\max\{\rho_L(\mathbf{y}) : \sum\nolimits_{i=1}^m y_i = C, \mathbf{y} \in \mathbb{Z}_+^m\}$$

with

$$\rho_L(\mathbf{y}) := \max\{\phi_L(\mathbf{b}, \mathbf{y}) : \mathbf{b} \in \bar{\mathbb{Z}}_+^m\}.$$

By the additivity of the function $\mathbf{b} \to \phi_L(\mathbf{b}, \mathbf{y})$ given in (2.26) it follows that

$$\rho_L(\mathbf{y}) = \sum_{i=1}^{m} \rho_i(y_i)$$

with

$$\rho_i(y_i) = \max\{\tau_i \mathbb{E}(\mathbf{N}_i(b_i)) - \theta \mathbb{E}(\,[\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - y_i]^+) : b_i \in \bar{\mathbb{Z}}_+\}.$$

Since the random variable $\mathbf{B}(\beta_i^s, \mathbf{N}_i(b))$ is bounded above by b and the function $b \to \tau_i \mathbb{E}(\mathbf{N}_i(b))$ is increasing, we can restrict the feasible region $\{b_i \in \bar{\mathbb{Z}}_+\}$ by adding the valid inequality $b_i \geq y_i$ and obtain

$$\rho_i(y_i) = \max\{\tau_i \mathbb{E}(\mathbf{N}_i(b_i)) - \theta \mathbb{E}(\,[\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i)) - y_i]^+) : b_i \geq y_i,\ b_i \in \bar{\mathbb{Z}}_+\}.$$

Observe that the above problem is in the form of the problem (P_T) presented in the previous section. Then, by using relation (2.15), the optimal solution of the above problem becomes

$$b_i^*(y_i) = \min\left\{ b \geq y_i : \mathbb{P}(\mathbf{B}(\beta_i^s, b) \geq y_i) > \frac{\tau_i}{\theta \beta_i^s} \right\}.$$

This yields

$$\rho_i(y_i) = \tau_i \mathbb{E}(\mathbf{N}_i(b_i^*(y_i)) - \theta \mathbb{E}(\,[\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i^*(y_i)) - y_i]^+). \tag{2.27}$$

Therefore, the problem (P_I^{LB}) boils down to a simple allocation problem

$$\max\left\{ \sum_{i=1}^{m} \rho_i(y_i) : \sum_{i=1}^{m} y_i = C, \mathbf{y} \in \mathbb{Z}_+^m \right\}$$

that can be solved by dynamic programming with a one-dimensional state space, where the stages correspond to the fare classes. The associated dynamic programming recursion can be formulated as follows: We consider for $j \in \{1, \ldots, m\}$ and $n \in \{0, 1, \ldots, C\}$, the parameterized optimization problems

$$R_j(n) = \max\left\{ \sum_{i=j}^{m} \rho_i(y_i) :\ \sum_{i=j}^{m} y_i = n,\ y_i \in \mathbb{Z}_+,\ i = j, \ldots, m \right\}. \tag{2.28}$$

By relation (2.28), the boundary condition for $n \in \{0, 1, \ldots, C\}$ becomes

$$R_m(n) = \rho_m(n).$$

Then, by the dynamic programming optimality principle, the recursive relation for every $j \in \{1, \ldots, m-1\}$ and $n \in \{0, 1, \ldots, C\}$ is given by

$$R_j(n) = \max \{\rho_j(y_j) + R_{j+1}(n - y_j) : \ y_j \leq n, \ y_j \in \mathbb{Z}_+\}.$$

Notice that this solution method requires evaluating the value of the function $\rho_i(y_i)$ given in (2.27) for all $i \in \{1, \ldots, m\}$ and $y_i \in \{0, 1, \ldots, C\}$. It is easy to find $b_i^*(y_i)$ using the recursive relation (2.16). Then, we need to efficiently calculate $\mathbb{E}(\,[\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i^*(y_i)) - y_i]^+)$ for all $y_i \in \{0, 1, \ldots, C\}$. To achieve this, we derive the distribution function of the bounded random variable $\mathbf{N}_i(b_i)$ and compute $P(\mathbf{B}(\beta_i^s, n) = k)$ for $n \in \{0, \ldots, b_i\}$ and $k \in \{0, \ldots, n\}$ using the following recursion:

$$P(\mathbf{B}(\beta_i^s, n) = k) = (1 - \beta_i^s) P(\mathbf{B}(\beta_i^s, n-1) = k) + \beta_i^s P(\mathbf{B}(\beta_i^s, n-1) = k-1)$$

with the boundary condition $P(\mathbf{B}(\beta_i^s, 0) = 0) = 1$.

We remark that the lower bounding problem (P_I^{LB}) has a nice interpretation. The decision maker first determines the y_i, $i = 1, \ldots, m$, values representing a partitioned allocation of the available capacity to each fare class. Then, the risk she takes is the possibility of observing that the total number of fare class i shows exceeds the preallocated capacity y_i, in which case she ends up paying a penalty cost. This means that a penalty is incurred even if a reservation occupies a preallocated seat belonging to a different fare class. With this interpretation, it is clear that by solving the problem (P_I^{LB}), we obtain a lower bound on the actual optimal expected total net revenue that would be secured by solving the actual problem (P_I).

As discussed in the beginning of this section, the practitioners prefer to use the partitioned booking limits in a nested way. Therefore, one can use the partitioned booking limits obtained by our lower and upper bounding models to calculate the nested booking limits, or equivalently, the nested protection levels that could be used in a dynamic setting. To be precise, the nested booking limit for fare class i is determined as $\sum_{j=1}^{i} b_j,\ i = 1, \dots, m$. In fact, this shall also be our approach in our computational study given in Section 2.3.

We also work on the error committed by solving (P_I^{LB}) or (P_I^{UB}) instead of the originally proposed problem (P_I). We partially answer this question in the case of the upper bounding problem by utilizing its continuous relaxation. We provide upper bounds on the error introduced by solving (P_I^{UB}). To derive these bounds we use the optimal function value of (P_I^{LB}) and the continuous relaxation of (P_I^{UB}) obtained by dropping the integrality restriction on the booking limits:

$$\max\{\sum_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) - \theta \left[\sum_{i=1}^{m} \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i)) - C\right]^+ \ :\ \mathbf{b} \in \bar{\mathbb{R}}_+^m\}, \tag{R_I^{UB}}$$

where $\bar{\mathbb{R}}_+ = \mathbb{R}_+ \cup \{\infty\}$ denotes the set of extended non-negative real numbers.

We first present some simple structural observations about the optimal solutions, then derive an exact analytical expression for them. This expression will allow us to obtain an upper bound on the error (introduced by solving (P_I^{UB})) solely in terms of the problem parameters.

Lemma 2.1.5 *Consider the index set* $I = \{i : \tau_i - \theta\beta_i^s \geq 0\}$ *and its complement* $I^C = \{1, \dots, m\} \setminus I$.

i. There exists optimal solutions $\mathbf{b}^*$ *and* $\mathbf{b}^{R*}$ *of the problems* (P_I^{UB}) *and* (R_I^{UB}) *such that* $b_i^* = b_i^{R*} = \infty$ *holds for every* $i \in I$.

ii. If $\sum_{i=1}^{m} \beta_i^s \mathbb{E}(\mathbf{D}_i) \leq C$ *holds, then* $\mathbf{b}^* = (\infty, \dots, \infty)$ *is an optimal solution of both* (P_I^{UB}) *and* (R_I^{UB}).

Proof. Since the proofs for (R_I^{UB}) are similar, we only prove the results for (P_I^{UB}). To show (i) we first observe that the objective function of (P_I^{UB}) can be written as follows:

$$\phi(\mathbf{b}) = \min\left(\sum_{i=1}^{m} (\tau_i - \theta\beta_i^s)\mathbb{E}(\mathbf{N}_i(b_i)) + \theta C, \quad \sum_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(b_i))\right). \tag{2.29}$$

Let $\hat{\mathbf{b}}$ be an arbitrary optimal solution of (P_I^{UB}) and consider the feasible solution $\mathbf{b}^* \in \bar{R}_+$ with

$$b_i^* = \begin{cases} \hat{b}_i & \text{if } i \in I^C \\ \infty & \text{if } i \in I. \end{cases}$$

Using the assumption that $\tau_i - \theta\beta_i^s \geq 0$ for all $i \in I$ we have

$$\begin{aligned} \textstyle\sum_{i=1}^m (\tau_i - \theta\beta_i^s)\mathbb{E}(\mathbf{N}_i(\hat{b}_i)) &\leq \textstyle\sum_{i\in I^C}(\tau_i - \theta\beta_i^s)\mathbb{E}(\mathbf{N}_i(\hat{b}_i)) + \sum_{i\in I}(\tau_i - \theta\beta_i^s)\mathbb{E}(\mathbf{D}_i) \\ &= \textstyle\sum_{i=1}^m (\tau_i - \theta\beta_i^s)\mathbb{E}(\mathbf{N}_i(b_i^*)). \end{aligned}$$

Similarly, by the positivity of the parameters τ_i, we have

$$\sum_{i=1}^m \tau_i \mathbb{E}(\mathbf{N}_i(\hat{b}_i)) \leq \sum_{i\in I^C} \tau_i \mathbb{E}(\mathbf{N}_i(\hat{b}_i)) + \sum_{i\in I} \tau_i \mathbb{E}(\mathbf{D}_i) = \sum_{i=1}^m \tau_i \mathbb{E}(\mathbf{N}_i(b_i^*)).$$

By plugging these inequalities into (2.29) we obtain $\phi(\hat{\mathbf{b}}) \leq \phi(\mathbf{b}^*)$, which shows that $\mathbf{b}^*$ is also an optimal solution, and proves our claim.

To show (ii) we observe that if $\sum_{i=1}^m \beta_i^s \mathbb{E}(\mathbf{D}_i) \leq C$ holds, the objective function of (R_I^{UB}) becomes $\sum_{i=1}^m \tau_i \mathbb{E}(\mathbf{N}_i(b))$. Since the coefficients τ_i are nonnegative and the mapping $b \mapsto \mathbb{E}(\mathbf{N}_i(b))$ is nondecreasing, our claim follows immediately. $\square$

In the next lemma, it is shown that an optimal solution of (R_I^{UB}) can also be easily obtained when $\sum_{i=1}^m \beta_i^s \mathbb{E}(\mathbf{D}_i) > C$.

Lemma 2.1.6 *Consider the continuous relaxation (R_I^{UB}) and assume that $\sum_{i=1}^m \beta_i^s \mathbb{E}(\mathbf{D}_i) > C$ holds.*

i. Suppose that the fare classes are indexed such that $\frac{\tau_1}{\beta_1^s} \geq \frac{\tau_2}{\beta_2^s} \geq \cdots \geq \frac{\tau_m}{\beta_m^s}$. If $\tau_i - \theta\beta_i^s \leq 0$ holds for all $i = 1, \ldots, m$, then an optimal solution is given by

$$b_i^{R*} = \begin{cases} \infty & \textit{if } 1 \leq i \leq k^* - 1 \\ \min\{b \in \bar{\mathbb{R}}_+ : \mathbb{E}(\mathbf{N}_{k^*}(b)) = y_{k^*}\} & \textit{if } i = k^* \\ 0 & \textit{if } k^* + 1 \leq i \leq m, \end{cases} \tag{2.30}$$

where

$$k^* = \min\{k \ : \ \sum_{i=1}^{k} \beta_i^s \mathbb{E}(\mathbf{D}_i) \geq C\}$$

and

$$y_{k^*} = \frac{C - \sum_{i=1}^{k^*-1} \beta_i^s \mathbb{E}(\mathbf{D}_i)}{\beta_{k^*}^s}.$$

We remark that $C \leq 0$ *implies* $\mathbf{b}^{R*} = (0, \ldots, 0)$*.*

ii. *As before, consider the index set* $I = \{i : \tau_i - \theta\beta_i^s \geq 0\}$ *and its complement* I^C*. There exists an optimal solution* $\mathbf{b}^{R*}$ *such that* $b_i^{R*} = \infty$ *holds for every* $i \in I$*, while* $(b_i^{R*})_{i \in I^C}$ *is an optimal solution of the following* residual problem*:*

$$\max\left\{\sum_{i\in I^c} \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) - \theta\left[\sum\nolimits_{i \in I^c} \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i)) - \hat{C}\right]^+ : b_i \in \bar{\mathbb{R}}_+, \ i \in I^c\right\}, \tag{2.31}$$

where $\hat{C} = C - \sum_{i \in I} \beta_i^s \mathbb{E}(\mathbf{D}_i)$*.*

Note that, since $\tau_i - \theta\beta_i^s \leq 0$ *holds for all* $i \in I^C$*, the residual problem has an optimal solution of the form described in part (i).*

Proof. (i) We first prove that the problem (R_I^{UB}) has an optimal solution $\mathbf{b}^{R*}$ which satisfies

$$\mu(\mathbf{b}^{R*}) := \sum\nolimits_{i=1}^{m} \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i^{R*})) = C. \tag{2.32}$$

Consider an arbitrary optimal solution $\hat{\mathbf{b}}$.

Case 1: $\mu(\hat{\mathbf{b}}) \leq C$. Since μ is continuous and nondecreasing on $\bar{\mathbb{R}}_+^m$, and $\mu\left((\infty, \ldots, \infty)\right) = \sum_{i=1}^m \beta_i^s \mathbb{E}(\mathbf{D}_i) > C$, there exists some $\mathbf{b}^{R*} \geq \hat{\mathbf{b}}$ satisfying $\mu(\mathbf{b}^{R*}) = C$. As the coefficients τ_i are nonnegative, and the mappings $b \mapsto \mathbb{E}(\mathbf{N}_i(b))$ are nondecreasing, we have

$$\phi_U(\mathbf{b}^{R*}) = \sum\nolimits_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(b_i^{R*})) \geq \sum\nolimits_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(\hat{b}_i)) = \phi_U(\hat{\mathbf{b}}).$$

It follows that $\mathbf{b}^{R*}$ is also an optimal solution, which proves our claim.

Case 2: $\mu(\hat{\mathbf{b}}) \geq C$. Similarly to the previous case, $\mu\left((0,\ldots,0)\right) = 0$ implies that there exists some $\mathbf{b}^{R*} \leq \hat{\mathbf{b}}$ satisfying $\mu(\mathbf{b}^{R*}) = C$. Using the assumption that $\tau_i - \theta\beta_i^s \leq 0$ for all $i = 1,\ldots,m$, we now have

$$\begin{aligned}
\phi_U(\mathbf{b}^{R*}) &= \sum\nolimits_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(b_i^{R*})) - \theta\left(\sum\nolimits_{i=1}^{m} \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i^{R*})) - C\right) \\
&= \sum\nolimits_{i=1}^{m} (\tau_i - \theta\beta_i^s)\mathbb{E}(\mathbf{N}_i(b_i^{R*})) + \theta C \\
&\geq \sum\nolimits_{i=1}^{m} (\tau_i - \theta\beta_i^s)\mathbb{E}(\mathbf{N}_i(\hat{b}_i)) + \theta C \\
&= \sum\nolimits_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(\hat{b}_i)) - \theta\left(\sum\nolimits_{i=1}^{m} \beta_i^s \mathbb{E}(\mathbf{N}_i(\hat{b}_i)) - C\right) = \phi_U(\hat{\mathbf{b}}).
\end{aligned}$$

It follows that $\mathbf{b}^{R*}$ is also an optimal solution, which proves our claim. By incorporating the valid equality (2.32) into (R_I^{UB}) we obtain the problem

$$\max\{\sum\nolimits_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) \ :\ \sum\nolimits_{i=1}^{m} \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i)) = C,\ \mathbf{b} \in \bar{\mathbb{R}}_+^m\}.$$

Since the mapping $b \mapsto \mathbb{E}(\mathbf{N}_i(b))$ is continuous and nondecreasing for all $i = 1,\ldots,m$, we can apply the change of variable $y_i = \mathbb{E}(\mathbf{N}_i(b_i))$. This leads to the continuous knapsack problem below, which has the same optimal objective value as (R_I^{UB}).

$$\max\left\{\sum\nolimits_{i=1}^{m} \tau_i y_i \ :\ \sum\nolimits_{i=1}^{m} \beta_i^s y_i = C,\ 0 \leq y_i \leq \mathbb{E}(\mathbf{D}_i)\right\}$$

According to the ordering of the indices specified in the statement of the lemma, the optimal solution of this problem is given by

$$y_i^* = \begin{cases} \mathbb{E}(\mathbf{D}_i) & \text{if } 1 \leq i \leq k^* - 1 \\ \dfrac{C - \sum_{i=1}^{k^*-1} \beta_i^s \mathbb{E}(\mathbf{D}_i)}{\beta_{k^*}^s} & \text{if } i = k^* \\ 0 & \text{if } k^* + 1 \leq i \leq m, \end{cases}$$

where $k^* = \min\{k :\ \sum_{i=1}^{k} \beta_i^s \mathbb{E}(\mathbf{D}_i) \geq C\}$. Applying the transformation $b_i^{R*} = \min\{b \in \bar{\mathbb{R}}_+ : \mathbb{E}(\mathbf{N}_i(b)) = y_i^*\}$ we obtain the optimal solution given in (2.30).

(ii) By part (ii) of Lemma 2.1.5, there exists an optimal solution $\mathbf{b}^{R*}$ such that $b_i^{R*} = \infty$ holds for every $i \in I$. Utilizing this result the problem (R_I^{UB}) becomes

$$\sum_{i\in I} \tau_i \mathbb{E}(\mathbf{D}_i) + \max\{\sum_{i\in I^c} \tau_i \mathbb{E}(\mathbf{N}_i(b_i)) - \theta \left[\sum_{i\in I^c} \beta_i^s \mathbb{E}(\mathbf{N}_i(b_i)) - \hat{C}\right]^+ : b_i \in \bar{\mathbb{R}}_+,\ i \in I^c\}.$$

By dropping the constant term $\sum_{i\in I} \tau_i \mathbb{E}(\mathbf{D}_i)$ we arrive at the residual problem (2.31).

□

As an immediate consequence of the above result, we obtain an analytical expression for the optimal objective value of (R_I^{UB}).

Corollary 2.1.1 *Suppose that the fare classes are indexed such that* $\frac{\tau_1}{\beta_1^s} \geq \frac{\tau_2}{\beta_2^s} \geq \cdots \geq \frac{\tau_m}{\beta_m^s}$. *Substituting the optimal solutions characterized in Lemma 2.1.6 into the objective function* ϕ_U *we obtain*

$$\phi_U(\mathbf{b}^{R*}) = \begin{cases} \sum_{i=1}^{m} \tau_i \mathbb{E}(\mathbf{D}_i) & \text{if } \sum_{i=1}^{m} \beta_i^s \mathbb{E}(\mathbf{D}_i) \leq C \\ \sum_{i\in I} (\tau_i - \theta\beta_i^s) \mathbb{E}(D_i) + \theta C & \text{if } \sum_{i\in I} \beta_i^s \mathbb{E}(\mathbf{D}_i) \geq C \\ \sum_{i=1}^{k^*-1} \tau_i \mathbb{E}(\mathbf{D}_i) + \tau_{k^*} y_{k^*} & \text{otherwise,} \end{cases} \tag{2.33}$$

where $k^* = \min\{k \ :\ \sum_{i=1}^{k} \beta_i^s \mathbb{E}(\mathbf{D}_i) \geq C\}$.

Lemma 2.1.6 states that according to the policy characterized by the solution of (R_I^{UB}), there is at most one fare class for which booking decisions are made based on a finite positive booking limit. The remaining fare classes are divided into two groups: for some classes all the booking requests are accepted, while for the others all the requests are rejected. Since the optimal solution of (R_I^{UB}) can have a fractional component associated with the index k^*, $\mathbf{b}^{R*}$ might not be a feasible solution to (P_I^{UB}). However, we can obtain a feasible solution by simple rounding:

$$\lfloor b_i^{R*} \rfloor := \begin{cases} \infty & \text{if } b_i^{R*} = \infty \\ \lfloor b_i^{R*} \rfloor & \text{otherwise.} \end{cases} \tag{2.34}$$

We next derive upper bounds on the error introduced by solving (P_I^{UB}) instead of (P_I).

LEMMA 2.1.7 *Suppose that* $\mathbf{b}^*$, $\mathbf{b}^{U*}$ *and* $\mathbf{b}^{R*}$ *denote the optimal solutions of the original problem (*P_I*), the upper bounding problem (*P_I^{UB}*), and the relaxed problem (*R_I^{UB}*), respectively. In addition,* $\lfloor \mathbf{b}^{R*} \rfloor$ *is defined as in* (2.34). *If* $\phi(\lfloor \mathbf{b}^{R*} \rfloor) > 0$*, then the following relations hold:*

$$0 \leq \frac{\phi_U(\mathbf{b}^{U*}) - \phi(\mathbf{b}^*)}{\phi(\mathbf{b}^*)} \leq \frac{\phi_U(\mathbf{b}^{R*}) - \phi(\mathbf{b}^*)}{\phi(\mathbf{b}^*)} \leq \frac{\phi_U(\mathbf{b}^{R*})}{\phi(\lfloor \mathbf{b}^{R*} \rfloor)} - 1. \tag{2.35}$$

Since $\phi_U(\mathbf{b}^{U^*}) \geq \phi(\mathbf{b}^*) \geq 0$, $\phi_U(\mathbf{b}^{U*}) \leq \phi_U(\mathbf{b}^{R*})$, and $\phi(\mathbf{b}^*) \geq \phi(\lfloor \mathbf{b}^{R*} \rfloor) > 0$, the assertion immediately follows.

Note that we have an analytical expression for the solution $\lfloor \mathbf{b}^{R*} \rfloor$ (see (2.30) and (2.34)). Thus, the upper bound given in Lemma 2.1.7 *depends solely on the problem data* and can be computed without performing optimization.

In our computational study, we have observed that $\phi(\lfloor \mathbf{b}^{R*} \rfloor)$ is positive for all the problem instances. If the condition $\phi(\lfloor \mathbf{b}^{R*} \rfloor) > 0$ is violated, we can utilize the optimal solution of (P_I^{LB}), which we denote by $(\mathbf{b}^{L*}, \mathbf{y}^{L*})$. Since $\phi_L(\mathbf{b}^{L*}, \mathbf{y}^{L*}) \geq 0$ and $\phi(\mathbf{b}^*) \geq \max\{\phi(\lfloor \mathbf{b}^{R*} \rfloor), \phi_L(\mathbf{b}^{L*}, \mathbf{y}^{L*})\}$, a generalized version of (2.35) becomes

$$0 \leq \frac{\phi_U(\mathbf{b}^{U*}) - \phi(\mathbf{b}^*)}{\phi(\mathbf{b}^*)} \leq \frac{\phi_U(\mathbf{b}^{R*}) - \phi(\mathbf{b}^*)}{\phi(\mathbf{b}^*)} \leq \frac{\phi_U(\mathbf{b}^{R*})}{\max\{\phi(\lfloor \mathbf{b}^{R*} \rfloor), \phi_L(\mathbf{b}^{L*}, \mathbf{y}^{L*})\}} - 1. \tag{2.36}$$

The remaining challenge is to compute $\phi(\lfloor \mathbf{b}^{R*} \rfloor)$ appearing in (2.35) and (2.36). Assuming that the random demands for fare classes, $\mathbf{D}_i$, $i = 1, \ldots, m$, are bounded and independent, for a given booking policy denoted by $\mathbf{b} \in \bar{\mathbb{Z}}_+^m$ we can numerically calculate the value of the $\phi(\mathbf{b})$ using the FFT method (see, e.g.,Tijms [83]). Basically, we need to compute numerically the distribution function of the bounded random variable

$$\mathbf{\Delta}(\mathbf{b}) := \sum\nolimits_{i=1}^m \mathbf{B}(\beta_i^s, \mathbf{N_i}(b_i)).$$

To achieve this, we compute the generating function of the random variable $\Delta(\mathbf{b})$. By the independence of the random demand variables $\mathbf{D}_i$, $i = 1, \ldots, m$, and hence, the independence of the random variables $\mathbf{N}_i(b_i), i = 1, \ldots, m$, and relation (2.2), we obtain the generating function as follows:

$$\begin{aligned}
\mathbb{E}(z^{\Delta(\mathbf{b})}) =& \Pi_{i=1}^{m} \mathbb{E}(z^{\mathbf{B}(\beta_i^s, \mathbf{N}_i(b_i))}) \\
=& \Pi_{i=1}^{m} \mathbb{E}\left((1 - \beta_i^s + \beta_i^s z)^{\mathbf{N}_i(b_i)}\right) \\
=& \Pi_{i=1}^{m} \mathcal{P}_i(1 - \beta_i^s + \beta_i^s z),
\end{aligned}$$

where $\mathcal{P}_i(w) := \mathbb{E}(w^{\mathbf{N}_i(b_i)})$. Notice that $\mathcal{P}_i(w)$ can be easily calculated for given distributions of the random demand variables $\mathbf{D}_i, i = 1, \ldots, m$. Since the random variable $\Delta(\mathbf{b})$ is bounded, we apply the standard FFT method for a finite sequence using $\mathbb{E}(z^{\Delta(\mathbf{b})})$ and obtain the distribution function of $\Delta(\mathbf{b})$. Then, we simply compute the challenging expectation $\mathbb{E}([\Delta(\mathbf{b}) - C]^+)$ appearing in the expected net revenue function ϕ.

2.2 Dynamic Overbooking Model

In this section, we discuss the dynamic overbooking problem, Since overbooking is allowed, the total number of reservations may exceed the actual capacity but the consequences, like denying boarding or departing with vacant seats, are faced at the time of departure. As time progresses during the reservation period the booking requests arrive randomly, and when a request arrives into the system we need to decide whether to accept or reject that request. The sequence of these accept or reject decisions leading to the highest net revenue is the optimal policy that we are after in this section.

2.2.1 Dynamics of The System

We introduce a discrete-time dynamic overbooking model, where time 0 represents the beginning of the reservation horizon and time T represents the departure time of the flight. The request arrivals only occur at discrete time points $t_k = kh$, $k = 1, \ldots, K-1$, with h being chosen sufficiently small, $T = Kh$, $K \in \mathbb{N}$, and $t_0 = 0$. At most one booking request occurs at each time period $I_k = [t_{k-1}, t_k)$.

A sample path of this discrete time arrival process is represented by a realization of a finite random vector $(\xi_1, \dots, \xi_{K-1})$, where $\xi_k = i$ designates that a request for fare class i arrives at time t_k, $i \in \{0, \dots, m\}$, $k = 1, \dots, K-1$. Note that a request for fare class 0 is also added to represent a no arrival at a given time point. The probability that a request for fare class i arrives at time t_k is $p_i(t_k) := \mathbb{P}(\xi_k = i)$, $i \in \{0, \dots, m\}$, $k = 1, \dots, K-1$. Clearly, $p_i(t_k) \geq 0$ and $\sum_{i=0}^{m} p_i(t_k) = 1$ for all time points $t_1, \dots, t_{K-1}$.

To model the cancellation process, we assume that each reservation, independently of other reservations, cancels in period I_k with probability $c(I_k)$, $k = 2, \dots, K$. Thus, the number of cancellations in period I_k, given that there are n accepted requests at time t_{k-1}, is a binomial distributed random variable $\mathbf{B}(c(I_k), n)$. Consequently, the number of accepted requests just before time t_k becomes $\mathbf{B}(1 - c(I_k), n)$. Observe that when

$$c(I_k) = 1 - \exp\left(-\lambda^c h\right),$$

the cancellation process is represented by a homogeneous Markovian death process with departure rate $\lambda^c > 0$, and hence, the cancellation probability does not depend on when the reservation was made. This property is coined as "forgetfulness property" and it is empirically confirmed to hold in practice [73].

As before r_i is the price of a fare class i ticket, $i = 1, \dots, m$. We also introduce $r_0 = 0$ to represent the price for the no-arrival case. Without loss of generality, we take $r_0 < r_1 < \dots < r_m$. We assume that each cancelled reservation receives a fixed refund of κ, and the airline incurs a fixed cost of θ for each denied boarding. At each time epoch t_k, we decide to accept or reject a possible request after the number of cancellations in the time interval I_k is realized. We might observe some no-shows just before the departure of the flight. It is assumed that the show-up probability of each reservation does not depend on its fare class, and it is denoted by β^s.

At this point we should note that some aspects of our model are covered by Subramanian et al. [76] and Chatwin [23]. Subramanian et al. consider the arrival of a cancellation, the arrival of a booking request and no-arrival of any type as a combined stream. That is, they assume that only a booking request, a cancellation or a null event (no booking request, no cancellation) can be realized at each time epoch. This implies that the arrival and cancellation events are dependent and hence the probability measure of the arrival process of requests depends implicitly on the total number of reservations. However, their discretization approach allows for the independence of these two stochastic processes up to a $o(h)$ error in the associated probabilities, where h is the length of each time interval. In other words, in the discrete time setting of their model the independence between the arrival and cancellation processes holds as h goes to zero. On the other hand, our approach avoids this technical issue by modeling the arrival and cancellation processes as two different streams and allows naturally the independence between these two stochastic processes. Moreover, our alternative modeling approach yields a simpler mathematical proof of the discrete concavity of the expected optimal net revenue as a function of the total number of reservations. Chatwin avoids the discretization approach and assumes that the overall arrival process of the requests is a continuous time homogeneous Poisson process, and the probabilities to identify the class of a request are independent of time. Under this assumption, the arrival processes of requests for different fare classes are independent homogeneous Poisson processes. Also he models the cancellation process as a homogenous Markovian death process, and therefore, (although Chatwin applies the Bellman-Jacobi differential approach) it is possible to use a regenerative approach to analyze his model. However, for nonhomogeneous stochastic processes it is more difficult to apply the Bellman-Jacobi or regenerative approach (essentially we need to use a two dimensional state space in our optimal control problem) and since the corresponding continuous optimal value equation needs to be solved by discretization, it seems to be more natural to start at the beginning with a discrete time nonhomogenous arrival process.

2.2.2 Analysis of The Proposed Model

We now present the detailed mathematical description of the proposed dynamic model. Let us denote by t_k^+ the time epoch just after an accept or reject decision for a request that arrives at time t_k, $k = 1, \ldots, K-1$. Similarly, the time epoch just after the departure of the flight is denoted by t_K^+. Let $J_k(n)$, $k = 1, \ldots, K-1$, denote the expected optimal net revenue from t_k^+ up to t_K^+ given that the number of reservations at t_k^+ is n. To determine $J_k(n)$, $n \in \mathbb{Z}_+$, $k = 1, \ldots, K-1$, we first observe that after an accept or reject decision at t_k yielding a total of n reservations at time t_k^+, the number of cancelled reservations in the interval I_{k+1} is a binomially distributed random variable $\mathbf{B}(c(I_{k+1}), n)$. Hence, the total number of reservations just before time t_{k+1} is $\mathbf{B}(1 - c(I_{k+1}), n)$. This implies that the total number of reservations just before the departure time is $\mathbf{B}(1 - c(I_K), n)$ and the total number of shows is given by $\mathbf{B}(\beta^s(1 - c(I_K)), n)$. Then, by $\mathbb{E}(\mathbf{B}(c(I_k), n)) = nc(I_k)$, the independence of the arrival and cancellation processes and the dynamic programming optimality principle we obtain for every $k = 1, \ldots, K-2$, and $n \in \mathbb{Z}_+$

$$
\begin{aligned}
J_k(n) &= -\kappa n c(I_{k+1}) + p_0(t_{k+1})\mathbb{E}\left(J_{k+1}(\mathbf{B}(1 - c(I_{k+1}), n))\right) \\
&\quad + \sum\nolimits_{i=1}^{m} p_i(t_{k+1})\mathbb{E}\left(\max\left\{r_i + J_{k+1}(\mathbf{B}(1 - c(I_{k+1}), n) + 1), J_{k+1}(\mathbf{B}(1 - c(I_{k+1}), n))\right\}\right)
\end{aligned}
\tag{P_{DM}}
$$

and the boundary condition

$$
J_{K-1}(n) = -\kappa n c(I_K) - \theta \mathbb{E}\left([\mathbf{B}(\beta^s(1 - c(I_K)), n) - C]^+\right). \tag{2.37}
$$

Clearly, for $n = 0$ we obtain $P(\mathbf{B}(1 - c(I_{k+1}), 0) = 0) = 1$, and the above recursion reduces to

$$
J_k(0) = p_0(t_{k+1})J_{k+1}(0) + \sum\nolimits_{i=1}^{m} p_i(t_{k+1}) \max\left\{r_i + J_{k+1}(1), J_{k+1}(0)\right\}.
$$

Next, we shall mention some results related to the discrete concavity (convexity) that are used in our analysis of the above model. We start with a definition.

Definition 2.2.1 *A function $f : \mathbb{Z}_+ \mapsto \mathbb{Z}$ is discrete concave if and only if the differences $n \mapsto f(n+1) - f(n)$ are decreasing. A function f is discrete convex if and only if -f is discrete concave.*

The proof of the following lemma is given by Lippman and Stidham [62].

LEMMA 2.2.1 *Let $r \geq 0$ and $f : \mathbb{Z}_+ \mapsto \mathbb{R}$ be a discrete concave function. Then the function $h : \mathbb{Z}_+ \mapsto \mathbb{R}$ given by $h(n) = \max\{r + f(n+1), f(n)\}$ is also discrete concave.*

We next obtain the optimal policy of the dynamic programming model (P_{DM}) by showing that the function $n \mapsto J_k(n)$ is a discrete concave function on $\mathbb{Z}_+$ for every $k = 1, \ldots, K-1$.

LEMMA 2.2.2 *The function $n \mapsto J_k(n)$ is discrete concave on $\mathbb{Z}_+$ for every $k = 1, \ldots, K-1$.*

PROOF. For ease of exposition we introduce the function $n \mapsto \Gamma_{k+1}(i, n)$ given by

$$\Gamma_{k+1}(i,n) := \begin{cases} \max\left\{r_i + J_{k+1}(n+1), J_{k+1}(n)\right\}, & \text{for } i \in \{1, \ldots, m\}; \\ \\ J_{k+1}(n), & \text{for } i = 0, \end{cases} \tag{2.38}$$

Then, the recursion of the dynamic model (P_{DM}) for every $k = 1, \ldots, K-2$, becomes

$$J_k(n) \quad = -\kappa n c(I_{k+1}) + \textstyle\sum_{i=0}^{m} p_i(t_{k+1}) \mathbb{E}\left(\Gamma_{k+1}\left(i, \mathbf{B}(1 - c(I_{k+1}), n)\right)\right). \tag{2.39}$$

Using Lemma 2.1.2, it follows that the function $n \mapsto J_{K-1}(n)$ listed in relation (2.37) is discrete concave on $\mathbb{Z}_+$. Suppose now for a given $k+1 < K$ that the function $n \mapsto J_{k+1}(n)$ is discrete concave on $\mathbb{Z}_+$. Our proof is then completed once we show that the function $n \mapsto J_k(n)$ is discrete concave on $\mathbb{Z}_+$. Applying our induction hypothesis and Lemma 2.2.1, we first obtain that the function $n \mapsto \Gamma_{k+1}(i, n)$ given in (2.38) is discrete concave for any $i \in \{0, 1 \ldots, m\}$. This implies using Lemma 2.1.2 that the function

$$n \mapsto \mathbb{E}\left(\Gamma_{k+1}\left(i, \mathbf{B}(1 - c(I_{k+1}), n)\right)\right)$$

is discrete concave on $\mathbb{Z}_+$ and by relation (2.39) the result follows. □

Let us now introduce

$$b_{ki} := \max \{ n \in \mathbb{Z}_+ : r_i \geq J_{k+1}(n) - J_{k+1}(n+1) \}.$$

Since a discrete concave function has decreasing differences by definition, it follows by Lemma 2.2.2 that the following dynamic booking limit policy is optimal:

"accept the request for fare class i at t_k $\Leftrightarrow$ total number of reservations $\leq b_{ki}$"

As the fares are assumed to be ordered, we then obtain the following nested structure:

$$b_{k1} \leq b_{k2} \leq \cdots \leq b_{km}.$$

2.3 Computational Experiments

We devote this section to a computational study for discussing different aspects of the models proposed in the previous sections. In particular, we conduct simulation experiments to benchmark the policies obtained with our lower bounding model (P_I^{LB}), upper bounding model (P_I^{UB}) and the dynamic model (P_{DM}) against some well-known approaches used in the literature [58, 57]. We next explain our simulation setup in detail and then present our numerical results.

2.3.1 Simulation Setup

We simulate the arrival of requests and cancellations over the discrete time points $t_k,\ k = 1, \ldots, K-1$. The probability that there is a request for fare class i at time point t_k is $p_i(t_k)$. If we accept a request for fare class i, then we generate a revenue of r_i. Without loss of generality, we take $r_0 < r_1 < \ldots < r_m$. Each accepted fare class i request cancels with probability $c_i(I_k)$ in period $I_k = [t_{k-1}, t_k),\ k = 2, \ldots, K$. Hence, the number of fare class i cancellations at time point t_k is binomially distributed with a success probability $c_i(I_{k+1})$. Each cancellation is refunded with an amount of $r_i\alpha_i,\ i = 1, \ldots, m$. At the end of the reservation period, each reservation shows up with probability β_i^s and the penalty cost of denying boarding to a reservation for fare class i is νr_i.

To generate these arrival and cancellation probabilities we shall mimic the actual stochastic processes. We assume that the booking requests arrive according to a homogeneous Poisson process with rate λ^a, and the cancellations for fare classes $i = 1, \ldots, m$, are modeled by a Markovian death process with departure rates λ_i^c. Then, we have for $k = 1, \ldots, K-1$

$$p_0(t_k) = \exp(-\lambda^a h)$$

and

$$c_i(I_k) = 1 - \exp\left(-\lambda_i^c h\right).$$

Given a request arrives at time t_k, this request is for fare class i with probability $f_i(t_k)$ satisfying, $f_i(t_k) \geq 0$ and $\sum_{i=1}^m f_i(t_k) = 1$. In other words, upon an arrival at time t_k, the different fare class requests are generated according to a multinomial selection scheme with probabilities $f_i(t_k)$, $i = 1, \ldots, m$, $k = 1, \ldots, K-1$. Assuming that in reality the lower fare class requests arrive more frequently in the early periods than the higher fare classes, we set the multinomial probabilities as

$$f_i(t_k) = \frac{\pi_i(t_k)}{\sum_{i=1}^m \pi_i(t_k)}, \ i = 1, \ldots, m,$$

where $\pi_i(t_k)$ are simple linear functions. This way of setting the multinomial probabilities complies with the desired demand pattern. As illustrated in Figure 2.1, we set

$$p_i(t_k) = f_i(t_k)(1 - p_0(t_k)), \quad i = 1, \ldots, m, \ k = 1, \ldots, K-1.$$

To obtain the optimal booking limits of static models, we need to compute the demand probabilities $P(\mathbf{D}_i = k)$ for all $i = 1, \ldots, m$ and $k = 0, \ldots, K-1$. Since the arrivals are independent across time periods, the total demand for fare class i is the sum of independent Bernoulli random variables with success probabilities $p_i(t_k)$, $k = 1, \ldots, K-1$. We obtain these demand distributions by applying the well-known Fast Fourier Transform (FFT) method [see, e.g., 83]. The distribution of the total demand

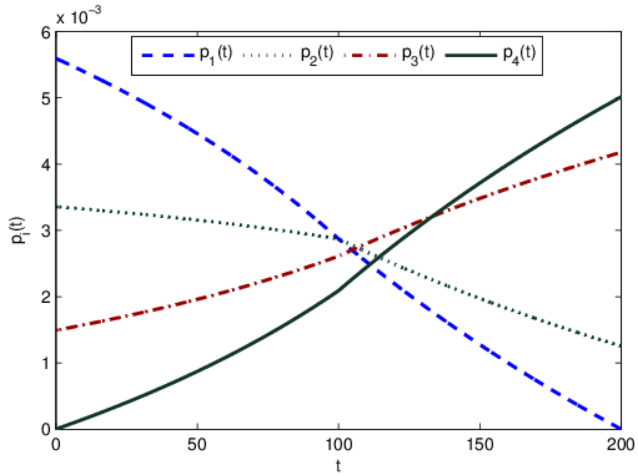


Figure 2.1: An example of the changes in multinomial probabilities over time

used by the EMSR-based heuristics is obtained by the FFT method as well, since the random demands for individual fare classes are also independent.

In our simulation setup, the following class-dependent parameters are given: fares (r_i), refund percentages (α_i), cancellation probabilities (β_i^c), and show-up probabilities (β_i^s). In order to test the performances of the booking policies against varying arrival intensities, we use the load factor parameter ρ, which is given by

$$\rho = \frac{(K-1)(1-\exp(-\lambda^a h))}{C}. \tag{2.40}$$

Observe that the numerator is the expected number of booking requests. To conform with our simulation setup, we tie the arrival rate to a given load factor and obtain λ^a by solving (2.40) for a specified value of ρ. When it comes to the cancellation rates, we assume that the behaviors of the customers towards cancellation are independent of whether they have reserved a ticket or not. Using this assumption and simple conditioning, we can relate the cancellation probabilities to the cancellation rates and acquire λ_i^c, $i = 1, \ldots, m$, from

$$\frac{\sum_{k=1}^{K-1}(1-\exp(-\lambda_i^c(T-t_k)))f_i(t_k)}{\sum_{k=1}^{K-1} f_i(t_k)} = \beta_i^c, \ i = 1, \ldots, m.$$

Then, we obtain the probabilities $p_i = \frac{\mathbb{E}(\mathbf{D}_i)}{\mathbb{E}(\mathbf{D})}$, $i = 1, \ldots, m$, denoting the fractions of the aggregate demand allocated to different fare classes.

Recall that in our dynamic model the cancellation and show-up probabilities do not depend on the fare classes. By applying a simple conditioning, we estimate the class-independent show-up and cancellation probabilities as

$$\beta^s = \sum_{i=1}^{m} \beta_i^s p_i \text{ and } \beta^c := \sum_{i=1}^{m} \beta_i^c p_i, \tag{2.41}$$

respectively. Using now the class-independent cancellation probability, we obtain the cancellation rate, λ^c by solving

$$\frac{\sum_{k=1}^{K-1}(1 - \exp(-\lambda^c(T - t_k)))}{K - 1} = \beta^c.$$

2.3.2 Numerical Results

In this section, we apply a benchmarking study including several approaches from the literature as well as our static and dynamic models. We also provide an experimental design, similar to the one in [86], for different parameters used in our simulation. All the contender methods that we use for benchmarking apply the EMSR-b heuristic but they mainly differ in terms of the way the virtual capacity is obtained:

- EMSR/Risk: Our total booking limit given by relation (2.15) is used as the virtual capacity.

- EMSR/MP: The virtual capacity is set according to the deterministic rule described by [11]. However, this rule requires a class-independent show up rate. Therefore, we use β^s as described at the end of the previous section and the virtual capacity is equal to C/β^s.

- EMSR/SL: The virtual capacity is based on a type-I service level constraint using the actual capacity. This constraint imposes that probability of overbooking is less than or equal to $1.0e - 3$ [69, Section 9.3].

- EMSR/NO: Overbooking is not allowed. Therefore, EMSR-b heuristic is applied with the actual capacity.

In the sequel, we simulate the arrival process for many replications and refer to the average revenues obtained by the optimal policies of our static models (P_I^{UB}) and (P_I^{LB}) as UB and LB, respectively. Likewise, we denote the average revenue of the dynamic policy obtained with our model (P_{DM}) by DM. We note once again that both of the static models provide partitioned booking limits but we use these limits in a nested way in all our simulations.

In all our numerical experiments, we set the capacity of the plane, the planning horizon, the discretization mesh lengths and the number of discrete time points to $C = 150$, $T = 200$, $h = 1.0e - 2$, $K = 20,000$, respectively. The refund percentages $(\alpha_1, \ldots, \alpha_m)$ and the cancellation probabilities $(\beta_1^c, \ldots, \beta_m^c)$ are evenly distributed in the intervals $[0.00, 0.30]$ and $[0.05, 0.17]$. For our dynamic programming implementation to solve the DP model, an upper bound sufficiently larger than C was imposed on the total number of reservations. This allows us to restrict the state space for computational purposes. In the implementation for solving the DP model, setting such an upper bound means that a booking request would be rejected if the total number of reservations reaches this upper bound. As required by formulation (2.21)-(2.25), we also need to impose an upper bound M_i on the booking limit b_i for each $i = 1, \ldots, m$. To serve this purpose, we choose sufficiently large M_i values by setting $\epsilon = 1.0e - 7$ in Lemma 2.1.4.

Our experimental design is based on various factors of the fares (r_i), the overbooking cost θ, the load factor ρ, the number of fare classes m, and the show-up probabilities (β_i^s). The lowest price is fixed to 50 and the prices of the other fare classes are evenly distributed in the interval $[50, \eta 50]$, where $\eta \in \{4, 7\}$ gives two sets of fares. For the proposed static and dynamic models the class-independent overbooking cost is determined by

$$\theta = \nu \sum_{i=1}^{m} r_i p_i,$$

where $\nu \in \{3, 5\}$ is used to create two factors indicating low and high overbooking costs. We use load factor values $\rho \in \{1.4, 1.8\}$ corresponding to medium and high loads. We also apply sensitivity analysis with respect to the number of fare classes selected as $m \in \{4, 8\}$. The last parameter set comes from the show-up probabilities $\boldsymbol{\beta}^s_\bullet := (\beta^s_1, \ldots, \beta^s_m)$. We give two sets of show-up probabilities to represent possibly low and high show-up rates. These are $\boldsymbol{\beta}^s_L := (0.95, 0.92, 0.80, 0.77)$ and $\boldsymbol{\beta}^s_H := (0.98, 0.95, 0.83, 0.80)$ for $m = 4$; $\boldsymbol{\beta}^s_L := (0.95, 0.93, 0.91, 0.89, 0.83, 0.81, 0.79, 0.77)$ and $\boldsymbol{\beta}^s_H := (0.98, 0.96, 0.94, 0.92, 0.86, 0.84, 0.82, 0.80)$ for $m = 8$. Under this setup, we evaluate the solutions of the contender approaches for all 32 test problem instances. Then, the policies obtained by these solutions are compared for each instance by taking 50 simulation runs.

Table 2.1 presents the optimal objective function values of (P_I^{UB}) and (P_I^{LB}), where $\mathbf{b}^{\mathbf{U}*}$ and $(\mathbf{b}^{\mathbf{L}*}, \mathbf{y}^{\mathbf{L}*})$ denote their optimal solutions, respectively. The last column gives the percentage gap between the objective function values of these two bounding problems. As seen from this table, the relative differences are mostly affected by the number of fare classes. Recall that (P_I^{LB}) partitions the actual capacity to each fare class and incurs a penalty even if a reservation occupies a preallocated seat belonging to a different fare class. This treatment of the capacity does not allow sharing the seats among the fare classes efficiently. Consequently, the performance of (P_I^{LB}) deteriorates more than that of (P_I^{UB}) and the percentage gap increases with a higher number of fare classes. The results also depict that the optimal objective function value of (P_I^{LB}) decreases slightly as the overbooking cost coefficient ν gets higher. On the other hand, the change in the optimal objective function value of (P_I^{UB}) is even less significant when the overbooking cost becomes higher. Consequently, the percentage gap tends to increase with ν; nonetheless, this change is quite minor. Regarding the impact of varying class fares, we observe that the optimal objective function values of both models increase as the parameter η becomes larger. However, the increase in the optimal objective function value is larger for (P_I^{LB}) compared to (P_I^{UB}). Therefore, the percentage gap decreases as η gets larger.

Table 2.1: The optimal objective function values of P_I^{UB} and P_I^{LB}

Instances					(a)	(b)	
m	ρ	$\boldsymbol{\beta}^s_\bullet$	η	ν	$\phi_L(\mathbf{b}^{\mathbf{L}*}, \mathbf{y}^{\mathbf{L}*})$	$\phi_U(\mathbf{b}^{\mathbf{U}*})$	$((b)-(a))/(a)\%$
4	1.4	$\boldsymbol{\beta}^s_H$	4	3	21,444.88	22,815.67	6.39%
			4	5	21,337.41	22,815.67	6.93%
			7	3	35,601.98	37,265.54	4.67%
			7	5	35,464.30	37,268.98	5.09%
		$\boldsymbol{\beta}^s_L$	4	3	21,654.65	23,071.37	6.54%
			4	5	21,528.50	23,071.38	7.17%
			7	3	35,834.72	37,527.54	4.72%
			7	5	35,702.62	37,527.54	5.11%
	1.8	$\boldsymbol{\beta}^s_H$	4	3	24,434.11	26,106.48	6.84%
			4	5	24,186.78	26,106.48	7.94%
			7	3	41,014.06	43,672.63	6.48%
			7	5	40,618.44	43,672.63	7.52%
		$\boldsymbol{\beta}^s_L$	4	3	24,904.36	26,674.35	7.11%
			4	5	24,622.59	26,674.35	8.33%
			7	3	41,714.30	44,537.86	6.77%
			7	5	41,277.23	44,537.86	7.90%
8	1.4	$\boldsymbol{\beta}^s_H$	4	3	20,403.45	22,657.20	11.05%
			4	5	20,215.49	22,657.20	12.08%
			7	3	33,653.33	36,990.55	9.92%
			7	5	33,396.48	36,990.55	10.76%
		$\boldsymbol{\beta}^s_L$	4	3	20,670.32	23,053.38	11.53%
			4	5	20,436.40	23,053.38	12.81%
			7	3	34,022.66	37,502.04	10.23%
			7	5	33,702.95	37,502.04	11.27%
	1.8	$\boldsymbol{\beta}^s_H$	4	3	23,141.81	25,606.24	10.65%
			4	5	22,873.36	25,606.24	11.95%
			7	3	38,817.54	42,726.61	10.07%
			7	5	38,399.12	42,726.61	11.27%
		$\boldsymbol{\beta}^s_L$	4	3	23,542.35	26,135.89	11.02%
			4	5	23,209.54	26,137.07	12.61%
			7	3	39,384.75	43,504.99	10.46%
			7	5	38,917.66	43,502.70	11.78%

Figures 2.2 to 2.5 present average net revenues over all simulation runs for the booking policies obtained by the different methods for varying factors. In these figures, we compare the performances of the booking policies obtained by our proposed models to those of the benchmarking methods with respect to high and low show-up probabili-

ties (denoted by H and L) and the overbooking cost coefficient, ν. The details of these figures are given in Table 2.2, where the revenue obtained by the dynamic model is used as a base approach to report the relative performances of the remaining approaches. Figures 2.6 and 2.7 depict the number of seats overbooked by various methods averaged over all simulation runs. In these figures, the number of fare classes is fixed, while the parameters $(\rho, \boldsymbol{\beta}^s_\bullet, \eta, \nu)$ vary.

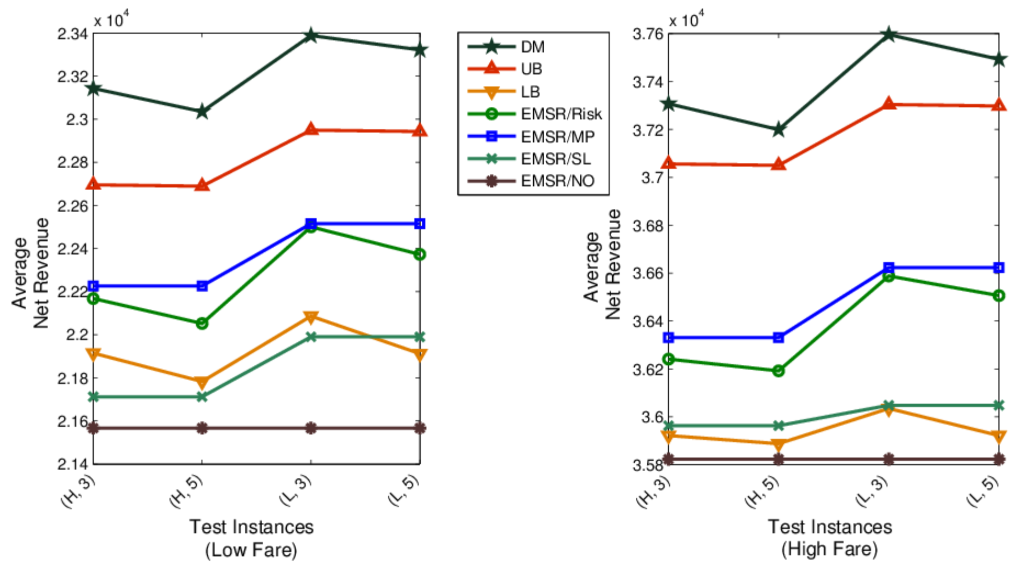


Figure 2.2: Average net revenues ($\rho = 1.4$, $m = 4$)

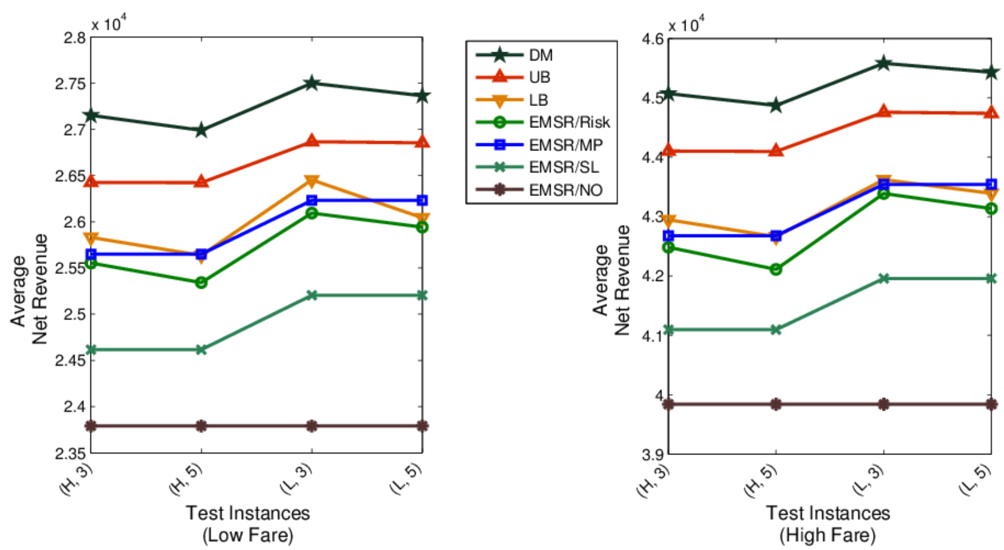


Figure 2.3: Average net revenues ($\rho = 1.8$, $m = 4$)

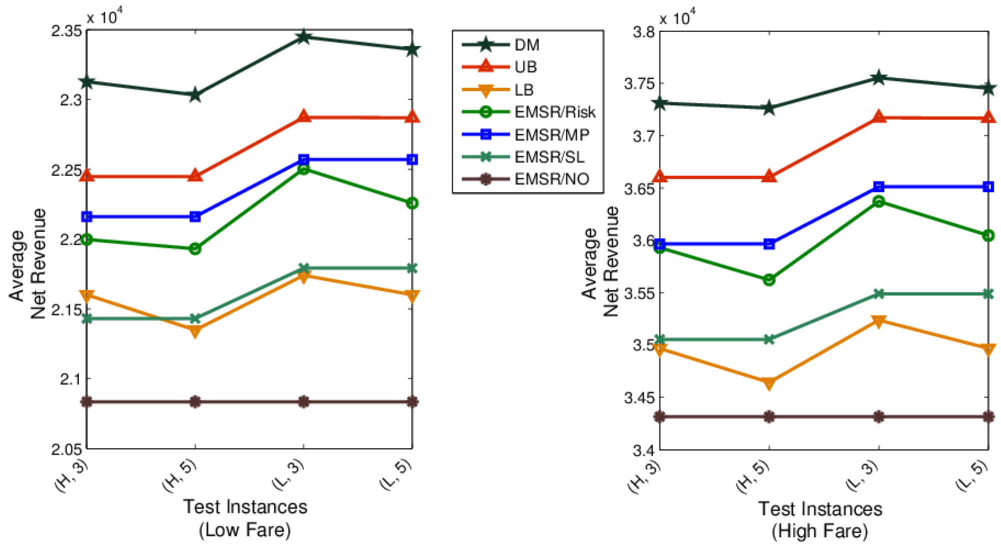


Figure 2.4: Average net revenues ($\rho = 1.4$, $m = 8$)

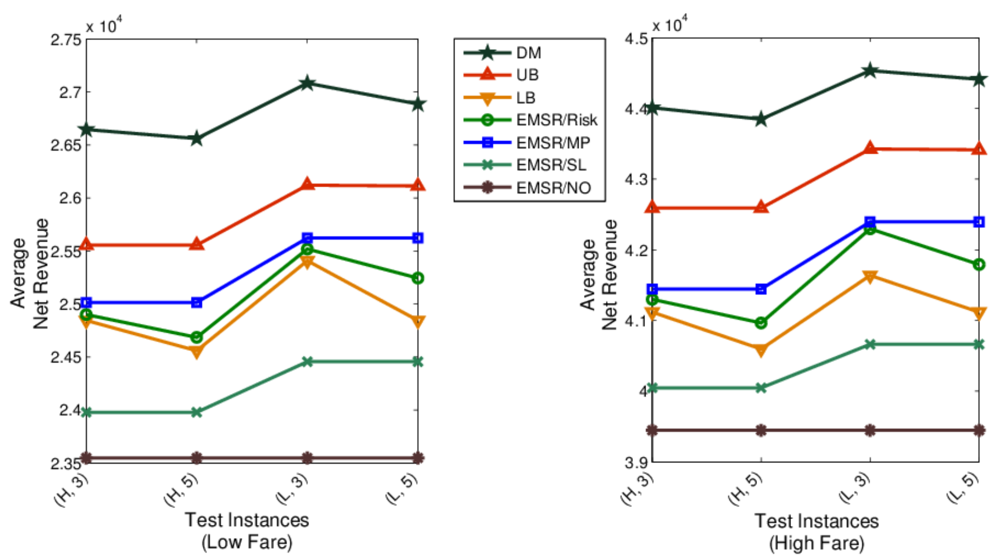


Figure 2.5: Average net revenues ($\rho = 1.8$, $m = 8$)

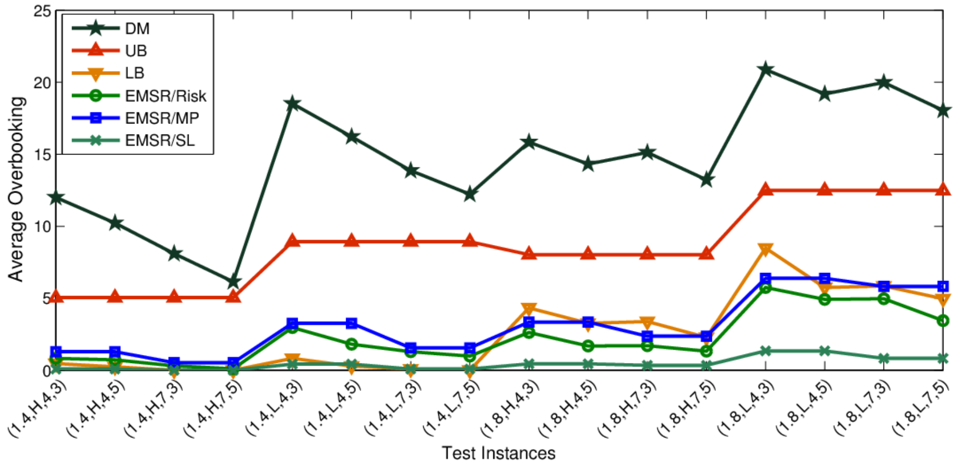


Figure 2.6: Average overbooking amount ($m = 4$)

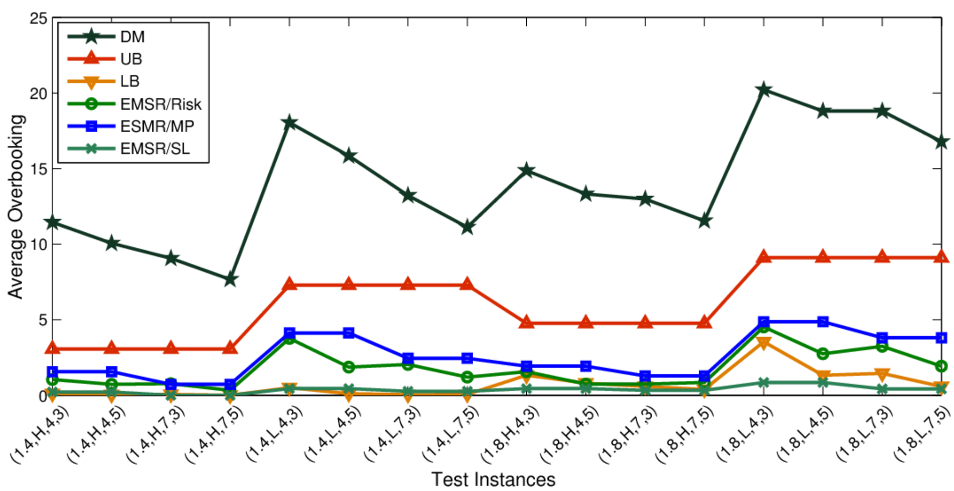


Figure 2.7: Average overbooking amount ($m = 8$)

The first observation we have is that the proposed upper bounding model (P_I^{UB}) performs better than all the EMSR-based heuristics for any combination of the parameters (Figures 2.2 to 2.5). On the other hand, Figures 2.6 and 2.7 show that the upper bounding model (P_I^{UB}) overbooks on average more seats than the other solution methods. However, this excess overbooking compensates for the revenue loss due to empty seats. We also observe the cases where the average revenues of the booking policies obtained by (P_I^{UB}) and (P_{DM}) can become relatively close. We caution the reader that these relatively small gaps between DM and UB implicitly demonstrates the importance of considering class-dependent show-up and cancellation probabilities. Lacking this consideration, the dynamic model treats all cancellations and no-shows the same way, and consequently, may fail to capture the actual dynamics of the system. As Figures 2.3-2.5 illustrate, the lower bounding problem (P_I^{LB}) performs slightly better when the load factor is high. As we mentioned before, (P_I^{LB}) is more conservative than the upper bounding problem and its overbooking policy is based on reserving more seats only for the expensive fare classes. Therefore, when the load-factor is high, it benefits from the increase in the number of booking requests for the expensive fare classes and it makes more overbooking. Comparing the plots for (P_I^{LB}) in Figures 2.2 and 2.4 with those in Figures 2.3 and 2.5, we note that the average revenue obtained by solving (P_I^{LB}) is closer to the revenue obtained by EMSR/SL for the lower load-factor value. However, it performs better and the average revenues as well as the number of overbooked seats stay close to those of EMSR/Risk and EMSR/MP when the load factor is high. As depicted in Figures 2.6 and 2.7, even there are instances when (P_I^{LB}) overbooks more seats than EMSR/MP and EMSR/Risk on average. These instances correspond to the cases where (P_I^{LB}) outperforms both EMSR/MP and EMSR/Risk. However, when the number of fare classes increases, the performance of (P_I^{LB}) deteriorates even if the load factor is high (see Figures 2.3 and 2.5).

When we look into the performances of the EMSR-based heuristics, we observe that EMSR/Risk and EMSR/MP are better than the remaining two heuristics. This difference is more striking when the load factor is high and the show-up probabilities are low as designated by Figures 2.3 and 2.5 (see also the rows corresponding to $\boldsymbol{\beta}_L^s$

in Table 2.2). This behavior can be attributed to the impact of overbooking. As illustrated by Figures 2.6 and 2.7, the differences between the average number of seats overbooked by the different EMSR-based heuristics are more significant when show-up probabilities are low. In those cases EMSR/Risk and EMSR/MP benefit more than EMSR/SL from the extra revenue gained by the overbooked seats. The average revenue obtained by EMSR/MP is slightly higher than that of EMSR/Risk. Unlike EMSR/Risk, EMSR/MP does not consider the overbooking penalty when determining the virtual capacity. Therefore, the difference between the average revenues of the policies obtained by these models increases with the overbooking cost factor. It turns out that the proposed weighted average of the class-dependent show-up rates given in relation (2.41) captures the nature of the show-up behavior accurately. We observe in our numerical study that EMSR/MP reserves slightly more seats than EMSR/RISK (at most 3 seats over all instances), and these additional seats are effective for collecting extra revenues from overbooking. This success of EMSR/MP is also in accordance with the observation made in [69, Section 9.3].

Table 2.2 and Figures 2.2 to 2.5 illustrate that, like our bounding models, the performances of the EMSR-based heuristics deteriorate with respect to the dynamic model with a higher number of fare classes. The deterioration in the performances of the EMSR-based heuristics can be explained by the fact that these heuristics are mainly based on comparing two fare classes. To obtain such a structure, each fare-class is compared against the aggregation of the classes with lower fares. As the number of fare-classes increases, the aggregation does not capture the stochastic nature of the problem well. It is also important to note that the percentage gaps between DM and the revenues of the remaining strategies are more striking when the load factor is high. This can be due to the reactions of the models to the low fare class requests, especially, in the early periods. As the load factor becomes higher, we observe many requests throughout the planning horizon. The dynamic policy then reacts in a more conservative way and rejects the early low fare requests. Such behaviour allows reserving seats for more expensive fare classes arriving in later periods, and hence, results with an increase in the total revenue. However, working with aggregate demands, the static models cannot react to the changes within different time intervals. Moreover, unlike the static models, the dynamic model adjusts the booking limits by taking into account the reservations and cancellations that have already taken place. It ends up overbooking more than the static solution methods, and consequently, the revenue loss due to empty seats is counteracted by the gains from the overbooked seats.

Table 2.2: Percentage differences relative to the expected net revenue of P_{DM}

Instances					DM *versus*					
m	ρ	$\boldsymbol{\beta}^s_\bullet$	η	ν	EMSR/NO	EMSR/SL	EMSR/MP	EMSR/Risk	LB	UB
4	1.4	$\boldsymbol{\beta}^s_H$	4	3	6.91%	6.29%	4.06%	4.31%	5.39%	2.02%
			4	5	6.46%	5.84%	3.59%	4.35%	5.52%	1.58%
			7	3	4.04%	3.67%	2.67%	2.93%	3.78%	0.73%
			7	5	3.76%	3.39%	2.39%	2.75%	3.59%	0.45%
		$\boldsymbol{\beta}^s_L$	4	3	7.88%	6.08%	3.82%	3.88%	5.66%	1.95%
			4	5	7.61%	5.80%	3.54%	4.14%	6.13%	1.69%
			7	3	4.79%	4.20%	2.65%	2.75%	4.23%	0.83%
			7	5	4.53%	3.93%	2.38%	2.71%	4.26%	0.57%
	1.8	$\boldsymbol{\beta}^s_H$	4	3	12.48%	9.43%	5.61%	5.96%	4.94%	2.75%
			4	5	11.96%	8.89%	5.04%	6.19%	5.10%	2.19%
			7	3	11.71%	8.92%	5.39%	5.83%	4.80%	2.23%
			7	5	11.32%	8.52%	4.97%	6.26%	5.01%	1.81%
		$\boldsymbol{\beta}^s_L$	4	3	13.58%	8.43%	4.67%	5.08%	3.87%	2.37%
			4	5	13.16%	7.98%	4.21%	5.28%	4.91%	1.94%
			7	3	12.69%	8.01%	4.54%	4.88%	4.37%	1.87%
			7	5	12.40%	7.71%	4.23%	5.12%	4.56%	1.59%
8	1.4	$\boldsymbol{\beta}^s_H$	4	3	10.01%	7.42%	4.25%	4.95%	6.67%	3.02%
			4	5	9.64%	7.04%	3.85%	4.86%	7.38%	2.61%
			7	3	8.09%	6.11%	3.66%	3.75%	6.32%	1.94%
			7	5	7.97%	5.98%	3.54%	4.45%	7.07%	1.82%
		$\boldsymbol{\beta}^s_L$	4	3	11.23%	7.14%	3.81%	4.09%	7.35%	2.54%
			4	5	10.90%	6.79%	3.45%	4.78%	7.59%	2.18%
			7	3	8.68%	5.54%	2.82%	3.18%	6.20%	1.07%
			7	5	8.43%	5.29%	2.55%	3.79%	6.67%	0.81%
	1.8	$\boldsymbol{\beta}^s_H$	4	3	11.72%	10.12%	6.22%	6.65%	6.84%	4.18%
			4	5	11.45%	9.83%	5.92%	7.16%	7.63%	3.88%
			7	3	10.46%	9.12%	5.93%	6.25%	6.66%	3.31%
			7	5	10.13%	8.78%	5.58%	6.68%	7.51%	2.95%
		$\boldsymbol{\beta}^s_L$	4	3	13.15%	9.80%	5.48%	5.85%	6.26%	3.63%
			4	5	12.51%	9.15%	4.79%	6.19%	7.67%	2.95%
			7	3	11.50%	8.78%	4.88%	5.09%	6.57%	2.55%
			7	5	11.27%	8.54%	4.63%	5.99%	7.50%	2.31%

We next report an encouraging result about the error we introduce by solving the upper bounding problem. As in Lemma 2.1.7, we denote the optimal solutions of the original static problem (P_I) and the continuous relaxation of upper bounding problem (R_I^{UB}) by $\mathbf{b}^*$ and $\mathbf{b}^{R*}$, respectively. Moreover, $\phi_U(\mathbf{b}^{R*})$ is the optimal objective value of the relaxed upper bounding problem. Table 2.3 shows the values of the upper bound on the optimality gap $(\phi_U(\mathbf{b}^{U*}) - \phi(\mathbf{b}^*))/\phi(\mathbf{b}^*)$ given in Lemma 2.1.7. These results indicate that the error bound is mostly affected by the overbooking penalty and the load factor. We observe that $\phi(\lfloor \mathbf{b}^{R*} \rfloor)$ is significantly smaller when ν is higher, and consequently, the error bound increases as ν gets higher. On the other hand, when the load factor is high, (P_I^{UB}) reacts in a more conservative way and reduces the booking limits of cheaper fare classes. Therefore, the resulting overbooking cost decreases, the revenue $\phi(\lfloor \mathbf{b}^{R*} \rfloor)$ increases, and hence, the error bound decreases as the load factor gets higher. We also observe that the error bound tends to decrease as η increases, since the relative increase in $\phi(\lfloor \mathbf{b}^{R*} \rfloor)$ is more than the increase in $\phi_U(\mathbf{b}^{R*})$.

We conclude the presentation of our numerical results by reporting the wall-clock times of the proposed solution methods. We used a computer with 2.4 GHz Intel Core 2 Quad processor and 3024 MB of RAM. The codes are written in MATLAB 7.6.0 running under Windows XP operating system. EMSR/NO, EMSR/SL, EMSR/MP, and EMSR/Risk heuristics require on average less than 0.1 seconds. It takes on average 1.10 and 0.40 seconds to solve the lower and the upper bounding problems, respectively. Thus, our heuristics are comparable to the widely-applied EMSR-based heuristics in terms of computational efficiency. The most computational effort is invested in finding the optimal policy of the dynamic model, which takes on average 2,260 seconds. Clearly, this time depends on the mesh-size parameter h and the length of the planning horizon T.

Table 2.3: Bound on error introduced by solving P_I^{UB}

Instances					$\phi_U(\mathbf{b}^{R*})$	$\phi(\lfloor \mathbf{b}^{R*} \rfloor)$	Error Bound
m	ρ	$\boldsymbol{\beta}^s_{\bullet}$	η	ν			
4	1.4	$\boldsymbol{\beta}^s_H$	4	3	22,818.62	21,269.03	0.073
			4	5	22,818.62	20,267.47	0.126
			7	3	37,274.78	34,842.09	0.070
			7	5	37,274.78	33,251.79	0.121
		$\boldsymbol{\beta}^s_L$	4	3	23,089.39	21,492.75	0.074
			4	5	23,089.39	20,440.34	0.130
			7	3	37,545.56	35,020.98	0.072
			7	5	37,545,56	33,349.93	0.126
	1.8	$\boldsymbol{\beta}^s_H$	4	3	26,139.73	24,669.71	0.060
			4	5	26,139.73	23,723.86	0.102
			7	3	43,735.93	41,406.30	0.056
			7	5	43,735.93	39,904.45	0.096
		$\boldsymbol{\beta}^s_L$	4	3	26,708.94	25,239.06	0.058
			4	5	26,708.94	24,282.21	0.100
			7	3	44,589.74	42,258.88	0.055
			7	5	44,589.74	40,739.57	0.094
8	1.4	$\boldsymbol{\beta}^s_H$	4	3	22,666.43	21,031.50	0.078
			4	5	22,666.43	19,947.70	0.136
			7	3	37,002.55	34,408.86	0.075
			7	5	37,002.55	32,687.73	0.132
		$\boldsymbol{\beta}^s_L$	7	3	23,059.60	21,391.50	0.078
			7	5	23,059.60	20,287.86	0.137
			7	3	37,513.66	34,868.28	0.076
			7	5	37,513.66	33,115.64	0.133
	1.8	$\boldsymbol{\beta}^s_H$	7	3	25,614.71	23,985.24	0.068
			7	5	25,614.71	22,943.83	0.116
			7	3	42,742.07	40,162.18	0.064
			7	5	42,742.07	38,508.38	0.110
		$\boldsymbol{\beta}^s_L$	4	3	26,146.71	24,525.39	0.066
			4	5	26,146.71	23,478.70	0.114
			7	3	43,525.34	40,956.54	0.063
			7	5	43,525.34	39,294.34	0.108

Chapter 3

SINGLE-LEG PROBLEM: DELAYED PURCHASE OPTION

In this chapter, we study delayed purchase decisions in single-leg revenue management. As mentioned in Chapter 1, airline reservation systems recently offer the contingent commitment option to customers. This option allows customers to reserve a seat for a certain duration of time with a small fee. Therefore, it has the potential to attract price sensitive customers as well as improve overall capacity utilization. However, it also creates another source of uncertainty leading to probable revenue loss due to empty seats.

As an example of a contingent commitment, consider a flight for which the airline offers the commitment option for all fare classes. Customers can still buy seats as usual. However, if a customer prefers to reserve a seat instead of buying it, then she can commit to a seat for a fixed non-refundable fee. Such a passenger would then be guaranteed a seat of the fare class until the end of a predetermined commitment period. The length of the commitment period is fixed by the airline. If the customer decides to purchase her committed ticket within this period, then she pays the ticket fare at the time of initial inquiry. Otherwise, she leaves the system without any reimbursement. In short, this option allows passengers to delay their purchase decision with seat and price guarantee for the length of the commitment period.

In practice, there are variants of the contingent commitment option. While some airline companies offer this option to all customers before they choose their flights, some other companies present this option right before customers purchase their tickets.

Figure 3.1 shows a typical screen shot from an airline reservation website that offers a contingent commitment option at the beginning of the booking operation. This reservation website serves 12 airline companies in the USA and Canada. We observe that the contingent commitment fee depends on the length of the commitment period and the option is offered until a week before the departure time of the flight. Although the commitment option resembles a typical travel insurance, there are two important differences. First, the commitment option holds the reservation for a fixed period of time, whereas the travel insurance is valid only until the departure date. Second, the contingent commitments allow passengers to cancel their reservations at any time within the commitment period. However, travel insurance allows free cancellation only if specific circumstances, like emergencies, arise. Therefore, a passenger is more likely to cancel her contingent commitment than canceling her travel insurance.

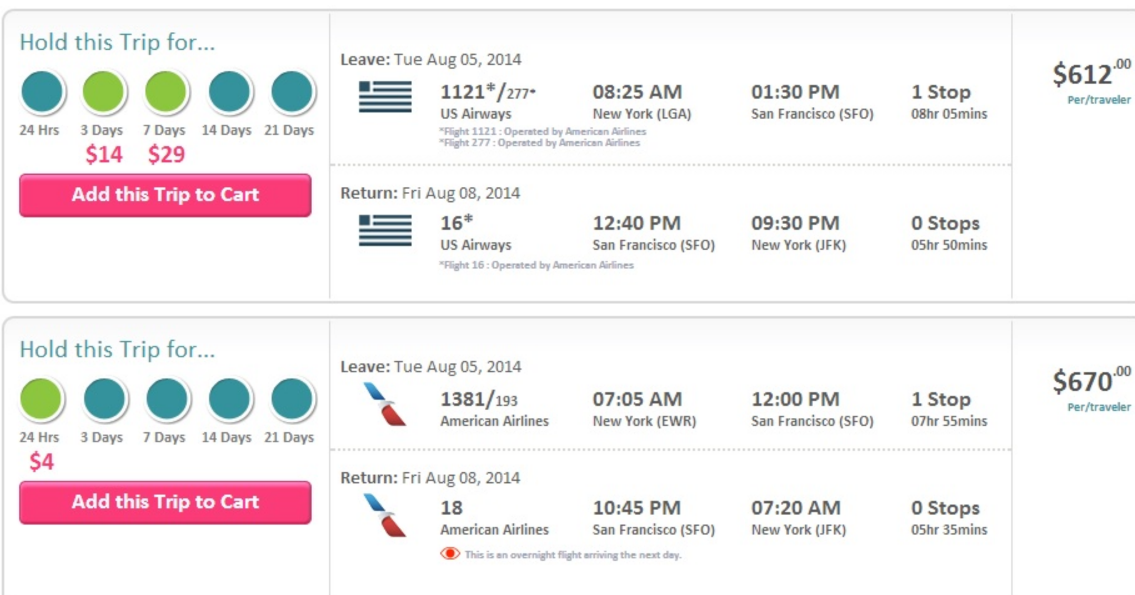


Figure 3.1: A screen shot of a contingent commitment option [91]

From an airline perspective, every committed seat provides an additional revenue from the non-refundable fee. However, reserving a seat, especially early in the reservation period, may result in rejecting a high fare class request at a later time, which in turn can lead to significant revenue losses. Therefore, the contingent commitment and the capacity control decisions should be simultaneously taken into consideration.

To simplify the discussion, we refer to immediately purchased seats as bookings in the subsequent part.

In this work, we address the joint problem of capacity allocation and commitment option for a single flight leg. We seek answers to the following questions: (i) How does offering commitment options to customers affect overall revenue? (ii) How do commitment options affect the optimal seat allocation? (iii) Should commitment options be offered throughout the planning horizon or just during a predefined time period? (iv) What is the effect of duration of commitment period on the optimal policy?

Our problem setting is based on two independent streams of events; arrivals of booking and commitment requests and cancellations of committed seats. At each time period, either a commitment request or a booking request can be realized independently. We need to decide whether to accept or reject each arriving request. We first introduce an exact dynamic programming formulation for this problem. However, this formulation requires keeping track of the remaining commitment time of each accepted contingent commitment request, and hence, makes use of a high-dimensional state variable. Consequently the dynamic program becomes intractable. As a remedy, we propose an approximation to the dynamic programming formulation that performs remarkably well as we demonstrate in our computational study. In addition to this approximation, we present deterministic linear programming models which provide upper bounds on the intractable exact dynamic programming formulation.

Although we use airline reservation systems as the primary application area of this research, the commitment option as we consider here is applicable to any industry selling fixed, perishable capacity, such as; cargo, hotel and car rental. To make our point clear, we note that hotel reservation systems and car rental agencies are already exercising some options similar to the commitment option here. Car rental and hotel systems offer both flexible and non-refundable products. While flexible products can be canceled without any penalty, non-refundable products are offered with various penalty options like charging the first day or the entire trip. For the non-refundable products the reservation systems present insurance policies for a fixed price. These insurance policies guarantee the refund of the whole reservation price, if the reservation is canceled. We will revisit these cases after presenting our models. Although, some of the problems in hotel and car rental industries are network based problems, the methods proposed in this thesis may also be applied in these applications since in practice single-leg decomposition methods are frequently applied to network problems.

To the best of our knowledge, the concept of contingent commitment option has not been previously studied in the literature. However, there exists a number of studies on the sale options used in the revenue management. Recently callable and flexible products have been introduced in airline industry. Callable products give airline the flexibility of accepting expensive fare class customers instead of low fare class customers. A buyer of such a product can be transferred to a later flight if there is no capacity left in the flight she has booked. In that case, airline pays a pre-specified recall price to the customer (Gallego et al., [39, 37]). Similarly, in flexible products airline is free to assign the flexible product buyer to any of the pre-specified alternatives (Gallego and Phillips [40]). Unlike the callable product, a flexible product guarantees a seat in those alternatives. Callable and flexible product options appeal to the customers who have low product valuation and flexible travel time. Gallego and Phillips show that offering the flexible product significantly increases the profitability. These options are also examined in the marketing science literature. Fay and Xie [34] work on the concept of probabilistic goods and selling. In their study, a probabilistic good corresponds to a set of multiple services that a buyer obtains with a probability. The probabilistic selling denotes the selling strategy where probabilistic goods and standard products are sold together. They examine the benefits of offering probabilistic goods. Similar to the flexible products, the opaque selling option is introduced in the travel industry. In opaque selling, product alternatives are concealed from a customer and she is unaware of the product she buys until the purchase. Anderson and Xie [3] present a recent study on the opaque selling option and examine the cases where opaque selling is offered with regular full information selling. They show that offering opaque selling with regular selling improves the customer segmentation, and hence, increases the revenues. Gallego and Stefanescu [42] give a nice overview of different options introduced in the service industries.

Lately, Sainam et al. [74] investigate the benefits of call options in sport events. This option allows sport fans to reserve a ticket for the final game until the teams playing in final are identified. If the option buyer decides to attend the game, she pays for the final. Otherwise, she cancels the ticket. Sainam et al. show that the call options provide extra revenue when they are offered with the advance purchase option. Balseiro et al. [9] extend the work of Sainam et al. by including pricing analysis of call options. They propose a two-stage optimization model. In the first stage, a pricing problem is solved and in the second stage, given the fixed prices, the capacity allocation problem is solved. The problem is intractable. Therefore, they propose a deterministic approximation. Gallego and Sahin [41] work on the partially refundable fares and show that offering partially refundable fares is more profitable than offering non-refundable and fully refundable fares. They propose an inter-temporal valuations model by considering both capacity provider and consumer. The commitment option that we discuss here can be considered as a special case of partially refundable fares where the passengers can get the refund, if they cancel during the commitment period. However, these contingent commitment options bring an additional source of complexity as they can be utilized only within a certain time window.

Contingent commitments in our study are somewhat similar to the options in the finance literature. That literature focuses on pricing and exercise time of options. An option pricing problem can be modeled as a Markov decision problem. However, the resulting problem is hard to solve due to curse of dimensionality. One approach is to use Monte Carlo simulation to generate good solutions (see for instance Board et al. [18] for the pricing of European options). Another approach is to apply approximate dynamic programming to give lower and upper bounds on the value of the option [65, 88, 45]. The pricing problem is also approximated by solving linear programming models [32]. We refer the reader to Trigeorgis [87] for the review of pricing models. Several researchers work on the optimal time of exercising the real option. McDonald and Siegel [66] work on the investment timing problem for an irreversible project and develop an investment rule when the value and the cost of the project are both stochastic. Rhys et al. [71] use first passage time approach to obtain expected waiting time to exercise an option.

Han and Park [44] develop a model to determine the exercise timing by considering the trade-off between early exercising and waiting.

This chapter is organized as follows. In Section 3.1, we present the problem and develop a dynamic programming model to make the capacity allocation and contingent commitment decisions over a single flight leg. Due to the curse of dimensionality this model is hard to solve. Therefore, in Section 3.2, we present an alternate tractable dynamic programming model that approximates the actual contingent commitments process. In Section 3.3, we introduce deterministic and randomized linear programming approximations that give upper bounds on the exact dynamic programming model. A lower bound is also obtained when the problem size becomes large in terms of capacities and the expected number of arrivals. Finally, in Section 3.4, we analyze the effects of offering contingent commitment option through computational experiments. We demonstrate that under certain conditions, offering this option will increase the expected revenue of the flight, though offering the contingent commitment options is not always in the best interest of the airline. In the same section, we introduce an alternate dynamic programming model for contingent commitment option. In this model, we assume that we have an additional information on the probability distribution of standard bookings and contingent commitments. We present several examples to show the effect of this information on the expected revenues.

3.1 Problem Formulation

We have a single flight leg with m fare classes and capacity C. The reservation horizon is partitioned into T time periods, and the flight departs at the beginning of period $T+1$. At each time period, a customer arrives into the system with a particular fare class in mind. If this fare class is open for purchase, then the customer books the ticket. After booking the ticket, she decides whether she wants to pay extra to purchase the flexibility provided by the contingent commitment option. A customer that is interested in fare class i arrives at time period t with probability α_{it}. Then, she either buys the commitment option with probability ν_i or book the seat with probability $(1-\nu_i)$. In other words, booking and commitment requests for fare class i arrive with probabilities

$p_{it} = \alpha_{it}(1 - \nu_i)$ and $q_{it} = \alpha_{it}\nu_i$, respectively. We assume that $\sum_{i=1}^{m} p_{it} + q_{it} \leq 1$ for all $t \in \{1, ..., T\}$ and denote the probability of having no arrival by $p_{0t} = 1 - \sum_{i=1}^{m} p_{it} + q_{it}$. A customer with a contingent commitment stays in the system for exactly s periods before making a final purchase decision. After s time periods, she buys the seat with probability p_b or leaves the system with probability $p_l = 1 - p_b$.

At each time period, we have to decide whether to accept or reject the arriving fare class requests. If we accept a booking request for fare class i, then we generate a revenue of f_i. When we accept a commitment request for fare class i, we gain a fixed non-refundable revenue f^c at the period of request. After s periods of commitment duration elapses, we generate a revenue of f_i with probability p_b, if the same customer decides to buy the ticket she committed to. Whether the accepted customer is charged at the time of reservation or later, Talluri and van Ryzin demonstrate that there is no difference in the total expected revenue [80, Section 4.4.2]. Therefore, the expected revenue of an accepted commitment request for fare class i is $\phi_i := f^c + p_b f_i$. Each type of request consumes unit capacity on the flight leg and the rejected requests or the canceled contingent commitments simply leave the system.

Next we give a dynamic programming formulation. We denote the total number of bookings and accepted commitments at a time period (decision epoch) t by x_t. To store the accepted contingent commitments between time periods $t - s$ and t, we designate an s-dimensional binary vector, $\mathbf{z}_t$. If there is an accepted commitment in one of the intervals $\{t - s, t - s + 1, \cdots, t - 1\}$, then the corresponding component of $\mathbf{z}_t$ equals to 1; otherwise, it is set to 0. The pair x_t and $\mathbf{z}_t$ represents the state in our dynamic programming model of the problem. Note that the first element of $\mathbf{z}_t$ shows if there is a commitment request by a customer s time periods ago. At each time period, we need to check if there is such a customer and determine whether she makes an actual purchase decision or not. Letting z_{1t} be the first element of $\mathbf{z}_t$, the leaving passenger without making an actual purchase decision is represented by a Bernoulli random variable $\mathbf{B}(z_{1t}, p_l)$ having a success probability of p_l. As we move from period t to $t+1$, the first element of $\mathbf{z}_t$ needs to be dropped, and $\mathbf{z}_{t+1}$ is constructed by appending a binary variable to the remaining $s - 1$ elements of $\mathbf{z}_t$. To denote this shifting operation, we define $\Gamma : \{0,1\}^{s+1} \mapsto \{0,1\}^s$ given by

$$\Gamma(\mathbf{z}, \zeta) = [\mathbf{0}\ I_s] \begin{bmatrix} \mathbf{z} \\ \zeta \end{bmatrix},$$

where $\mathbf{0}$ is an s-dimensional column vector consisting of zeros, I_s is an $s \times s$ identity matrix, and $\zeta \in \{0,1\}$. Using now this notation, if we accept the commitment request at time $t+1$, then $\mathbf{z}_{t+1} = \Gamma(\mathbf{z}_t, 1)$; otherwise, $\mathbf{z}_{t+1} = \Gamma(\mathbf{z}_t, 0)$.

We capture the decisions at time period t by m-dimensional binary vector $\mathbf{u}_t = [u_{1t}, u_{2t}, \cdots, u_{mt}]^\intercal$ where u_{it} takes value 1 if we accept the arriving reservation request at time period t, and takes value 0 if we reject the arriving reservation request at time period t. Since our accept-reject decision depends on the available capacity, the set of feasible decisions at time period t is given by

$$\mathcal{U}_t(x_t) = \{\mathbf{u}_t \in \{0,1\}^m : x_t + u_{it} \leq C, \quad i = 1, 2, \cdots, m\}.$$

We are ready to formulate the problem as a dynamic program. Let $J_t(x_t, \mathbf{z}_t)$ denote the expected optimal revenue from t up to T given that at time period t, the total number of bookings and commitments is x_t and the commitment history for s periods is $\mathbf{z}_t$. By the independence of the arrival and the commitment processes and the dynamic programming optimality principle, we obtain for every $1 \leq x_t \leq C$, $\mathbf{z}_t \in \{0,1\}^s$ and $t = 1, 2, \cdots, s$ that

$$\begin{aligned} J_t(x_t, \mathbf{z}_t) = \max_{\mathbf{u}_t \in \mathcal{U}(x_t)} \Bigg\{ & \sum_{i=1}^{m} p_{it} \Big\{ f_i u_{it} + J_{t+1}(x_t + u_{it}, \Gamma(\mathbf{z}_t, 0)) \Big\} + \\ & \sum_{i=1}^{m} q_{it} \Big\{ \phi_i u_{it} + J_{t+1}(x_t + u_{it}, \Gamma(\mathbf{z}_t, u_{it})) \Big\} + p_{0t} J_{t+1}(x_t, \Gamma(\mathbf{z}_t, 0)) \Bigg\} \end{aligned} \tag{3.1a}$$

and for $s < t \leq T$,

$$J_t(x_t, \mathbf{z}_t) = \max_{\mathbf{u}_t \in \mathcal{U}(x_t)} \Bigg\{ \sum_{i=1}^{m} p_{it} \Big\{ f_i u_{it} + \mathbb{E} J_{t+1}(x_t + u_{it} - \mathbf{B}(z_{1t}, p_l), \Gamma(\mathbf{z}_t, 0)) \Big\} + \sum_{i=1}^{m} q_{it} \Big\{ \phi_i u_{it} + \mathbb{E} J_{t+1}(x_t + u_{it} - \mathbf{B}(z_{1t}, p_l), \Gamma(\mathbf{z}_t, u_{it})) \Big\} + p_{0t} \mathbb{E} J_{t+1}(x_t - \mathbf{B}(z_{1t}, p_l), \Gamma(\mathbf{z}_t, 0)) \Bigg\}. \tag{3.1b}$$

The boundary condition is simply $J_{T+1}(x_{T+1}, \mathbf{z}_{T+1}) = 0$. Since a contingent commitment makes the purchase decision at the end of the commitment period, we do not observe any commitment purchase decisions during the first s time periods. After time period s due to the random cancellations of contingent commitments, we need to compute the expectation of optimal value functions over a Bernoulli event. This means for $z_{1t} = 1$ that

$$\mathbb{E} J_{t+1}(x_t + u_{it} - \mathbf{B}(z_{1t}, p_l), \Gamma(\mathbf{z}_t, 0)) = p_b J_{t+1}(x_t + u_{it}, \Gamma(\mathbf{z}_t, 0)) + p_l J_{t+1}(x_t + u_{it} - 1, \Gamma(\mathbf{z}_t, 0)).$$

Clearly, $J_1(0, \mathbf{0})$ gives the optimal expected total revenue at the beginning of the planning horizon, where $\mathbf{0}$ represents the fact that we start with no commitments.

3.2 Approximate Model

We note that the state variable $\mathbf{z}_t$ in the exact model may involve many dimensions in actual applications. Thus, solving the recursive equation through standard dynamic programming tools can be computationally demanding. Therefore, we propose an alternate dynamic programming formulation based on geometric approximation. Our approximation hinges on the assumption that each commitment, independently of other commitments, can buy, leave or retain *at each time period* with probabilities q_b, q_l, and q_r, respectively. In other words, each commitment makes buy-leave decision with probability $(q_b + q_l)$ or stays in the system with probability q_r. Since we want to approximate the amount of time that a contingent commitment stays in the system with s periods, we calibrate these probabilities such that $1/(1 - q_r) = s$. Assuming a contingent commitment results in a final purchasing decision with probability p_b^a, we obtain that $p_b^a = q_b + q_b q_r + q_b q_r^2 + \ldots$ Once we choose q_r and q_b in this fashion, we find $q_l = 1 - q_r - q_b$. Furthermore, given that there are y accepted commitments, the random numbers of bought, $\mathbf{M_b}(y)$, cancelled, $\mathbf{M_l}(y)$ and retained, $\mathbf{M_r}(y)$ commitments in period t follow collectively the multinomial distribution with parameters q_b, q_l, q_r, and number of trials y. Note that under this probabilistic setting, a committed passenger may stay in the system until the departure time. Since each accepted commitment request *eventually* buys the ticket with probability p_b^a, the expected revenue obtained from fare class i commitment request is given by $\phi_i^a := f^c + p_b^a f_i$. We also assume that an accepted commitment request cannot make buy or leave decision in the time period she is accepted. We believe that this assumption is more realistic since in practice the duration of a time period is quite short. An appealing feature of this modelling approach is that it avoids the necessity to keep track of how long each accepted contingent commitment has been in the system since a contingent commitment makes a decision to buy, cancel or keep the commitment at each time period independently. In this case, the state variable in the dynamic programming formulation of the commitment problem collapses to two scalars; the number of bookings and the number of accepted contingent commitments.

Let x_t and y_t be the total number of reservations (including both contingent commitments and bookings) and contingent commitments at time period t, respectively. Then, the recursive equations for the proposed approximate dynamic programming model is given by

$$\begin{aligned} V_t(x_t, y_t) = \max_{\mathbf{u}_t \in \mathcal{U}(x_t)} \Bigg\{ & \sum_{i=1}^{m} p_{it} \Big\{ f_i u_{it} + \mathbb{E} V_{t+1}(x_t + u_{it} - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)) \Big\} + \\ & \sum_{i=1}^{m} q_{it} \Big\{ \phi_i^a u_{it} + \mathbb{E} V_{t+1}(x_t + u_{it} - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + u_{it}) \Big\} + \\ & p_{0t} \mathbb{E} V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)) \Bigg\}. \end{aligned} \tag{3.2}$$

Again, the boundary condition is simply $V_{T+1}(x_{T+1}, y_{T+1}) = 0$. In this formulation, $x_t - \mathbf{M_l}(y_t)$ and $\mathbf{M_r}(y_t)$ represent the remaining number of reservations and commitments, given the state of reservations at the beginning of time period t is (x_t, y_t) and we do not accept anybody during that time period. On the other hand, if we accept a commitment request at time period t, the state of the system becomes $(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + 1)$, since we assume that commitments cannot cancel within the period they are accepted.

Note that in both models, the purchase probability of a contingent commitment is class independent. In case of the exact model (3.1a)-(3.1b), we could have relaxed this assumption and worked with class dependent purchase probabilities. Then, we would have needed to store the fare class of each accepted commitment, which would have required holding even a larger state space. In case of the approximate model (3.2), however, q_b and q_r values are class independent by definition. We could have used weighted averages to set both probabilities. In fact this was the approximation used in overbooking problem defined in Chapter 2. If we had used such an approach, then we would have added one more level of approximation to our dynamic programming model. Therefore, we avoided this kind of construction and decided to work with a purchase probability that is class independent.

Before we discuss the optimal policy, let us note that the way we use the commitment option in the approximate dynamic model reminds similar options offered in the service industry. For instance, the insurance policies are also commonly offered to guarantee reservations. In this case, the customers can cancel at any time until they receive the service. However, this default option in insurance policies is just an assumption in our approximate model.

The optimal policy of problem (3.2) can be summarized as follows: Given the state variables (x_t, y_t) at time period t, the optimal decisions at time period t are given by

$$u_{it}^* = \begin{cases} 1, & \text{if} \quad (1-\nu_i)(f_i + V_{t+1}(x_t+1, y_t)) + \nu_i(\phi_i^a + V_{t+1}(x_t+1, y_t+1)) \geq V_{t+1}(x_t, y_t); \\ 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Next, we present that optimal decisions have a nested structure under certain conditions.

Proposition 3.2.1 *Suppose the probability of a request being a commitment is class independent; that is, $\nu_1 = \nu_2 = \cdots = \nu_m$. Then, given the fare ordering $f_1 \geq f_2 \geq ... \geq f_m$, and hence, the ordering of the expected commitment revenues, $\phi_1^a \geq \phi_2^a \geq ... \geq \phi_m^a$, we have $u_{1t}^* \geq u_{2t}^* \geq ... \geq u_{mt}^*, t = 1, ..., T$.*

Proof. For given x and y, to maximize $V_t(x_t, y_t)$ we accept a booking or commitment request ($u_{it}^* = 1$) if

$$\begin{aligned} p_{it}(f_i + &\mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t))) + q_{it}(\phi_i^a + \mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + 1)) \\ &\geq p_{it}\mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)) + q_{it}\mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)). \end{aligned}$$

Let $\nu := \nu_1 = \nu_2 = \cdots = \nu_m$. Then, by using $p_{it} = (1-\nu)\alpha_{it}$ and $q_{it} = \nu\alpha_{it}$, we obtain

$$\begin{aligned} (1-\nu)(f_i + &\mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t))) + \nu(\phi_i^a + \mathbb{E}V_{t+1}(x_t + 1 - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + 1)) \\ &\geq \mathbb{E}V_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)). \end{aligned} \tag{3.4}$$

Since $f_{i-1} \geq f_i$ and $\phi_{i-1} \geq \phi_i$, if relation (3.4) holds for fare class i request ($u_{it}^* = 1$), then it also holds for the fare class $i-1$ request $u_{(i-1)t}^* = 1$. Similarly, if relation (3.4) does not hold for the expensive fare class $i-1$, then it does not hold for the cheaper fare class i either. This means that, if $u_{(i-1)t}^* = 0$ then $u_{it}^* = 0$. Therefore, we obtain the desired result. □

The assumption in Proposition 3.2.1 seems crucial as we can give a simple counter example where the optimal policy does not have a nested structure. Figure 3.2 illustrates such an example. In this example, we set the problem parameters as follows; $m = 2$, $C = 25$, $f_1 = 26$, $f_2 = 25$, $f^c = 10$, $s = 2$, and $p_b = 0.7$. Given that there are $x = 1$ reservations and $y = 0$ commitments at the beginning of each time period, the optimal decisions are computed. The optimal policy table is given in the lower part of the figure. As this table shows, although a request for the low fare class is accepted, the expensive fare class request is rejected for the first two periods.

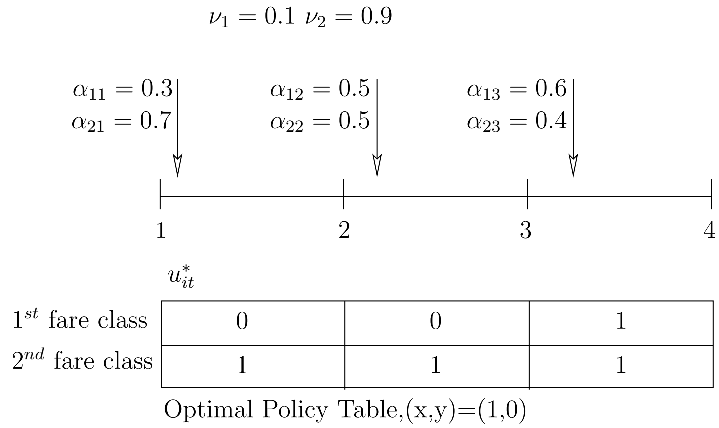


Figure 3.2: A counter example when the assumption in Proposition 3.2.1 does not hold

3.3 Linear Programming Approach

An alternate approximation approach is to model a linear program that corresponds to the exact dynamic programming model (3.1a)-(3.1b). Deterministic linear program is a well-known method to solve the network capacity allocation problem [93, 77]. It assumes that all random quantities are known in advance and they take on their expected values. Talluri and van Ryzin [78] propose a randomized version of DLP for the network problem that randomizes the realizations of demand. The main advantage of linear programming models is the computational efficiency. Due to its computational performance, these approaches are popular in practice. However, an important shortcoming of these methods is that they do not capture the temporal dynamics of the problem as dynamic models do.

3.3.1 Deterministic Linear Program

In this section, we introduce the deterministic model for the contingent commitment problem. To formulate this linear program, let w_{it} be the number of the bookings and commitments that we plan to accept for the fare class i at time period t. Since an arriving customer either buys the commitment option with probability ν_i or books the seat with probability $(1 - \nu_i)$, the expected number of booked and committed fare class i seats at time period t are given by $(1 - \nu_i)w_{it}$ and $\nu_i w_{it}$, respectively. Consequently, the deterministic linear program has the following form:

$$\begin{aligned}
\text{maximize} \quad & \sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)w_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} \phi_i \nu_i w_{it} && (3.5)\\
\text{subject to} \quad & \upsilon_1 = C, && (3.6)\\
& \upsilon_t = \upsilon_{t-1} - \sum_{i=1}^{m} w_{i(t-1)}, \qquad 2 \le t \le s+1, && (3.7)\\
& \upsilon_t = \upsilon_{t-1} - \sum_{i=1}^{m} w_{i(t-1)} + \sum_{i=1}^{m} \nu_i w_{i(t-s-1)} p_l, \qquad s+2 \le t \le T, && (3.8)\\
& \upsilon_{T+1} = \upsilon_T - \sum_{i=1}^{m} w_{iT} + \sum_{k=T-s}^{T}\sum_{i=1}^{m} \nu_i w_{ik} p_l, && (3.9)\\
& w_{it} \le \alpha_{it}, \quad i = 1, ..., m; t = 1, ..., T, && (3.10)\\
& \upsilon_t, w_{it} \ge 0, \quad i = 1, ..., m; t = 1, ..., T, && (3.11)
\end{aligned}$$

where υ_t is the remaining capacity at the beginning of time period $t \in \{1, ..., T+1\}$. Constraints (3.6)-(3.9) keep track of the remaining capacity at each time period. Constraints (3.10) ensure that the reservation requests that we plan to accept do not exceed the expected number of arrivals. Note that by substituting constraints (3.6)-(3.8) into constraint (3.9), we obtain $\sum_{t=1}^{T}\sum_{i=1}^{m}(1-\nu_i)w_{it} + (1-p_l)\sum_{t=1}^{T}\sum_{i=1}^{m}\nu_i w_{it} \le C$. Therefore, the total expected number of assigned seats may exceed the capacity. In this model, this excess amount depends on the commitment cancellation probability. We denote the optimal objective function value of (3.5)-(3.11) by Z^*_{DLP}.

There are two uses of DLP. First, it gives a policy to accept or reject the product requests. Let $\{w^*_{it}, \forall i, t\}$ be the optimal value of the decision variables in problem (3.5)-(3.11). Then, according to the policy dictated by DLP, a booking or a commitment request is accepted with probability w^*_{it}/α_{it}. Second, its optimal objective value provides an upper bound on the maximum expected revenue over the whole planning horizon, as we show in the next proposition.

Proposition 3.3.1 *The optimal objective value of the DLP model gives an upper bound on the exact dynamic programming model. That is,* $J_1(0, \mathbf{0}) \le Z^*_{DLP}$.

Proof. Suppose the random variables W_{it}, $\forall i, t$ denote the number of reservations accepted over the planning horizon under the optimal policy of the exact dynamic programming model. Each accepted reservation for fare class i either buys the contingent commitment option with probability ν_i or books the seat with probability $(1-\nu_i)$. Let X_{it} and Z_{it} be the random numbers of bookings and commitments accepted for fare class i at time period t, respectively. Since an accepted commitment request can cancel with probability p_l, we also let S_{it} and L_{it} be the binary random numbers denoting the sale and cancellation of the commitments, respectively. That is, S_{it} takes value 1, if there is a commitment reservation for fare class i at time period t and that reservation decides to buy, and L_{it} takes value 1 if this commitment reservation cancels.

Let now D_{it} be the random number of reservation requests for fare class i at time period t. Then, we have,

$$\mathcal{V}_1 = C, \tag{3.12}$$

$$\mathcal{V}_t = \mathcal{V}_{t-1} - \sum_{i=1}^{m} X_{i(t-1)} - \sum_{i=1}^{m} Z_{i(t-1)}, \qquad 2 \leq t \leq s+1, \tag{3.13}$$

$$\mathcal{V}_t = \mathcal{V}_{t-1} - \sum_{i=1}^{m} X_{i(t-1)} - \sum_{i=1}^{m} Z_{i(t-1)} + \sum_{i=1}^{m} L_{i(t-s-1)}, \quad s+2 \leq t \leq T, \tag{3.14}$$

$$\mathcal{V}_{T+1} = \mathcal{V}_T - \sum_{i=1}^{m} X_{iT} - \sum_{i=1}^{m} Z_{iT} + \sum_{k=T-s}^{T} \sum_{i=1}^{m} L_{ik}, \tag{3.15}$$

$$X_{it} + Z_{it} \leq D_{it}, \qquad i = 1, ..., m; t = 1, ..., T, \tag{3.16}$$

where (3.12)-(3.15) ensure that the balance equations in each time period holds, (3.16) ensures that total number of bookings and commitments that we accept under the optimal policy do not exceed the reservation requests. Consequently, the total revenue under the optimal policy of the exact dynamic programming is

$$\sum_{t=1}^{T} \sum_{i=1}^{m} f_i X_{it} + \sum_{t=1}^{T} \sum_{i=1}^{m} f^c Z_{it} + f_i S_{it}.$$

By conditioning on W_{it} we trivially obtain $\mathbb{E}(Z_{it}) = \nu \mathbb{E}(W_{it})$. Since $X_{it} = W_{it} - Z_{it}$, we have $\mathbb{E}(X_{it}) = (1-\nu_i)\mathbb{E}(W_{it})$. Similarly, conditioning on Z_{it} leads to $\mathbb{E}(S_{it}) = p_b \nu_i \mathbb{E}(W_{it})$. Therefore, the total expected revenue is given by

$$J_1(0, \mathbf{0}) = \sum_{t=1}^{T} \sum_{i=1}^{m} f_i (1-\nu_i)\mathbb{E}(W_{it}) + \sum_{t=1}^{T} \sum_{i=1}^{m} f^c \nu_i \mathbb{E}(W_{it}) + f_i p_b \nu_i \mathbb{E}(W_{it}).$$

Taking the expectations (3.12)-(3.16) and noting that $\mathbb{E}(D_{it}) = \alpha_{it}$, the solution given by $w_{it} = \mathbb{E}(W_{it})$ and $\upsilon_t = \mathbb{E}(\mathcal{V}_t)$ is feasible for the DLP model (3.5)-(3.11). Therefore, we have

$$Z^*_{DLP} \geq J_1(0, \mathbf{0}) = \sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)\mathbb{E}(W_{it}) + \sum_{t=1}^{T}\sum_{i=1}^{m} \phi_i\nu_i\mathbb{E}(W_{it}),$$

and the desired result holds. □

Next, we focus on obtaining an asymptotic lower bound. To obtain this bound, we make use of another upper bounding problem. Note that problem (3.5)-(3.11) ensures that the remaining capacity at each time period, υ_t, is non-negative. By relaxing this constraint, we can give an upper bound on the DLP model (3.5)-(3.11) as follows:

$$\text{maximize} \quad \sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)w_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} \phi_i\nu_i w_{it} \tag{3.17}$$

$$\text{subject to} \quad \sum_{t=1}^{T}\sum_{i=1}^{m}(1-\nu_i)w_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} p_b\nu_i w_{it} \leq C, \tag{3.18}$$

$$w_{it} \leq \alpha_{it}, \qquad i = 1, ..., m; t = 1, ..., T, \tag{3.19}$$

$$w_{it} \geq 0, \qquad i = 1, ..., m; t = 1, ..., T. \tag{3.20}$$

We denote the optimal objective function of this model by Z^*_{DLP-UB}. Here, constraint (3.18) is obtained by substituting constraints (3.6)-(3.8) into constraint (3.9).

Remark 3.3.1 *When there is no commitment option ($s = 0$) or probability of buying the committed seat equals to 1 ($p_b = 1$), DLP given by (3.5)-(3.11) boils down to the standard capacity allocation problem. Furthermore, when $Z^*_{DLP-UB} = Z^*_{DLP}$, the dual variables corresponding to constraints (3.6)-(3.9) in problem (3.5)-(3.11) are all equal. Therefore, the dual of problem (3.5)-(3.11) can be reduced to a one-dimensional unconstrained problem, and it can be solved very efficiently by any variant of the bisection method.*

Now we are ready to obtain an asymptotic lower bound on the distance between the optimal objective function value of DLP and the optimal expected revenue of the exact dynamic programming model. Our analysis follows a similar approach as in Gallego et al. [38]. However, in our case, we need to consider the cancellation explicitly. Let us first explain the setting. We introduce a sequence of problems $\{\mathcal{P}^\kappa : \kappa \in \mathbb{Z}_+\}$ indexed by parameter κ. Problem $\mathcal{P}^\kappa$ has κT time periods in the planning horizon and the capacity of the flight is

κC. Moreover, the probability of reservation request for fare class i at time period t is given by $\alpha_{i\lceil t/\kappa \rceil}$, where operator $\lceil . \rceil$ rounds up the values passed to it. We note that the problem described in Section 3.1 is $\mathcal{P}^1$. The flight capacity in problem $\mathcal{P}^\kappa$ is κ times the capacity of the flight in problem P^1. Similarly, the length of the booking horizon in problem $\mathcal{P}^\kappa$ is κ times the length of the booking horizon in problem $\mathcal{P}^1$. In addition, the arrival probabilities $\alpha_{i\lceil t/\kappa \rceil}$ at time periods $\{\kappa(t-1)+1, ..., \kappa t\}$ in problem $\mathcal{P}^\kappa$ are the same as the arrival probabilities at time period t in problem $\mathcal{P}^1$. Hence, the expected total booking demand and the expected total commitment demand for fare class i in problem $\mathcal{P}^\kappa$ is

$$\sum_{t=1}^{\kappa T} \alpha_{i\lceil t/\kappa \rceil} = \kappa \sum_{t=1}^{T} \alpha_{it}.$$

This implies that the expected numbers of reservation requests in problem $\mathcal{P}^\kappa$ are κ times larger than those in problem $\mathcal{P}^1$. Consequently, problem $\mathcal{P}^\kappa$ is a scaled version of problem $\mathcal{P}^1$.

We consider the linear programming model (3.17)-(3.20) for problem $\mathcal{P}^\kappa$. Let Z^κ_{DLP-UB} denote the optimal objective value of the upper bound on DLP for the scaled problem $\mathcal{P}^\kappa$. Likewise, Z^κ_{DLP} denotes the optimal objective value of the scaled deterministic linear program given by problem (3.5)-(3.11) and $J^\kappa_1(0, \mathbf{0})$ stands for the optimal expected total revenue for the scaled problem $\mathcal{P}^\kappa$ that we obtain by solving the corresponding dynamic program. Proposition 3.3.1 shows that the optimal objective value of the deterministic linear program provides an upper bound on the optimal expected total revenue. Thus, we have $Z^\kappa_{DLP} \geq J^\kappa_1(0, \mathbf{0})$. Since $Z^\kappa_{DLP-UB} \geq Z^\kappa_{DLP}$, we also have $Z^\kappa_{DLP-UB} \geq J^\kappa_1(0, \mathbf{0})$.

To prove the asymptotic bound, we first define a lower bound on the rate of convergence. Let d_{ib} and d_{ic} denote the random numbers of total fare class i requests for bookings and commitments i respectively. Then, the expected booking and contingent commitment demands for fare class i are computed as

$$\mu^b_i := \mathbb{E}(d_{ib}) = (1-\nu_i) \sum_{t=1}^{T} \alpha_{it} \;\text{ and }\; \mu^c_i := \mathbb{E}(d_{ic}) = \nu_i \sum_{t=1}^{T} \alpha_{it}.$$

Likewise, σ_{ib} and σ_{ic} denote the corresponding standard deviations. Then, the coefficient of variation of the number of requests for bookings and commitments are given as

$$CV_i^b = \frac{\sqrt{\sigma_{ib}^2}}{\mu_i^b} \text{ and } CV_i^c = \frac{\sqrt{\sigma_{ic}^2}}{\mu_i^c}, \text{ for } i = 1, ..., m.$$

We also define

$$CV = \max_{1 \le i \le m} \Big\{ CV_i^b, CV_i^c \Big\},$$

as the maximum coefficient of variation.

Proposition 3.3.2 *Let CV denote the maximum coefficient of variation over bookings and commitments for all fare classes. Then for $\epsilon \in [1 - p_b, 1]$, we have*

$$J_1(0, \mathbf{0}) \ge \Big(1 - \epsilon - \frac{CV^2}{\epsilon^2}\Big) Z^*_{DLP-UB}.$$

Proof. Let $\{w^*_{it} : \forall i, t\}$ be the optimal value of the decision variables in problem (3.17)-(3.20). We consider a policy π that accepts at most $(1-\epsilon)(1-\nu_i)\sum_{t=1}^{T} w^*_{it}$ booking requests and $(1-\epsilon)\nu_i \sum_{t=1}^{T} w^*_{it}$ contingent commitment requests for fare class i for $\epsilon \in (0,1)$. Due to the capacity constraint (3.18) in DLP-UB model, the policy π is feasible if $(1-\epsilon) \le p_b$. The expected revenue $\mathcal{P}^\pi$ is given by

$$\begin{aligned}\mathcal{P}^\pi = \mathbb{E}[\sum_{i=1}^{m} f_i \min(d_{ib}, (1-\epsilon)\sum_{t=1}^{T}(1-\nu_i)w^*_{it}) + \sum_{i=1}^{m} f^c \min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}) \\ + \sum_{i=1}^{m} f_i \mathcal{S}(\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}))],\end{aligned}$$

where $\mathcal{S}(k)$ is a binomial random variable with k independent trials with success probability p_b and it gives the number of purchased committed seats. A lower bound to the generic term in the expression for $\mathcal{P}^\pi$ is then given by

$$\mathbb{E}[\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it})] \ge (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}\mathbf{P}(d_{ic} \ge (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}) \tag{3.21}$$

$$\ge (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}\mathbf{P}(d_{ic} \ge (1-\epsilon)\sum_{t=1}^{T}\nu_i \alpha_{it}) \tag{3.22}$$

$$\ge (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}\Big(1 - \frac{CV_i^{c2}}{CV_i^{c2} + \epsilon^2}\Big) \tag{3.23}$$

$$\ge (1-\epsilon)\sum_{t=1}^{T}\nu_i w^*_{it}\Big(1 - \frac{CV^2}{\epsilon^2}\Big). \tag{3.24}$$

The inequality (3.22) holds since $\nu_i w_{it}^* \le \nu_i \alpha_{it}$, (3.23) follows from the Marshall's inequality and (3.24) holds due to the definition of CV. Since $\mathbb{E}[\mathcal{S}(\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^*))] = p_b \mathbb{E}[\min(d_{ic}, (1-\epsilon)\sum_{t=1}^{T} \nu_i w_{it}^*)]$, we can give a lower bound to $\mathcal{P}^\pi$ by using the inequality (3.24) as follows:

$$\begin{aligned}
\mathcal{P}^\pi &\ge (1-\epsilon)\Big(1 - \frac{CV^2}{\epsilon^2}\Big)(\sum_{i=1}^{m}\sum_{t=1}^{T} f_i(1-\nu_i)w_{it}^* + \sum_{i=1}^{m}\sum_{t=1}^{T} f^c \nu_i w_{it}^* + \sum_{i=1}^{m}\sum_{t=1}^{T} f_i p_b \nu_i w_{it}^*) \\
&\ge (1-\epsilon)\Big(1 - \frac{CV^2}{\epsilon^2}\Big) Z_{DLP-UB}^* \\
&\ge \Big(1 - \epsilon - \frac{CV^2}{\epsilon^2}\Big) Z_{DLP-UB}^*
\end{aligned}$$

This implies

$$J_1(0, \mathbf{0}) \ge \mathcal{P}^\pi \ge \Big(1 - \epsilon - \frac{CV^2}{\epsilon^2}\Big) Z_{DLP-UB}^*.$$

□

To tighten the lower bound in the above inequality, we maximize it over ϵ and obtain

$$\epsilon^* = \max\{(2CV^2)^{1/3}, 1 - p_b\}.$$

Since $\epsilon \in [1-p_b, 1]$, this tighter bound is only obtained when $2CV^2 < 1$. Consequently we have,

$$J_1(0, \mathbf{0}) \ge \mathcal{P}^\pi \ge \Big(1 - \epsilon^* - \frac{CV^2}{\epsilon^{*2}}\Big) Z_{DLP-UB}^*.$$

Next we examine the structure of the lower bound as the problem size gets large.

Proposition 3.3.3 *Given $\epsilon \in [1-p_b, 1]$ and $\kappa > 0$, we have*

$$Z_{DLP-UB}^\kappa \ge Z_{DLP}^\kappa \ge J_1^\kappa(0, \mathbf{0}) \ge \left(1 - \epsilon - \frac{CV^2}{\kappa\epsilon^2}\right) Z_{DLP-UB}^\kappa,$$

where CV denotes the maximum coefficient of variation over bookings and commitments for all fare classes. Therefore,

$$p_b \le \lim_{\kappa\to\infty} \frac{J_1^\kappa(0, \mathbf{0})}{Z_{DLP}^\kappa} \le 1.$$

Proof. We observe that if $\{w_{it}^* : \forall i, t\}$ is an optimal solution to problem (3.17)-(3.20). Then $\{w_{i\lceil t/\kappa\rceil}^* : \forall i, t\}$ is an optimal solution for the scaled problem. Thus, it follows that $Z^{\kappa}_{DLP-UB} = \kappa Z_{DLP-UB}$. For the scaled problems, the expected demand and the variance are scaled with κ. If μ and σ^2 denote the mean demand and variance for problem $\mathcal{P}^1$, then the mean demand is $\kappa\mu$ and the variance is $\kappa\sigma^2$ for the problem $\mathcal{P}^{\kappa}$. Therefore, the maximum coefficient of variation of the scaled problem is

$$CV^{\kappa} = \max_{1\leq i\leq m}\left\{\frac{\sqrt{\kappa\sigma^2_{ib}}}{\kappa\mu^b_i}, \frac{\sqrt{\kappa\sigma^2_{ic}}}{\kappa\mu^c_i}\right\} = \frac{CV\sqrt{\kappa}}{\kappa}$$

By following the result of Proposition 3.3.2 and replacing CV^{κ} with $\frac{CV\sqrt{\kappa}}{\kappa}$, we have

$$J^{\kappa}_1(0,\mathbf{0}) \geq \left(1 - \epsilon - \frac{CV^2}{\kappa\epsilon^2}\right) Z^{\kappa}_{DLP-UB}.$$

When κ goes to infinity, the expression $\left(1 - \epsilon - \frac{CV^2}{\kappa\epsilon^2}\right)$ approaches to $(1-\epsilon)$. Since $\epsilon \in [1-p_b, 1]$, this bound is tighter when $\epsilon = 1 - p_b$. Therefore, as p_b goes to 1, the upper bound obtained from Z^{*}_{DLP-UB} becomes asymptotically tight. Following the result of Proposition 3.3.2, we obtain the following convergence rate

$$\kappa Z_{DLP-UB} \geq Z^{\kappa}_{DLP} \geq J^{\kappa}_1(0,\mathbf{0}) \geq \left(1 - \epsilon - \frac{CV^2}{\kappa\epsilon^2}\right)\kappa Z_{DLP-UB},$$

Dividing the chain of inequalities by κZ_{DLP-UB} and taking the limit as κ goes to infinity, we get

$$p_b \leq \lim_{\kappa\to\infty}\frac{J^{\kappa}_1(0,\mathbf{0})}{Z^{\kappa}_{DLP-UB}} \leq \lim_{\kappa\to\infty}\frac{Z^{\kappa}_{DLP}}{Z^{\kappa}_{DLP-UB}} \leq 1$$

which implies,

$$p_b \leq \lim_{\kappa\to\infty}\frac{J^{\kappa}_1(0,\mathbf{0})}{Z^{\kappa}_{DLP}} \leq 1.$$

□

The first part of this proposition gives a lower bound for the scaled problems of DLP and the exact dynamic programming model. The asymptotic result in the second part implies that the optimal objective function value of the DLP is at most $1/p_b$ multiple of the exact model as the problem size gets large in terms of the capacity and the expected demand. This limiting behavior also shows that DLP becomes asymptotically tight as p_b becomes closer to 1. This is an obvious observation, since the commitment problem becomes a standard capacity allocation problem when $p_b = 1$ [77].

3.3.2 Randomized Linear Program

Talluri and van Ryzin [78] propose a randomized linear program for the network capacity control problem and show that it provides a tighter upper bound on the optimal expected revenue compared to the one obtained by the deterministic linear program. Kunnumkal et al. [51] extend the work of Talluri and van Ryzin by considering both the capacity control and the overbooking decisions. We follow their approach for our problem and propose a randomized linear program to obtain an upper bound on the optimal expected total revenue.

Let D_{it} be the random demand for fare class i at time period t, and $d = \{d_{it} : \forall i, t\}$ be a realization of the random variable $D = \{D_{it} : \forall i, t\}$. Note that we have $\mathbb{E}\{D_{it}\} = \alpha_{it}$. Under a realization of arrivals, we propose solving the following model:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)w_{it} + \sum_{t=1}^{T}\sum_{i=1}^{m} \phi_i \nu_i w_{it} \\
\text{subject to} \quad & \upsilon_1 = C, \\
& \upsilon_t = \upsilon_{t-1} - \sum_{i=1}^{m} w_{i(t-1)}, && 2 \le t \le s+1, \\
& \upsilon_t = \upsilon_{t-1} - \sum_{i=1}^{m} w_{i(t-1)} + \sum_{i=1}^{m} p_l \nu_i w_{i(t-s-1)}, && s+2 \le t \le T, \\
& \upsilon_{T+1} = \upsilon_T - \sum_{i=1}^{m} w_{iT} + \sum_{k=T-s}^{T}\sum_{i=1}^{m} p_l \nu_i w_{ik}, \\
& w_{it} \le d_{it}, \qquad i = 1, ..., m; t = 1, ..., T, \\
& \upsilon_t, w_{it} \ge 0, \qquad i = 1, ..., m; t = 1, ..., T.
\end{aligned}
$$

We define $Z^*_{RLP}(d)$ to denote the optimal total revenue obtained when we make the accept or reject decisions for the booking and commitment requests requests after observing a realization d of the demands. Next, we show that randomized linear program provides an upper bound on the optimal expected total revenue and this bound is tighter than the one obtained from the deterministic linear program.

Proposition 3.3.4 *Let $Z^*_{RLP} = \mathbb{E}\{Z^*_{RLP}(D)\}$, then we have $J_1(0, \mathbf{0}) \leq Z^*_{RLP} \leq Z^*_{DLP}$.*

Proof. Let $w(d) = \{w_{it}(d) \in \{0, 1\} : \forall i, t\}$ denote the number of accepted reservation requests under the optimal policy of the exact dynamic programming model. The number of accepted requests depends on the realization $d = \{d_{it}\ \forall i, t\}$. Then we have

$$\begin{aligned} J_1(0, \mathbf{0}) =& \mathbb{E}\{\sum_{t=1}^{T}\sum_{i=1}^{m} f_i(1-\nu_i)w_{it}(D) + \sum_{t=1}^{T}\sum_{i=1}^{m}(f^c \nu_i w_{it}(D) + f_i p_b \nu_i w_{it}(D))\} \\ \leq& \mathbb{E}\{Z^*_{RLP}(D)\} = Z^*_{RLP}. \end{aligned}$$

The first equality follows from the optimality of the policy. The first inequality holds, since the solution $w(d)$ is feasible for the randomize model but not necessarily optimal. Since $Z^*_{RLP}(d)$ is a piecewise linear, nonincreasing and concave. Using the Jensen's inequality, we obtain the result

$$Z^*_{RLP} = \mathbb{E}\{Z^*_{RLP}(D)\} \leq Z^*_{RLP}(\mathbb{E}\{D\}) = Z^*_{DLP}.$$

□

3.4 Computational Experiments

In this section, we conduct simulation experiments to evaluate the effects of offering the contingent commitment option. We also provide sensitivity analysis with respect to various parameters. Moreover, we compare the performance of our dynamic model against other benchmark strategies. We begin by describing the benchmark strategies listed below.

Approximate Dynamic Model (ADM): This is the solution method that we introduced in Section 3.2. That is, we solve the dynamic program in (3.2) to obtain the optimal policy. Then, we use the decision rule (3.3) as our accept-reject policy for booking and commitment requests.

Standard Booking Strategy (SBS): This policy ignores the commitment requests and only accepts the standard booking requests. Therefore, in this policy no arrival probability at time period t becomes $(1 - \sum_{i=1}^{m} p_{it})$. The optimal booking policy is then determined by solving the problem as a standard capacity allocation problem [80, Section 2.5].

Deterministic Linear Program (DLP): This is the solution method described in Section 3.3.1. We solve the problem (3.5)-(3.11) to obtain the optimal values of the variables $\{w^*_{it}, \forall i, t\}$. Provided that there is sufficient remaining capacity, we accept a reservation request for fare class i with probability w^*_{it}/α_{it} at time period t.

In the sequel, we refer to the average revenue obtained by the optimal policy of the exact dynamic model given by (3.1a)-(3.1b) as EDM. Recall that the exact dynamic model is computationally intractable for long commitment periods. Hence, we test the models with respect to EDM for only small instances. We simulate the arrival of requests and cancellations over discrete time periods $\{1, ..., T\}$. At each time period, we first generate an arrival request and then apply the corresponding policy. While an accepted booking request for fare class i generates a revenue of f_i, an accepted commitment request generates a revenue of f^c. After the arrival process, we check whether there is a commitment made s periods ago and simulate a purchase or leave decision. Each commitment passenger in fare class i buys the ticket with probability p_b generating an additional revenue of f_i, or leaves the system. In our numerical experiments, we set the purchase probability of committed seat (p^a_b) in approximate dynamic model equal to the purchase probability of committed seat (p_b) in exact dynamic model.

To test the performances of the booking policies against varying arrival intensities, we use the load factor parameter ρ. Noting that the total expected demand for the flight is $\sum_{t=1}^{T} \sum_{i=1}^{m} (p_{it} + p_b q_{it})$, the load factor is given by

$$\rho = \frac{\sum_{t=1}^{T} \sum_{i=1}^{m} (p_{it} + p_b q_{it})}{C}.$$

In all our numerical experiments, we set the capacity of the plane, the length of the planning horizon and the number of fare classes to $C = 100$, $T = 300$ and $m = 4$, respectively. The fares are evenly distributed between 250 and $1,000$.

3.4.1 Benchmarking Study

We first present the performance of the approximate dynamic model (ADM). Our experimental design is based on various factors of the load factor (ρ), the commitment period (s), the commitment fee (f^c), the probability of buying the committed seat (p_b), and the splitting probability of commitment arrivals (ν). We use load factor values $\rho \in \{1.2, 1.6\}$ corresponding to low and high loads. We select the commitment period lengths from the set $\{5, 25, 50\}$ to represent short, medium and long commitment intervals. The commitment fees $f^c \in \{40, 80\}$ are used to represent low and high fees. We also test the models for varying buy probabilities $p_b \in \{0.4, 0.7\}$. The last parameter set comes from the splitting probability of contingent commitments (ν_i values). We give two sets of values to represent low and high commitment arrivals. These are $\nu_L := (0.10, 0.15, 0.20, 0.25)$ and $\nu_H := (0.40, 0.45, 0.50, 0.55)$ where the values in each set are ordered from expensive to cheap fare class. We label our test problems by using all combinations of these parameters.

As mentioned in Section 3.3.1, DLP provides an upper bound on the maximum total expected revenue obtained by the dynamic model over the time periods $\{1, ..., \tau\}$. Moreover, we also show that the optimal objective function value of DLP is at most $1/p_b$ multiple of the dynamic model as the problem size gets large in terms of the capacity and the expected demand. Table 3.1 shows the upper bound obtained by DLP for different test instances. The first four columns indicate the characteristics of the test instances. The next two columns give the optimal objective values of EDM and DLP, respectively. The last column gives the percentage gaps between EDM and DLP. The results show that upper bound provided by DLP is significantly tight. For the test instances with high load factor and low commitment demand, the percentage gap is around 1%. Moreover, the quality of the upper bound is mostly affected by the tightness of the room capacities.

Table 3.1: Upper bound on the maximum total expected revenue ($s = 5$)

Instances						% Gap with EDM
ρ	$\nu_\bullet$	f^c	p_b	EDM	DLP	DLP
1.2	ν_H	40	0.4	65,657	66,570	1.39%
		40	0.7	64,390	65,270	1.37%
		80	0.4	68,251	69,213	1.41%
		80	0.7	66,563	67,484	1.38%
	ν_L	40	0.4	63,547	64,390	1.33%
		40	0.7	62,899	63,711	1.29%
		80	0.4	64,308	65,170	1.34%
		80	0.7	63,622	64,449	1.30%
1.6	ν_H	40	0.4	75,598	76,449	1.13%
		40	0.7	74,469	75,247	1.05%
		80	0.4	78,072	78,939	1.11%
		80	0.7	76,563	77,351	1.03%
	ν_L	40	0.4	73,582	74,296	0.97%
		40	0.7	73,002	73,702	0.96%
		80	0.4	74,259	74,974	0.96%
		80	0.7	73,649	74,349	0.95%

Next, we test the performances of the models against the dynamic model given by (3.1a)-(3.1b). We estimate the net expected revenues by simulating the arrivals of booking and commitment requests over 5,000 sample paths. We set the length of the commitment period to 5 for the instances where we compare our models with respect to EDM. Table 3.2 shows the average total revenues and percentage gaps between EDM and the remaining solution methods. The first four columns in Table 3.2 show the characteristics of the test instances. The next four columns give the expected total revenues obtained by EDM, ADM, SBS, and DLP, respectively. The last three columns give the percentage gaps between EDM and the remaining solution methods. Comparing the percentage gaps under this setup, we observe that the performances of EDM and ADM are very close, especially for high values of buy probability (see the rows corresponding to high p_b). As discussed with model (3.17)-(3.20), the commitment problem can be transformed to a standard capacity allocation problem when the probability of buying the committed seat is equal to 1 (see Remark 3.3.1). Therefore as expected, the percentage gaps between policies of the dynamic models decrease when p_b is high. Even for $p_b = 0.4$, the gap between ADM and EDM is less than one-tenth of a percent.

Moreover as the load factor increases, the percentage gap between EDM and ADM decreases. When arrival intensity is high, models can compensate the revenue loss due to empty seats. On the other hand, we observe that there is a noticeable performance gap between ADM and SBS. The performance of SBS improves slightly when the load factor is high and splitting probability is low. However even in this case, it performs worse than ADM. A noteworthy observation is the relatively large difference between ADM and SBS even when the load factor is high ($\rho = 1.6$) and the splitting probability is low (ν_L). Because in this case there is ample booking requests to use the full capacity of the flight. However, ADM still performs better than SBS by considering the commitment option. In addition, it is important to note that offering commitment option is most beneficial when the purchase probability and the commitment fee are high.

Table 3.2: Computational results for the test problems ($s = 5$)

Instances								% Gap with EDM		
ρ	$\nu_\bullet$	f^c	p_b	EDM	ADM	SBS	DLP	ADM	SBS	DLP
1.2	ν_H	40	0.4	65,748	65,732	51,455	63,692	0.024%	21.738%	3.127%
		40	0.7	64,443	64,437	42,685	62,159	0.010%	33.764%	3.544%
		80	0.4	68,298	68,294	51,455	66,247	0.007%	24.660%	3.004%
		80	0.7	66,459	66,453	42,685	64,210	0.009%	35.772%	3.384%
	ν_L	40	0.4	63,543	63,541	60,362	61,462	0.003%	5.007%	3.275%
		40	0.7	62,883	62,883	58,056	60,691	0.001%	7.677%	3.486%
		80	0.4	64,313	64,308	60,362	62,201	0.008%	6.143%	3.283%
		80	0.7	63,595	63,595	58,056	61,421	0.000%	8.710%	3.419%
1.6	ν_H	40	0.4	75,579	75,569	63,738	73,645	0.014%	15.667%	2.559%
		40	0.7	74,435	74,434	56,267	72,336	0.002%	24.409%	2.820%
		80	0.4	78,049	78,043	63,738	76,038	0.009%	18.336%	2.576%
		80	0.7	76,543	76,541	56,267	74,397	0.003%	26.490%	2.804%
	ν_L	40	0.4	73,547	73,545	71,411	71,588	0.003%	2.905%	2.664%
		40	0.7	72,964	72,963	69,289	71,032	0.001%	5.036%	2.647%
		80	0.4	74,221	74,220	71,411	72,243	0.001%	3.787%	2.665%
		80	0.7	73,607	73,606	69,289	71,660	0.002%	5.867%	2.645%

Next, we report our results for larger values of the commitment period, s. For comparison, we also present the results obtained when s is 5. Table 3.3 presents the performances of the benchmark strategies with respect to various test instances. The columns have the same interpretation as in Table 3.2. To emphasize the effect of commitment decision in these experiments, we fix the commitment fee to the highest value, $f^c = 80$. As depicted in Table 3.3, the total expected revenues decrease as the length of the commitment period increases since the revenues obtained from the contingent commitments decrease with the length of the commitment period. However, this loss can be compensated with the later arrivals of the booking requests. Thus, the decrease in revenue is more striking when the arrival intensity is low ($\rho = 1.2$). On the other hand, the results indicate that ADM consistently provides the highest total expected revenues. However as the length of the commitment period increases, the percentage gaps between ADM and the other solution methods decrease. This behavior can be attributed to the impact of commitment period on retain, buy and leave probabilities. Recall that $q_r = (s-1)/s$; so, q_r increases as length of the commitment period increases. Therefore, when s is high, our proposed dynamic model presumes that each accepted commitment request stays until the departure time (overestimates the commitment period). Consequently, it may fail to capture the actual dynamics of the system and its performance deteriorates. We note the performance difference between ADM and SBS with respect to different lengths of the commitment period. As the length of the commitment period increases, offering commitment option becomes less profitable. Moreover, as the length of the commitment period increases, there exist instances where the performances of SBS and DLP are significantly close. Recall that our problem can be transformed to a standard capacity allocation problem when $p_b = 1$. Thus, the performance of SBS improves as p_b increases and ν decreases.

3.4.2 Sensitivity Analysis

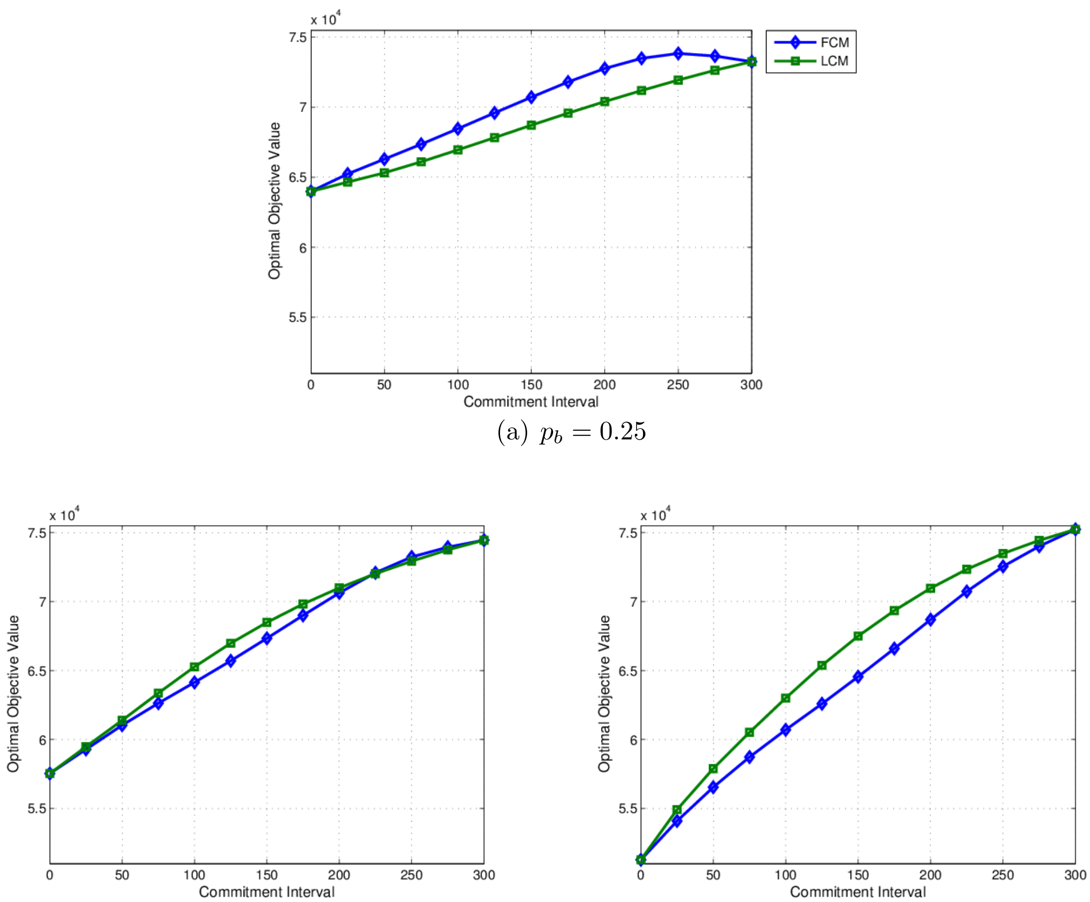
In the first set of simulations, we investigate the effects of the contingent commitment option. We set the load factor (ρ) to 1.6, the length of the commitment period (s) to 50, and the commitment fee (f^c) to 80. The splitting probabilities of commitments for all fare classes are set to the same value of 0.5. Initially, we study the potential revenue improvements of offering the commitment option relative to offering only standard bookings. We examine two models which accept contingent commitment requests during a certain time period. While the first model allows commitment arrivals only in the first t periods, the second model allows them only in the last t. These models

Table 3.3: Computational results for the test problems ($f^c = 80$)

Instances							% Gap with ADM	
ρ	$\nu_\bullet$	s	p_b	ADM	SBS	DLP	SBS	DLP
1.2	ν_H	5	0.4	68,294	51,455	66,247	24.66%	3.00%
		5	0.7	66,453	42,685	64,210	35.78%	3.37%
		25	0.4	67,583	51,455	66,000	23.86%	2.34%
		25	0.7	66,280	42,685	64,331	35.56%	2.94%
		50	0.4	66,541	51,455	65,079	22.67%	2.20%
		50	0.7	65,831	42,685	63,997	35.16%	2.79%
	ν_L	5	0.4	64,308	60,362	62,201	6.13%	3.27%
		5	0.7	63,595	58,056	61,421	8.71%	3.42%
		25	0.4	64,139	60,362	62,001	5.89%	3.33%
		25	0.7	63,514	58,056	61,482	8.59%	3.20%
		50	0.4	63,885	60,362	61,933	5.51%	3.05%
		50	0.7	63,401	58,056	61,379	8.43%	3.19%
1.6	ν_H	5	0.4	78,043	63,738	76,038	18.33%	2.57%
		5	0.7	76,541	56,267	74,397	26.49%	2.80%
		25	0.4	76,819	63,738	75,084	17.03%	2.26%
		25	0.7	76,067	56,267	74,167	26.03%	2.50%
		50	0.4	75,264	63,738	73,370	15.31%	2.52%
		50	0.7	75,385	56,267	73,492	25.36%	2.51%
	ν_L	5	0.4	74,220	71,411	72,243	3.79%	2.66%
		5	0.7	73,606	69,289	71,660	5.86%	2.64%
		25	0.4	73,943	71,411	72,098	3.42%	2.49%
		25	0.7	73,497	69,289	71,652	5.72%	2.51%
		50	0.4	73,540	71,411	71,695	2.89%	2.51%
		50	0.7	73,303	69,289	71,497	5.48%	2.46%

are denoted by FCM and LCM, respectively. Figure 3.3 shows the optimal objective values of these models with respect to different purchase probabilities. In this figure, the horizontal axis namely commitment interval shows t time periods during which the commitment requests are processed. For instance, $t = 10$ means that FCM accepts commitments during only the first 10 periods and LCM accepts them only in the last 10 periods. On the other hand, when $t = T$, both FCM and LCM boil down to ADM model where commitment arrivals are allowed during the whole reservation horizon.

As Figure 3.3 shows, FCM performs better than LCM when the probability of purchase is low. Due to the high retain probability (q_r) and low purchase probability (p_b), offering the commitment option later in the reservation horizon may result in empty seats. Since FCM accepts contingent commitment requests early in the reservation period, it can compensate the empty seats resulting from the commitment cancellations with the late booking arrivals. Therefore, accepting commitment requests up to a certain time period is more profitable than allowing them towards the end of the reservation horizon. On the other hand, as purchase probability increases, the performance of LCM improves. Since expensive fare class customers arrive later than the low-fare customers and the retain probability is high, FCM rejects the early commitment requests to keep seats for expensive fare class customers. Hence, it loses the potential revenue obtained from commitment reservations. Moreover, as Figure 3.3(b) and 3.3(c) depicts, allowing commitment arrivals during the whole reservation period is advantageous when the probability of purchasing a committed seat is not low.



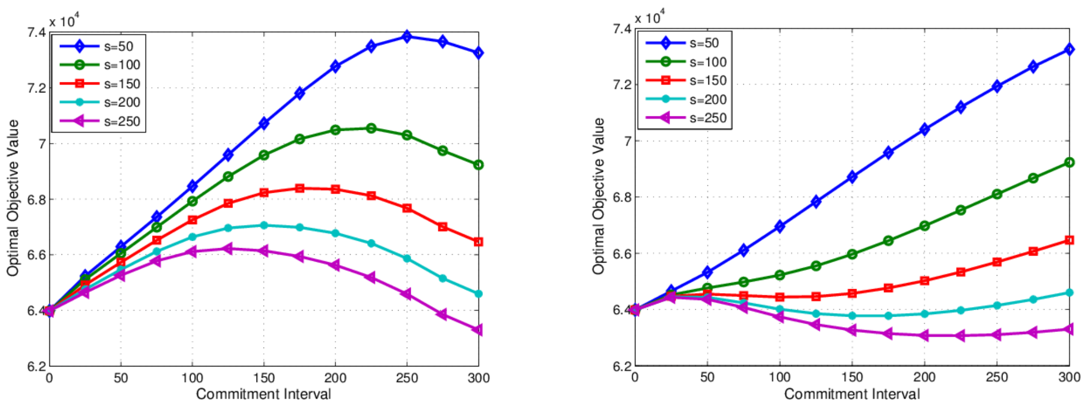
(a) $p_b = 0.25$

(b) $p_b = 0.50$

(c) $p_b = 0.75$

Figure 3.3: The effect of commitments on the total expected revenue for various buy probabilities

Next, we investigate the effect of the commitment period on total expected revenue. Figure 3.4 plots the changes in total expected revenues of FCM and LCM with respect to different lengths of commitment interval and the commitment period when p_b is low. Let t^* denote the commitment interval value at which the maximum total expected revenue is obtained either by FCM or LCM in Figure 3.4. As the length of the commitment period (s) increases, the value of t^* for FCM shifts to the beginning of the reservation period. Recall that retain probability is positively correlated with the length of the commitment period. Therefore, as s increases, the probability of waiting until the end of the reservation horizon (retain probability of a contingent commitment) also increases. As a result when p_b is low, it becomes more profitable for FCM to accept the commitment requests early in the reservation period and reserve seats for the late arrivals of expensive fare class customers. Similarly when s is high, it is more profitable for LCM to allow less commitment arrivals and t^* for LCM shifts towards the end of the reservation horizon. However, even in this case, offering contingent commitment options can provide additional revenue compared to only offering standard booking products (when commitment interval is 0).



(a) Total expected revenue of FCM (b) Total expected revenue of LCM

Figure 3.4: Change in the total expected revenue with respect to different s values (p_b=0.25)

In summary, Figure 3.3 depicts that accepting the commitment requests up to a certain time period is more profitable than accepting them during the whole reservation period when purchase probability p_b of the contingent commitment option is low. As purchase probability increases, allowing commitment arrivals during the whole period becomes more advantageous. In addition, offering commitment option towards the end of the reservation period is more beneficial than offering it at the beginning of the reservation period when purchase probability is high. Moreover, Figure 3.4 shows the effect of length of the commitment period on total revenue. We observe that as the length of the commitment period increases, it becomes more profitable to decrease the length of the time period during which the commitment option is offered.

Note that the results of our approximate model depend on the buy probability parameter, p_b^a. We next consider what happens if this parameter is not accurately specified. To answer this question, we simulate several scenarios, where p_b^a is either over- or under-estimated. We set the length of the commitment period, the commitment fee and the exact buy probability to $s = 5$, $f^c = 80$, and $p_b = 0.4$, respectively. We vary $p_b^a \in \{0.2, 0.4, 0.6\}$. Figure 3.5 presents the percentage gaps between the revenues obtained by the approximate dynamic model and the revenues of the exact dynamic model. In this figure, test instances are denoted by the triplet (ρ, ν, p_b^a). H and L indicate the high and low splitting probability (ν) which are defined at the beginning of Section 3.4.1. Moreover, the dashed lines depict the exact values of the buy probability.

An interesting result we want to point out is the change in the average percentage gaps. The percentage gaps decrease when the load factor (ρ) is high. When p_b^a is underestimated, ADM accepts more reservation requests from the low fare classes, since the estimated cancellation probability is high. This treatment results in empty seats when load factor is low. On the other hand, when p_b^a is overestimated, ADM rejects the reservation requests coming from the low fare classes. Therefore, it may lose the extra fee obtained from those rejected customers. When the load factor is high, it compensates this loss with the late arrivals.

3.4.3 An Alternate Simulation

In this simulation study, we relax the assumption related to the purchase decision of the committed seats. In our proposed models, we assumed that the customers who committed to a seat make the buy or leave decision at the end of the commitment

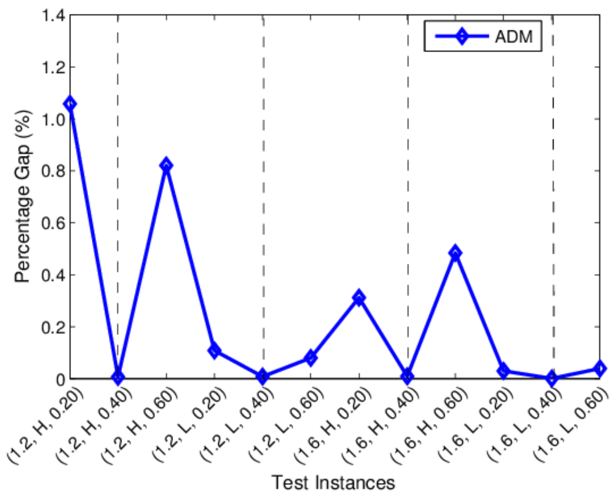


Figure 3.5: Effect of p_b^a estimation

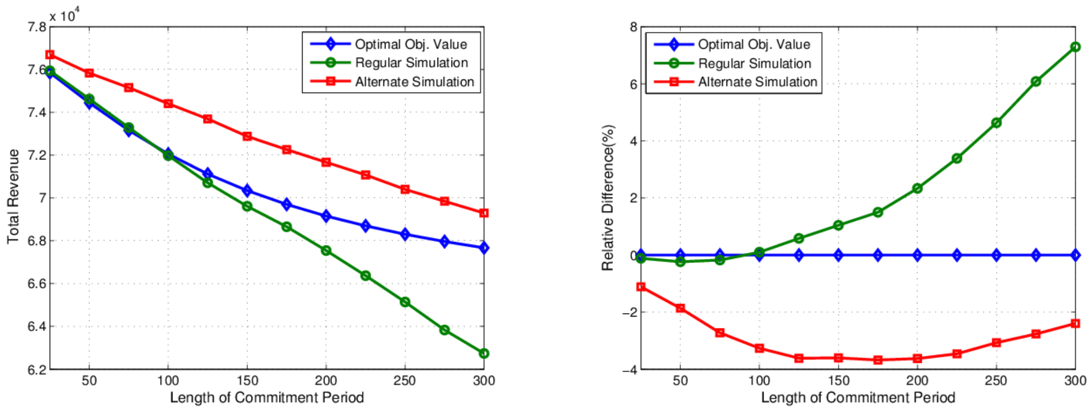
period. Although this is quite often the case, sometimes those customers may purchase the seat or cancel their reservation at any time within the commitment period. In this section, we simulate such a setting. We assume that after committing to a seat, it is equally likely for a customer to make a decision in each one of the s periods. Since we compare the benchmark strategies for long commitment periods, we were not able to solve the EDM model in this analysis.

Our results are summarized in Table 3.4. The commitment fee is set to $f^c = 80$. Comparing the total expected revenues in Table 3.3 against those given in Table 3.4, we notice that the total net revenues obtained by the policies of all solution methods slightly improve in this alternate simulation. Since frequent cancellations occur in the alternate simulation, the expected revenues obtained from the commitments increase. It is important to note that the percentage gaps between ADM and the other solution methods tend to increase when we allow frequent commitment cancellations. ADM adjusts the booking limits by taking into account the reservations and cancellations that have already taken place. Therefore, it ends up accepting more reservation requests from lower fare classes than the deterministic model, and consequently, the revenue loss due to empty seats is counteracted by the gains from the committed seats.

Table 3.4: Computational results for the test problems in the alternate simulation ($f^c = 80$)

Instances							% Gap with ADM	
ρ	$\nu_\bullet$	s	p_b	ADM	SBS	DLP	SBS	DLP
1.2	ν_H	5	0.4	68,352	51,455	66,261	24.72%	3.06%
		5	0.7	66,512	42,685	64,225	35.82%	3.44%
		25	0.4	67,967	51,455	66,365	24.29%	2.36%
		25	0.7	66,456	42,685	64,457	35.77%	3.01%
		50	0.4	67,376	51,455	66,014	23.63%	2.02%
		50	0.7	66,216	42,685	64,399	35.54%	2.74%
	ν_L	5	0.4	64,328	60,362	62,209	6.17%	3.29%
		5	0.7	63,595	58,056	61,426	8.71%	3.41%
		25	0.4	64,237	60,362	62,096	6.03%	3.33%
		25	0.7	63,568	58,056	61,528	8.67%	3.21%
		50	0.4	64,103	60,362	62,209	5.84%	2.95%
		50	0.7	63,501	58,056	61,510	8.57%	3.14%
1.6	ν_H	5	0.4	78,127	63,738	76,082	18.42%	2.62%
		5	0.7	76,578	56,267	74,406	26.52%	2.84%
		25	0.4	77,506	63,738	75,533	17.76%	2.55%
		25	0.7	76,375	56,267	74,363	26.33%	2.63%
		50	0.4	76,613	63,738	74,481	16.81%	2.78%
		50	0.7	76,014	56,267	73,988	25.98%	2.67%
	ν_L	5	0.4	74,242	71,411	72,249	3.81%	2.68%
		5	0.7	73,618	69,289	71,664	5.88%	2.65%
		25	0.4	74,125	71,411	72,208	3.66%	2.59%
		25	0.7	73,587	69,289	71,704	5.84%	2.56%
		50	0.4	73,918	71,411	72,018	3.39%	2.57%
		50	0.7	73,487	69,289	71,667	5.71%	2.48%

We also analyze how our approximation performs. Figure 3.6 presents the gap between the optimal objective value of the approximate dynamic model and the average revenue obtained by its policy when regular and alternate simulations are run. The first observation we have is that the percentage gaps are small when the length of commitment period is short. The intuition behind this result is that, as the length increases, ADM fails to predict the dynamics of the commitment process. As a result, the number of empty seats increases and hence, its performance deteriorates. We caution the reader to the performances under two simulations. As Figure 3.6(a) depicts total revenue obtained in the alternate simulation always higher than the one obtained in regular simulation. Moreover, as the length of the commitment period increases, the performance of ADM worsens more than we expected. This result was more striking with our regular simulation. This behavior can be attributed to the structure of the alternate simulation. Since our approximation allows contingent commitment cancellation at any time, it performs better in the alternate simulation.



(a) Total expected revenue (b) Percentage gap

Figure 3.6: The results related to optimal objective value of ADM and the average revenue obtained by the optimal policy of ADM

3.4.4 An Alternate Dynamic Programming Formulation

Dynamic model (3.2) assumes that a customer decides whether she wants to purchase the contingent commitment option after she books the ticket. Therefore, the decisions to accept or reject a request for a standard booking and a commitment are identical. We can also model this problem by assuming that booking and commitment decisions are independent. In this approach, a customer arrives into the system with a particular fare class in mind and with the intention to purchase the contingent commitment option. If the customer is interested in the contingent commitment option and the requested fare class is open with this option, then she purchases the option. Similarly, if the customer is not interested in purchasing the option and the requested fare class is open without a contingent commitment option, then she makes the booking.

To model this approach, we use $u_{it} \in \{0, 1\}$ and $w_{it} \in \{0, 1\}$ to represent the decisions for standard booking and contingent commitment option, respectively. Then, the sets of feasible decisions at time period t are

$$\mathcal{U}_t(x_t) = \{\mathbf{u}_t \in \{0,1\}^m : x_t + u_{it} \leq C, \quad i = 1, 2, \cdots, m\},$$

and

$$\mathcal{W}_t(x_t) = \{\mathbf{w}_t \in \{0,1\}^m : x_t + w_{it} \leq C, \quad i = 1, 2, \cdots, m\}.$$

As in the model (3.2), let x_t and y_t be the total number of reservations (including both contingent commitments and bookings) and contingent commitments at time period t, respectively. Then, the recursive equations for the alternate dynamic programming model is given by

$$\begin{aligned}\bar{V}_t(x_t, y_t) = \max_{\mathbf{u}_t \in \mathcal{U}(x_t)} & \Big\{ \sum_{i=1}^{m} p_{it}\{f_i u_{it} + \mathbb{E}\bar{V}_{t+1}(x_t + u_{it} - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t))\} \Big\} + \\ & \max_{\mathbf{w}_t \in \mathcal{W}(x_t)} \Big\{ \sum_{i=1}^{m} q_{it}\{\phi_i^a w_{it} + \mathbb{E}\bar{V}_{t+1}(x_t + w_{it} - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t) + w_{it})\} \Big\} + \\ & p_{0t}\mathbb{E}\bar{V}_{t+1}(x_t - \mathbf{M_l}(y_t), \mathbf{M_r}(y_t)).\end{aligned}$$

Again, the boundary condition is simply $\bar{V}_{T+1}(x_{T+1}, y_{T+1}) = 0$.

We compare the performance of this approach with our approximate dynamic model on various test instances in alternate simulation. We refer to the average revenue obtained by the optimal policy of this alternate model as AF. Our results are presented in Table 3.5. Comparing the relative gaps, we observe that AF performs better than ADM. This is an expected result, since AF uses additional arrival probability information while determining the optimal policy. A noteworthy observation is that the percentage gaps are more significant when the contingent commitment probability is high and the purchase probability of the committed seat is low. Since AF discriminates between commitment and standard booking decisions, it adjusts the booking limits by considering the low purchase probability of contingent commitments. As a result, it can benefit from the non-refundable commitment fee by improving its capacity utilization. On the other hand, when the load factor is high and the commitment arrival intensities are low, the average percentage difference is around 0.22%. In this case, both models benefit from the increase in the number of booking requests for the expensive fare classes.

Table 3.5: Computational results for the alternate dynamic programming model ($f^c = 80$)

Instances						AF *versus*
ρ	$\nu_\bullet$	s	p_b	AF	ADM	ADM
1.2	ν_H	5	0.4	69,359	68,352	1.45%
		5	0.7	67,106	66,512	0.88%
		25	0.4	68,919	67,967	1.38%
		25	0.7	67,087	66,456	0.94%
		50	0.4	68,380	67,376	1.47%
		50	0.7	66,852	66,216	0.95%
	ν_L	5	0.4	64,746	64,328	0.64%
		5	0.7	63,974	63,595	0.59%
		25	0.4	64,665	64,237	0.66%
		25	0.7	63,925	63,568	0.56%
		50	0.4	64,540	64,103	0.68%
		50	0.7	63,872	63,501	0.58%
1.6	ν_H	5	0.4	78,785	78,127	0.84%
		5	0.7	76,949	76,578	0.48%
		25	0.4	78,089	77,506	0.75%
		25	0.7	76,723	76,375	0.45%
		50	0.4	77,221	76,613	0.78%
		50	0.7	76,355	76,014	0.45%
	ν_L	5	0.4	74,447	74,242	0.27%
		5	0.7	73,737	73,618	0.16%
		25	0.4	74,327	74,125	0.27%
		25	0.7	73,723	73,587	0.19%
		50	0.4	74,104	73,918	0.25%
		50	0.7	73,606	73,487	0.16%

We conclude the presentation of our numerical results by reporting the CPU times of the solution methods. We used a computer with 2.13 GHz Intel Pentium P6200 processor and 2 GB of RAM. The codes are written in MATLAB R2012b running under Windows 7 operating system. Exact dynamic model requires on average 230 seconds for the problems where length of commitment period is 5. It takes on average 120 seconds to solve the approximate dynamic model. DLP requires on average less than a second.

Chapter 4

NETWORK PROBLEM: HOTEL REVENUE MANAGEMENT

In this chapter, we work on the capacity allocation policies in hotel industry. Hotel revenue management can be considered as an airline network revenue management by treating each day as a flight leg and each room as a flight seat. Generally successful airline RM techniques such as booking control and overbooking are directly applied to the hotel problems. However, unlike airline RM, the hotel problem has a different structure. Customers have different arrival dates, their length of stays overlap with each other and more importantly long-term stay is very common. While a customer can stay seven consecutive days in a hotel, a flight itinerary rarely includes more than three flight legs. Therefore, network effects can be more significant compared to the airline network problem. In addition, the demand structure of the hotel problem is different. While airline passengers make advance reservations, a good portion of the hotel customers are walk-ins and even early bookings can cancel for free depending on the hotel policy. Moreover, hotel customers can easily change their length of stay (extend or leave early) at a moment's notice. These features create unique challenges in hotel capacity allocation, which we address in the thesis.

Network structure allows staying single or multiple days in a hotel that increases the interaction between products. Room allocation problem is the control of total room capacity when the customer demand is characterized by the length of stay and the room types. Dynamic control of this problem is a challenge for analysis and optimization since the state space becomes basically the Cartesian product of daily capacities in the hotel network. To overcome this difficulty, approximation methods based on decomposition are proposed. The main idea behind this type of decomposition is to partition the network problem into independent subproblems. This approach has been widely adopted by the airline industry. However, these decomposition methods may undermine the network effects of shared daily capacities between different products (reservation types). To improve decomposition methods, recent studies focus on protecting network information [53, 96, 4]. Weatherford [90] points out that methods taking the length of stay into account in hotel revenue management generate greater expected revenues of up to 2.9% over the traditional single-day (or leg-based) methods.

Throughout this chapter, we focus on the problem of managing the room allocation decisions in case of stochastically arriving customers in a hotel. These customers are classified as advance bookings, stay-overs and walk-ins. While advance bookings make room reservations before they arrive at the hotel, walk-ins show up without any reservation during the service period. Stay-overs are the customers who had advance bookings and ask for extension on their reservations during their stay in the hotel. We build our model upon this terminology. We focus on network decomposition approaches proposed for airline RM and migrate the methods to the hotel RM context by eliminating all drawbacks in implementation.

We begin by reviewing the related work proposed in the hotel revenue management context and then explain decomposition approaches applied to the network problem. As mentioned in Chapter 1, there is limited research in hotel revenue management compared with the airline literature. Ladany [55] works on the single day model with two types of resources. The aim of the model is to find an allocation policy in order to maximize the expected revenue per day. He develops a dynamic programming formulation and obtains the allocation policy by using sequential decision process. Williams [92] works on the single-day model during the peak demand period. In this model, he assumes that demand arrives from three different sources; stayovers, advanced reservations and walk-ins. He computes the reservation policy for each customer type by comparing the underbooking and overbooking costs. Bitran and Mondschein [15] develop a dynamic programming model for single-day problem with multiple products.

Since the resulting model is computationally intractable for the real size problems, they utilize some heuristics when searching for the optimal allocation policy. Weatherford [90] focuses on the effect of length of stay. He proposes a heuristic method based on a static model and compares this method with the other booking policies developed for single-day. Bitran and Gilbert [17] work on a single-day and single-room problem. They assume that during the service day, three type of customers show-up; customers with guaranteed reservations, customers with reservations and walk-ins. They develop a dynamic model and propose a heuristic method to obtain the room allocation policy. Baker and Collier [7] extend the study of Weatherford and Bitran and Mondschein by allowing cancellations, overbooking and stay-overs. They develop two heuristics that integrate overbooking with the capacity allocation decisions. They compare the performance of these heuristics with the other booking control policies in the literature under different operating environments and discuss the advantages of each policy for each environment.

Later studies focus on the multi-product and multi-day stay problem. Chen [24] presents a general formulation for the deterministic problem and discusses that it can be transferred to a network flow problem. Moreover, he shows that optimal solution of the linear program is always integral. Goldman et al. [43] propose deterministic and stochastic linear programming models to find nested booking limits and bid prices for the multi-day stay problem. They utilize from the work of Weatherford [90] to develop the deterministic model. For the stochastic model, they extend the work of De Boer et al. [19] on airline revenue management problem. However, unlike Weatherford's and De Boer et al.'s model, they use the booking control policies over a rolling horizon of decision periods. Lai and Ng [56] work on a stochastic programming formulation for multi-day problem. They apply robust optimization techniques to solve the problem on a scenario-basis. They also consider the risk aversion of the decision maker and use mean absolute value to measure the revenue deviation risk. Koide and Ishii [50] work on the optimal room allocation policies for a single day by considering early discounts, cancellations and overbookings. They examine the properties of the expected revenue function and show that it is unimodal on the number of allocated rooms for early discount and overbooking. As with Lai and Ng, Liu et al. [64] present revenue optimization models for multi-day stay problem by considering the revenue risk. They propose stochastic programming model with semi-absolute deviations to measure the risk.

The problem considered in this study builds on the literature on decomposition methods in revenue management. The output of the decomposition methods is used to construct various strategies, such as bid-prices and nested booking limits. Adelman [1] develops an approximation method to compute dynamic bid prices. He first formulates the problem as a dynamic model and due to the curse of dimensionality, he derives a standard linear program by approximating the dynamic programming value functions. This approach provides an upper bound on the optimal expected revenue. Zhang [96] proposes a non-linear non-separable functional approximation to dynamic programming model that leads to a tighter upper bound. Topaloglu [84] focuses on the capacity dependent bid-prices and proposes a Lagrangian relaxation method to decompose the network problem by the flight legs. Erdelyi and Topaloglu [33] work on the overbooking problem in airline network and develop separable approximations to decompose the problem by flight legs. This approach constructs capacity dependent bid prices. However, it is quite difficult to compute the value functions for each leg. To reduce the computational burden, Kunnumkal and Topaloglu [54] develop stochastic approximation algorithm which provides capacity independent bid prices. In this method, they formulate the total expected profit as a function of bid prices and use the stochastic gradients to obtain a good bid price policy. Recently, Kunnumkal and Topaloglu [53] propose a new leg-based decomposition method for airline revenue management with customer choice behaviour. In this method, they allocate revenues of each itinerary among the covered legs and to incorporate network effect, they define a penalty term. They view the revenue allocations and penalty terms as decision variables and use subgradient search to find the optimal solution. Although this solution approach is manageable in small size networks, it will not be practical for the hotel problems since the network size can be larger compared to the airline problems. Hotel network problems are also tackled with decomposition methods. Zhang and Weatherford [97] work on the dynamic pricing problem in hotel network. They generalize the approximation method of Zhang [96] and decompose the problem into independent single-days by approximating the value functions with nonlinear non-seperable functions. They test the proposed approach by using data from a US hotel. Aslani et al. [4] propose a decomposition method for pricing problem in hotel revenue management. They develop an approach to estimate the effective arrival rate of each day by considering the stock-outs and customer losses due to high price levels. They decompose the network problem into single-day subproblems with these daily arrival rates.

In this chapter, we work on the hotel capacity allocation problem with walk-in and stay-over customers. We aim to develop practical models that provide good approximations to the hotel network problem. We first concentrate on the day-based decomposition method by considering the model of Kunnumkal and Topaloglu [53]. We propose a dynamic model including walk-in customers and formulate the problem as a linear programming model. We observe that this problem has a block angular structure and it can be efficiently solved by Dantzig-Wolfe decomposition. However, it may still be slow to solve larger-size network problems. Therefore, we propose an alternative solution method and show that this method provides an upper bound on the model of Kunnumkal and Topaloglu. We also discuss several well-known models from the literature and generate benchmark strategies to test the performances of these solution approaches.

To manage stay-over requests, we need to keep track of the number of reservations in each booking type. However, day-based methods decompose the network problem into independent days which results in a loss of information on the number of customers in each booking type. To overcome this difficulty, we work on pair-based decomposition methods. We concentrate on the decomposition approach proposed by Birbil et al. [29] and extend the model to include stay-overs. In our numerical experiments, we observe that the objective function of the stay-over model is discrete concave and hence, we propose a method to replace it with a piece-wise linear concave function. This enables us to rewrite the problem as a linear programming model.

This chapter is organized as follows. In Section 4.1, we describe the problem and present dynamic programming model to coordinate room allocation decisions of a hotel. We then discuss the single-day decomposition methods in Section 4.2.1. We start by explaining the work of Kunnumkal and Topaloglu [53]. We then introduce our linear programming approach and analyze its properties. We also present some alternative solution methods. In Section 4.2.3, we consider stay-over extension of the room allocation model and explain the check-in and check-out pair-based decomposition method. Furthermore, based on our observations, we discuss our solution approaches. Finally, in Section 4.3, we present computational experiments.

4.1 Model Formulation

We consider a hotel network with daily capacity C_i. Customers may book for a single day or up to n multiple-days. The days and booking fares can be combined to produce products (length of reservation and room price combinations). Figure 4.1 shows the structure of a hotel network with multiple-day stays. In this figure, each node presents the possible check-in and check-out days. The set of service days in the hotel network is denoted by $\mathcal{L}$ and the set of products is denoted by $\mathcal{K}$. We use $\mathcal{L}_k$ to denote the set of days that are used by product k and $\mathcal{K}_i$ to denote the set of products involving day i. Throughout, we index the service days by i and products by k.

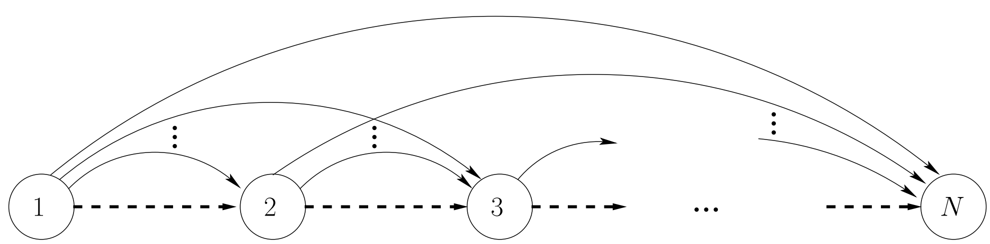


Figure 4.1: The hotel network with multiple time intervals

The planning period $\mathcal{T} = \{1, ..., \tau\}$ consists of two sub-periods, namely reservation and service. While in the reservation period product request arrives, accepted reservations occupy their rooms in the service period. During the planning period, we observe three type of customer arrivals. These arrivals are classified as advance bookings, walk-ins and stay-overs. Advance bookings are the customers who arrive with a reservation on the check-in date. On the other hand, walk-ins and stay-overs are the unexpected customer arrivals in the service period. While walk-ins show-up without any reservations, stay-overs request an extension on their stay. In this study, we first work on the room allocation model with walk-in customers. Then, we discuss the possible ways to incorporate stay-over requests.

Advance booking and walk-in requests can arrive for any of the products, and accepted reservations occupy the reserved rooms at the service period. At each period t, a reservation request for product k arrives with probability p_{kt} and we have to decide whether to accept or reject the booking requests. If we accept a reservation request for product k, then we generate a revenue of f_k. Each accepted product k request consumes one unit of capacity on the day $i \in \mathcal{L}_k$.

We assume that there are no reservations in the system at time $t = 0$, and at most one reservation requests arrive at each time period. We define a_{ik} to represent the amount of resource i required by product k customer. In other words, if day i is used by product k, $a_{ik} = 1$ and $a_{ik} = 0$ otherwise. We also define e_i as an $|\mathcal{L}|$-dimensional vector with a one in the element corresponding to day i. To give a dynamic programming formulation for this problem, we need to store the number of reservations in each day. We use z_{it} to denote the total number of reservations on day i at time period t and the vector $\mathbf{z}_t = \{z_{it} : i \in \mathcal{L}\}$ represents the state in our dynamic programming model.

We are ready to formulate the problem as a dynamic program. Let $J_t(\mathbf{z}_t)$ denote the expected optimal revenue from t up to τ given that the state of the reservations at the beginning of time period t is $\mathbf{z}_t$. For every $1 \leq t \leq \tau$, we can find the optimal policy by computing the value functions through the optimality equation

$$J_t(\mathbf{z}_t) = \sum_{k \in \mathcal{K}} p_{kt} \max\{f_k + J_{t+1}(\mathbf{z}_t + \sum_{i \in \mathcal{L}} a_{ik} e_i), J_{t+1}(\mathbf{z}_t)\} + (1 - \sum_{k \in \mathcal{K}} p_{kt}) J_{t+1}(\mathbf{z}_t) \quad (4.1)$$

where the boundary condition is simply $J_{\tau+1}(\mathbf{z}_{\tau+1}) = 0$ for all $\mathbf{z}_{\tau+1}$.

Unfortunately, this model is intractable because the state variable $\mathbf{z}$ may involve many dimensions in practical applications and hence, solving the complete dynamic model is not possible even for small-scale problems [80].

4.2 Decomposition Methods

In this section, we discuss the approaches to approximate the dynamic programming formulation in (4.1). We first concentrate on day-based decomposition methods and then work on the check-in and check-out pair-based methods.

4.2.1 Day-Based Decomposition

Recently, Kunnumkal and Topaloglu [53] have proposed an approximate dynamic programming formulation for the customer choice problem in airline network revenue management. This method decomposes the airline network into single-leg subproblems. To decompose the network, they allocate the revenue of each product to the flight legs that it uses. In other words, they define leg-based price $\{\alpha_{ikt} : i \in \mathcal{L}_k, k \in \mathcal{K}, t \in \mathcal{T}\}$ for each product as follows:

$$\sum_{i \in \mathcal{L}_k} \alpha_{ikt} = f_k \quad \forall k \in \mathcal{K}, t \in \mathcal{T}. \tag{4.2}$$

Moreover, they introduce penalty terms $\{\beta_{ikt} : i \in \mathcal{L}_k, k \in \mathcal{K}, t \in \mathcal{T}\}$ for each product to coordinate the network decisions. Day-based penalties for a product should satisfy

$$\sum_{i \in \mathcal{L}_k} \beta_{ikt} = 0 \quad \forall k \in \mathcal{K}, t \in \mathcal{T}. \tag{4.3}$$

This condition on penalties guarantees that if we make the same decision for a product at each subproblem, then the overall penalty for that product is equal to zero. In other words, if we accept a reservation request of product k for each day it uses, then the collected revenue is $\sum_{i \in \mathcal{L}_k} \alpha_{ikt} = f_k$ and the collected penalty is $\sum_{i \in \mathcal{L}_k} \beta_{ikt} = 0$.

By applying this idea to our model, we can transfer the multi-day hotel problem to single-day subproblems. In this case, if we accept a booking request for product k at time period t on day i, we generate a revenue of α_{ikt} and incur a penalty of β_{ikt} and consume the unit capacity of day i. We use z_{it} to denote the total number of reservations on day i at the beginning of time period t. Then, we can formulate the single-day problem as a dynamic program as

$$\begin{aligned} V_{it}(z_{it}|\alpha,\beta) = &\sum_{k\in\mathcal{K}} p_{kt} \max\{\alpha_{ikt} + \beta_{ikt} + V_{it+1}(z_{it} + a_{ik}), V_{it+1}(z_{it})\} \\ &+ (1 - \sum_{k\in\mathcal{K}} p_{kt}) V_{it+1}(z_{it}), \end{aligned} \tag{4.4}$$

for every $1 \leq t \leq \tau$. The boundary conditions simply become $V_{it}(C_i) = 0$ and $V_{it_i}(z_{it_i}) = \sum_{k\in\mathcal{K}_i} p_{kt_i} \max\{\alpha_{ikt_i} + \beta_{ikt_i}, 0\}$ for all $z_{it_i} = 1, \cdots, C_i - 1$, where t_i is the last day we can accept customers for day i; i.e., $t_i = t_1 + i - 1$. Kunnumkal and Topaloglu [53] show that this approximation provides an upper bound on the optimal expected revenue for customer choice model. This result can be easily extended to our model.

Proposition 4.2.1 *(Kunnumkal and Topaloglu [53], Proposition 1) For all $z_{it} \leq C_i$ and $1 \leq t \leq \tau$, we have $J_t(z_t) \leq \sum_{i\in\mathcal{L}} V_{it}(z_{it}|\alpha,\beta)$ such that (α,β) satisfies conditions (4.2) and (4.3).*

Since $\sum_{i\in\mathcal{L}} V_{i1}(0|\alpha,\beta)$ gives an upper bound on $J_1(\mathbf{0})$, we can obtain the tightest bound by solving the following problem.

$$\text{minimize} \quad \sum_{i\in\mathcal{L}} V_{i1}(0|\alpha,\beta) \tag{4.5}$$

$$\text{subject to} \quad \sum_{i\in\mathcal{L}_k} \alpha_{ikt} = f_k, \quad \forall k \in \mathcal{K}, t \in \mathcal{T} \tag{4.6}$$

$$\sum_{i\in\mathcal{L}_k} \beta_{ikt} = 0, \quad \forall k \in \mathcal{K}, t \in \mathcal{T} \tag{4.7}$$

$$\alpha_{ikt} \geq 0, \quad \forall k \in \mathcal{K}, i \in \mathcal{L}, t \in \mathcal{T} \tag{4.8}$$

Optimal objective value of problem (4.5)-(4.8) provides an upper bound on the maximum expected revenue over the whole planning horizon. Moreover, it gives the optimal values of the revenue allocations and the penalties that we can use to construct a control policy for the dynamic model (4.4).

The reservation policy for the network problem can be summarized as follows: Given the allocations $\{(\alpha_{ikt}, \beta_{ikt}), \forall i \in \mathcal{L}, k \in \mathcal{K}, t = 1, ..., \tau\}$ and the state variables $\{z_{it}, \forall i \in \mathcal{L}\}$ at time period t, we then accept a reservation request for product k if we have,

$$f_k \geq \sum_{i \in \mathcal{L}_k} ((V_{it}(z_{it}|\alpha, \beta) - V_{it}(z_{it} + 1|\alpha, \beta)) .$$

Next, we explain the steps that can reduce problem (4.5)-(4.8) to a linear program. Although value function $V_{it}(z_{it}|\alpha, \beta)$ depends on the whole set of $\{z_{it} : i \in \mathcal{L}\}$ due to the structure of problem (4.4), it actually depends only on z_{it} and $z_{it} + 1$. This result enables us to model the problem (4.5)-(4.8) as a tractable linear program.

For ease of expression, we replace z_{it} with z to denote the total number of reservations on day i at time period t. To rewrite the recursion let us first define

$$x^i_{ktz} := \max \{\alpha_{ikt} + \beta_{ikt} + V_{i,t+1}(z + 1) - V_{i,t+1}(z), 0\} \tag{4.9}$$

and

$$x^i_{kt_i} := \max \{\alpha_{ikt_i} + \beta_{ikt_i}, 0\} . \tag{4.10}$$

Then, we can rewrite (4.4) as

$$\begin{aligned} V_{it}(z) &= \sum_{k \in \mathcal{K}_i} p_{kt} x^i_{ktz} + V_{i,t+1}(z) \\ &= \sum_{k \in \mathcal{K}_i} p_{kt} x^i_{ktz} + \sum_{k \in \mathcal{K}_i} p_{k,t+1} x^i_{k,t+1,z} + V_{i,t+2}(z) \end{aligned}$$

If we continue in the same fashion, we obtain

$$V_{it}(z) = \sum_{k \in \mathcal{K}_i} p_{kt} x^i_{ktz} + \cdots + p_{k,t_i-1} x^i_{k,t_i-1,z} + p_{kt_i} x^i_{kt_i}.$$

We can then simplify the difference

$$\begin{aligned}
V_{it}(z+1) - V_{it}(z) &= \sum_{k \in \mathcal{K}_i} p_{kt}(x^i_{k,t,z+1} - x^i_{ktz}) + \cdots + p_{k,t_i-1}(x^i_{k,t_i-1,z+1} x^i_{k,t_i-1,z}) \\
&= \sum_{k \in \mathcal{K}_i} \sum_{s=t+1}^{t_i-1} p_{ks}(x^i_{k,s,z+1} - x^i_{ksz}).
\end{aligned}$$

This formulation enables us to linearize our model. Moreover, it provides an alternative simple proof for the convexity of the model (4.5)-(4.8).

As we mentioned before, we assume that at most one customer arrives at each time period. Therefore, the total number of accepted customers on day i at time period t is at most $\min\{C_i, t\}$. Let θ_t denote this bound on the total number of reservations. Together with (4.9) and (4.10), we have to introduce, for each $k \in \mathcal{K}$, $i \in \mathcal{L}_k$, $t = 1, \cdots, t_i - 1$ and $z = 1, \cdots, \theta_t - 1$, the following constraint into our linear programming problem:

$$x^i_{ktz} \geq \alpha_{ikt} + \beta_{ikt} + \sum_{\kappa \in \mathcal{K}_i} \sum_{s=t+1}^{t_i-1} p_{\kappa s}(x^i_{\kappa,s,z+1} - x^i_{\kappa sz}).$$

Then, we can reformulate the problem (4.5)-(4.7) as

$$\text{minimize} \quad \sum_{k\in\mathcal{K}}\sum_{i\in\mathcal{L}_k}\sum_{t=1}^{\tau} p_{k1}x^i_{kt0} \tag{4.11}$$

$$\text{subject to} \quad x^i_{ktz} \geq \alpha_{ikt} + \beta_{ikt} + \sum_{l\in\mathcal{K}}\sum_{s=t+1}^{t_i-1} a_{il}p_{lt+1}(x^i_{lsz+1} - x^i_{lsz}),$$
$$\forall k \in \mathcal{K}, \forall i \in \mathcal{L}_k, t = 1, .., t_i - 1, z = 0, ..., \theta_t - 1 \tag{4.12}$$

$$x^i_{kt_iz} \geq \alpha_{ikt_i} + \beta_{ikt_i}, \qquad \forall k \in \mathcal{K}, \forall i \in \mathcal{L}_k, z = 1, ..., C_i - 1 \tag{4.13}$$

$$x^i_{ktC_i} = 0, \qquad \forall i \in \mathcal{L}, \forall k \in \mathcal{K}, t \in \mathcal{T} \tag{4.14}$$

$$\sum_{i\in\mathcal{L}_k} \alpha_{ikt} = f_k, \qquad \forall k \in \mathcal{K}, t \in \mathcal{T} \tag{4.15}$$

$$\sum_{i\in\mathcal{L}_k} \beta_{ikt} = 0, \qquad \forall k \in \mathcal{K}, t \in \mathcal{T} \tag{4.16}$$

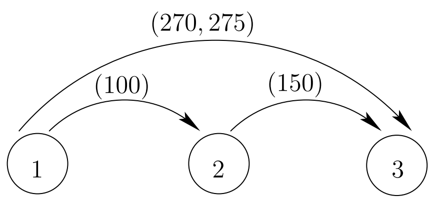
$$x^i_{ktz}, \alpha_{ikt} \geq 0, \qquad \forall i \in \mathcal{L}, \forall k \in \mathcal{K}, t \in \mathcal{T}, z = 0, ..., \theta_t - 1 \tag{4.17}$$

Problem (4.11)-(4.17) has $|\mathcal{L}||\mathcal{K}|(\tau - 1)\tau/2 + 2\tau|\mathcal{K}|$ constraints which may increase the solution time as time period increases. We discuss the solution algorithms in the following section.

Next, we examine the fare structure of the subproblems. We accept the arriving request of product k if its revenue is higher than the total marginal values of consuming one unit of capacity on the days traversed. Considering the optimal policy of exact dynamic model (4.1), it is optimal to accept a request for product k at time period t whenever

$$f_k \geq J_t(\mathbf{z}_t) - J_t(\mathbf{z}_t + \sum_{i\in\mathcal{L}} a_{ik}e_i).$$

Given the ordering $f_k > f_{k-1}$ between two products which use the same days, if we accept product $k-1$ request, then it is also optimal to accept product k request. However, we may not preserve this relation in each day, when we decompose the problem into single-days. We present a simple counter example where the ordering between the classes of the same pair is not preserved at each day. Figure 4.2 presents the network of this example. We have four different products and product 3 and 4 use the same days. Although the fares of the products 3 and 4 are ordered as $f_4 > f_3$, when we solve the problem (4.11)-(4.17) we obtain an interesting result. As it is seen in Figure 4.2, while the net revenue of product 3 on day 1, $(\alpha_{131} + \beta_{131})$, is greater than the bid price of that day, the net revenue of product 4 on day 1, $(\alpha_{141} + \beta_{141})$, is lower than that value. This means that the ordering between fare class on each day can deteriorate with the decomposition. In the next section, we will discuss a fare allocation procedure which also preserves the ordering between products consume capacities of the same days.



	Products 1	2	3	4
(i,j)	(1.2)	(2.3)	(1.3)	(1.3)
$E(D_k)$	2.40	2.05	4.45	1.15
f_k	100	150	270	275

(a) Network

	Products 3	4
$(\alpha_{ikt}, \beta_{ikt})$	$(270, -166.90)$	$(275, -174.06)$
Decision	Accept	Reject
$V_{it+1}(z_{it}) - V_{it+1}(z_{it}+1)$	100.99	

$(t = 1, i = 1, z_{it} = 0)$

(b) Results

Figure 4.2: A counter example $(C = 3, \tau = 10)$

4.2.2 Solution Approaches

In this section, we discuss the solution approaches to problem (4.5)-(4.8). LP formulation given by (4.11)-(4.17) has block angular structure where independent blocks are linked by coupling constraints. In this problem, the blocks correspond to the different days and coupling constrains correspond to the fare allocation decisions. In other words, constraints (4.12)-(4.14) belong to the subproblems and constraints (4.15) and (4.16) belong to the master problem. Therefore, it is suitable to use the decomposition method of Dantzig and Wolfe [31]. We can easily solve the problem (4.11)-(4.17) with column generation method in a reasonable time. However, as the network size increases, the solution time of this linear programming problem may degrade.

Now, we will explain alternative solution approaches which may be more practical in the larger-scale networks. We test the performances of these strategies in our numerical section.

Iterative Heuristic This approach provides an alternative heuristic solution method to problem (4.11)-(4.17). Instead of solving the problem (4.11)-(4.17) from scratch, we solve it for each time period starting from the last time period τ. We define $\bar{V}_t(z_{it}|\bar{\alpha},\bar{\beta})$ to denote the expected optimal revenue from t up to τ of the alternative solution approach. To compute the optimal value of $\bar{V}_t(z_{it}|\bar{\alpha},\bar{\beta})$ and $(\bar{\alpha},\bar{\beta})$ at any time period t, we use the optimal values of the value functions, $\bar{V}^*_{it+1}(z_{it}|\bar{\alpha}^*,\bar{\beta}^*)$, at time period $t+1$. In other words, at each time period t (starting from τ to 1), we solve the following problem:

$$
\begin{aligned}
\text{minimize} \quad & \sum_{i\in\mathcal{L}} \sum_{z_{it}=0}^{C_i-1} \bar{V}_{it}(z_{it}|\bar{\alpha},\bar{\beta}) \\
\text{subject to} \quad & \sum_{i\in\mathcal{L}_k} \bar{\alpha}_{ikt} = f_k, \quad \forall k\in\mathcal{K}, \\
& \sum_{i\in\mathcal{L}_k} \bar{\beta}_{ikt} = 0, \qquad \forall k\in\mathcal{K}, \\
& \bar{\alpha}_{ikt} \geq 0 \qquad \forall k\in\mathcal{K}, i\in\mathcal{L}
\end{aligned}
$$

where

$$
\begin{aligned}
\bar{V}_{it}(z_{it}|\bar{\alpha},\bar{\beta}) = & \sum_{k\in\mathcal{K}} p_{kt} max\{\bar{\alpha}_{ikt} + \bar{\beta}_{ikt} + \bar{V}^*_{it+1}(z_{it}+1|\bar{\alpha}^*,\bar{\beta}^*), \bar{V}^*_{it+1}(z_{it}|\bar{\alpha}^*,\bar{\beta}^*)\} \\
& + (1-\sum_{k\in\mathcal{K}} p_{kt})\bar{V}^*_{it+1}(z_{it}|\bar{\alpha}^*,\bar{\beta}^*)
\end{aligned}
$$

We refer this solution approach to decomposition problem as IHM, where the acronym stands for iterative heuristic method. This process continues in a backward recursion until we reach the initial time period 1. This method divides the problem (4.11)-(4.17) into τ smaller problems which are computationally very efficient to solve. Moreover, this approach provides an upper bound on the problem (4.5)-(4.8).

PROPOSITION 4.2.2 *Let (α^*, β^*) and $(\bar{\alpha}^*, \bar{\beta}^*)$ denote the optimal solutions to problem (4.11)-(4.17) and iterative heuristic method, respectively. Then, for all $z_{i1} \leq C_i$, we have* $\sum_{i\in\mathcal{L}} V_{i1}(z_{i1}|\alpha^*, \beta^*) \leq \sum_{i\in\mathcal{L}} \bar{V}_{i1}(z_{i1}|\bar{\alpha}^*, \bar{\beta}^*)$.

PROOF. Let $\bar{\alpha}_t^* = \{\bar{\alpha}_{ikt} : \forall k \in \mathcal{K}, \forall i \in \mathcal{L}_k\}$ and $\bar{\beta}_t^* = \{\bar{\beta}_{ikt} : \forall k \in \mathcal{K}, \forall i \in \mathcal{L}_k\}$ are the sets of optimal values of decision variables for IHM for time period $t \in \mathcal{T}$. Notice that $(\bar{\alpha}_t^*, \bar{\beta}_t^*)$ satisfies conditions (4.2) and (4.3) for all time periods and for all products and hence, it is feasible but not necessarily an optimal to problem (4.11)-(4.17). Therefore, we have $\sum_{i\in\mathcal{L}} V_{i1}(z_{i1}|\alpha^*, \beta^*) \leq \sum_{i\in\mathcal{L}} \bar{V}_{i1}(z_{i1}|\bar{\alpha}^*, \bar{\beta}^*)$. □

Deterministic Linear Program An alternative solution to capacity allocation problem in hotel revenue management is to solve a deterministic linear program (DLP). As we discussed in Chapter 3, DLP is formulated under the assumption that the arrivals of the product requests take on their expected values. Let x_k be the number of reservations that we plan to accept for product k. Then, the deterministic linear program has the following form:

$$z_{DLP} = \text{maximize} \quad \sum_{k\in\mathcal{K}} f_k x_k \tag{4.18}$$

$$\text{subject to} \quad \sum_{k\in\mathcal{K}} a_{ik} x_k \leq C_i, \qquad \forall i \in \mathcal{L}, \tag{4.19}$$

$$x_k \leq \sum_{t=1}^{\tau} p_{kt}, \qquad \forall k \in \mathcal{K}, \tag{4.20}$$

$$x_k \geq 0, \qquad \forall k \in \mathcal{K}. \tag{4.21}$$

DLP is a well-known solution method for the network revenue management problem which provides an upper bound on the optimal total expected revenue [80]. Moreover, the dual solution of DLP can be used to construct a policy to accept or reject the product requests. An important shortcoming of this model is that it ignores the uncertainty in arrival process.

Talluri and van Ryzin [77] discuss that the upper bound optained by DLP is asymptotically optimal as the capacities on the days (flight legs) and the expected numbers of product requests increase linearly with the same rate. Kunnumkal and Topaloglu [53] show that the model (4.5)-(4.8) provides a tighter upper bound than the one obtained by DLP. Therefore, the upper bound provided by the model (4.5)-(4.8) is also asymptotically optimal.

Fare Allocation Heuristic Now, we present a revenue allocation approach that decomposes the network into single days. As we previously mentioned in Section 4.2.1, as long as the fare allocations $\{\alpha_{itk} : i \in \mathcal{L}, t \in \mathcal{T}, k \in \mathcal{K}\}$ satisfy the condition $\sum_{i\in\mathcal{L}_k} \alpha_{ikt} = f_k$ for all $k \in \mathcal{K}$ and $t \in \mathcal{T}$, day-based decomposition with fare allocation provides an upper bound on the optimal expected revenue. By utilizing this property, we generate a benchmark strategy which also provides an upper bound.

The primary disadvantage of decomposition methods is that in the process of decomposition, important network effects in the subproblem approximations may be lost. We utilize the idea of prorated EMSR to retain network information. Williamson [93] proposes prorated EMSR to allocate a portion of the revenue of each product to the days traversed by the product and the EMSR model is then applied to each subproblem to obtain bid prices. She works on the several methods to obtain prorated revenues in airline problems. These allocation methods are based on mileage, ratio of the fare on each flight leg and number of days. Williamson point out that these fixed proration approaches are not efficient to respond dynamic network problem.

Each day has different demand load and hence, while the remaining capacity on some days is abundant, others can be congested. Therefore, insufficient fare allocation of a product to the congested days may result in rejecting the request. Since accept-reject decision of a product depends on the opportunity cost of a room on the days it uses, our decision changes with the remaining capacity and the remaining time. By considering these conditions, we propose the following approach. First we compute the load factor of each day depending on the remaining time and capacity. Second, we compute the weights of each product on each day by using the dynamic load factor. Finally, we allocate fares of each product to the days by considering these weights. This allocation method also guarantees that the fare class ordering between the products traversing the same days is protected.

Let w_{ikt} denote the portion of the contribution of day i the to total revenue of product k at time period t. We compute these weights by using the load factor of each day at each time period. For a given time period t, the load factor of day i is equal to the ratio of total expected demand on day i and the remaining capacity. Let R_{it} denote the remaining capacity on day i at time period t, for $1 \leq t \leq \tau$ the load factor is given by

$$\rho_{it}(R_{it}) = \frac{\sum_{l=t}^{\tau} \sum_{k \in \mathcal{K}} a_{ik} p_{kl}}{R_{it}}$$

Then, the weights for product k are computed as

$$w_{ikt} = \frac{\rho_{it}(R_{it})}{\sum_{i \in \mathcal{L}_k} \rho_{it}(R_{it})}, \qquad i \in \mathcal{L}_k, t \in \mathcal{T}.$$

These weights allocate higher portion of the product k revenue to the day with higher load factor. Moreover, due to the time dependent load factor, it adjusts revenues dynamically. The main advantage of this approach is that it is computationally very efficient. However, an important shortcoming of this approximation is that it assumes the remaining capacity on each day are equal. We can compute the decomposed revenues of advance booking and walk-in requests as follows,

$$f_{ikt} = w_{ikt} f_k, \qquad i \in \mathcal{L}_k, t \in \mathcal{T},$$

which guarantees that $f_k = \sum_{i \in \mathcal{L}_k} f_{ikt}$ for all t. Note that this is nothing but solving a single day problem with C_i capacities. Then, for every $1 \leq t \leq \tau$ we obtain that

$$\nu_{it}(z_{it}) = \sum_{k \in \mathcal{K}} p_{kt} \max\{f_{ikt} + \nu_{it+1}(z_{it} + a_{ik}), \nu_{it+1}(z_{it})\} + (1 - \sum_{k \in \mathcal{K}} p_{kt}) \nu_{it+1}(z_{it}) \qquad (4.22)$$

with the boundary conditions $\nu_{it}(C_i) = 0$ for all t and $\nu_{i\tau}(z_{i\tau}) = 0$ for all $z_{i\tau} \leq C_i$. The single-day problem (4.22) is similar to the model (4.4). However, in this case we ignore the penalty incurred by accepting the demand of a product.

Since revenue allocations $\{f_{ikt} : k \in \mathcal{K}, i \in \mathcal{L}_k, t \in \mathcal{T}\}$ satisfies condition (4.2), the summation of the value functions $\nu_{it}(z_{it})$ over all days at any time period provides an upper bound on the maximum expected revenue denoted by $J_t(z_t)$.

Corollary 4.2.1 *For all $z_{it} \leq C_i$ and $1 \leq t \leq \tau$, we have $J_t(z_t) \leq \sum_{i \in \mathcal{L}} \nu_{it}(z_{it})$.*

The proof of this corollary follows from the Proposition 4.2.2 with $\bar{\alpha}_{ikt} = f_{ikt}$ and $\bar{\beta}_{ikt} = 0$, for all $i \in \mathcal{L}$, $k \in \mathcal{K}$, and $t \in \mathcal{T}$.

4.2.3 CICO Pair-Based Decomposition

In this section, we present an alternative method that decomposes the dynamic program in (4.1) by the check-in and check-out (CICO) pairs. As we described in Section 4.1, customers can stay multiple days in hotel. The combination of check-in and check-out days produces product pairs in our hotel network. The objective of this model is to find the optimal room allocation policy for each check-in and check-out pair. Since customer stays are overlapping, allocation of the capacity among the pairs is to be determined considering the total hotel capacity. In other words, the rooms should be allocated in such a way that the total capacity is never exceeded at any time. Recently, Birbil et al. [29] propose an origin-destination (OD-pair) based decomposition method for airline revenue management problem. OD-pair corresponds to the chek-in and check-out pair in the hotel RM. The proposed approach consists of two stages. While in the first stage network capacities are allocated to each OD-pair, in the second stage seat allocation policy is determined within each OD-pair. In this way, we only need to solve single-leg problem for each pair. Next, we discuss the CICO pair-based decomposition for the generic model including walk-in customers.

Let the set of pairs in the hotel network is given by $\mathcal{S}$ and x_s be the amount of capacity allocated to pair s. Suppose pair s has m_s room types and the set of room types for pair s is denoted by $\mathcal{J}_s$. The probability that a request for product j of pair s arrives into the system at time period t is p^s_{jt}. We use p^s_{0t} to denote the no arrival probability at time period t. If we accept the arriving request, then we generate a revenue of r_{js} and consume one unit of the allocated capacity on pair s. Now we are ready to formulate the problem. Let $\varrho^t_s(x_s)$ denote the expected optimal revenue of pair s from t up to τ given that the remaining capacity is x_s at time period t. Then, the dynamic decomposition model for a pair s is as follows:

$$\begin{array}{lll} \text{maximize} & \sum_{s\in\mathcal{S}} \varrho^1_s(x_s) & \\ \text{subject to} & \sum_{s\in\mathcal{S}} a_{is}x_s \leq C_i, & i \in \mathcal{L}, \\ & x_s \in \mathbb{Z}_+, & s \in \mathcal{S}, \end{array} \tag{4.23}$$

In this problem, the objective function $\varrho^t_s(x_s)$ for a given x_s is an optimization problem itself, and the dynamic programming recursion is given by

$$\varrho^t_s(x_s) = \sum_{j\in\mathcal{J}_s} p^s_{jt} \max\{r_{js} + \varrho^{t+1}_s(x_s - 1), \varrho^{t+1}_s(x_s)\} + p^s_{0t}\varrho^{t+1}_s(x_s),$$

with the boundary conditions $\varrho_s^t(0) = 0$ for all t and $\varrho_s^{\tau+1}(x_s) = 0$ for all $x_s \geq 0$. It computes the optimal seat allocation policy for each pair $s \in \mathcal{S}$ with the objective of maximizing revenue. Although the objective function is dynamic, the overall problem (4.23) is not. As static models do, the allocated capacity to each pair is determined at the beginning of the planning period. However, given the allocated room capacity, reservation policy changes with the remaining time and capacity. Therefore, we refer this problem as partially dynamic programming model.

Lippman and Stidham [62] show that the objective function $x_s \mapsto \varrho_s^t(x_s)$ is discrete concave. Therefore, we can always replace it by a piece-wise linear concave function and then, rewrite the overall problem as a linear programming problem. For each pair s, we define the maximum available capacity as

$$B_s = \min\{C_i : a_{is} = 1\}$$

which gives the bound on the number of pieces for the concave function. We also define the function $x \longmapsto \min_{1 \leq l \leq B_s}\{\gamma_{sl} x + \rho_{sl}\}$ with $\gamma_{s1} \geq \gamma_{s2} \geq .... \geq \gamma_{sB_s}$ and $0 := \rho_{s1} \leq \rho_{s2} \leq .... \leq \rho_{sB_s}$ satisfying

$$\varrho_s^1(x_s) = \min_{1 \leq l \leq B_s}\{\gamma_{sl} x_s + \rho_{sl}\}.$$

Then, problem (4.23) can be rewritten as

$$\max\left\{\sum_{s \in \mathcal{S}} \min_{1 \leq l \leq B_s}\{\gamma_{sl} x_s + \rho_{sl}\} : \sum_{s \in \mathcal{S}} a_{is} x_s \leq C_i, i \in \mathcal{L}, x_s \in \mathbb{Z}_+, s \in \mathcal{S}\right\}.$$

This problem requires solving the independent subproblem of each pair s for all integer values of allocated capacity x_s from 1 to B_s. However, optimal policy table of the partial dynamic programming model already stores the optimal solution for all values, $x_s \in \{1, ..., B_s\}$ and hence, we only need to construct the problem for each pair once. Next, we present that problem (4.23) provides a lower bound on the maximum expected revenue over the whole planning horizon.

PROPOSITION 4.2.3 *The optimal objective value of the problem (4.23) gives a lower bound on the optimal expected revenue of dynamic programming model given in (4.1). That is, we have* $\sum_{s \in \mathcal{S}} \varrho_s^1(x_s) \leq J_1(0)$.

Proof. Suppose that $X_{js}, \forall j, s$ denote the random number of reservations accepted for class j in pair s over the planning horizon under the optimal policy of the decomposed dynamic programming model (4.23). Then we have

$$\sum_{j \in \mathcal{J}_s} X_{js} = X_s, \quad s \in \mathcal{S},$$
$$\sum_{s \in \mathcal{S}} a_{is} X_s \leq C_i, \quad i \in \mathcal{L}.$$

Then, the total expected revenue under the optimal policy of the decomposed dynamic programming model is

$$\sum_{s \in \mathcal{S}} \varrho_s^1(X_s) = \sum_{s \in \mathcal{S}} \sum_{j \in \mathcal{J}_s} r_{js} \mathbb{E}(X_{js}).$$

This solution is clearly feasible. However, the exact dynamic model considers all possible combinations of capacity allocations over each product and hence, decomposed dynamic model provides a lower bound on the optimal expected revenue. Therefore, we have

$$\sum_{s \in \mathcal{S}} \sum_{j \in \mathcal{J}_s} r_{js} \mathbb{E}(X_{js}) = \sum_{s \in \mathcal{S}} \varrho_s^1(x_s) \leq J_1(0).$$

□

Birbil et al. [29] show that several models proposed in the literature can also be modeled in this generic framework. In the next section, we discuss the inclusion of stay-overs to hotel room allocation problem.

4.2.4 Stay-Over Requests

Unlike the airline capacity allocation problems, customers with reservations can request an extension on their booking days in hotel systems. These customers are classified as stay-overs in hotel literature. To formulate these requests, we need to keep track of the accepted reservations in each pair. Since single-day decomposition only considers the total number of reservations in each day, it may perform poor for this problem. Therefore, we concentrate on the OD-based decomposition method proposed by Birbil et al. [29].

In this problem formulation, we assume that each accepted booking of pair s can request one day extension at the beginning of check-out day with probability π_s. If we accept the stay-over request of pair s, then we generate a revenue of θ_s. We let $a_{is} = 1$ if day i is used by the stay-over customer and $a_{is} = 0$ otherwise. We define x_s as the reserved capacity for advanced reservations and walk-in requests of pair s. We also define w_s as the reserved capacity for stay-over requests of pair s. Let y_s be the remaining capacity at the check-out day of pair s. Due to the assumption that the stay-over requests of different reservations are independent of each other, the total number of stay-over requests are represented by a Binomial random variable $\mathbf{B}(\pi_s, x_s - y_s)$ having a success probability of π_s.

The problem is to determine which of the product requests to accept during the reservation period and to determine which of the confirmed reservations to accept for stay-over during the service period. If $\phi_s^t(y_s|x_s, w_s)$ is the expected optimal revenue for pair s with reserved capacities x_s and w_s from period t to τ given that the remaining capacity is y_s, then the dynamic model within the proposed decomposition approach becomes

$$\begin{aligned}
\text{maximize} \quad & \sum_{s \in \mathcal{S}} \phi_s^1(y_s|x_s, w_s) && && (4.24)\\
\text{subject to} \quad & \sum_{s \in \mathcal{S}} a_{is}x_s + a_{is}w_s \leq C_i && \forall i \in \mathcal{L} && (4.25)\\
& w_s \leq x_s && \forall s \in \mathcal{S} && (4.26)\\
& x_s, w_s \in \mathbb{Z}_+ && \forall s \in \mathcal{S} && (4.27)
\end{aligned}$$

where,

$$\begin{aligned}
\phi_s^t(y_s|x_s, w_s) = & \sum_{j \in \mathcal{J}_s} p_{jt} \max\{r_{js} + \phi_s^{t+1}(y_s - 1|x_s, w_s), \phi_s^{t+1}(y_s|x_s, w_s)\} \\
& + p_{0t}\phi_s^{t+1}(y_s|x_s, w_s), \qquad (4.28)
\end{aligned}$$

for $1 \leq t \leq t_s$ where t_s is the check-out day of pair s. The boundary condition of dynamic model is $\phi_s^{t_s}(y_s|x_s, w_s) = \theta_s \mathbb{E}[min(w_s, \mathbf{B}(\pi_s, x_s - y_s))]$.

Note that the approach of Birbil et al. [29] is applicable when the objective function is discrete concave. To check this condition for given data, one needs to compute the differences, $x_s \mapsto \phi^t(y_s|x_s + 1, w_s) - \phi^t(y_s|x_s, w_s)$, $w_s \mapsto \phi^t(y_s|x_s, w_s + 1) - \phi^t(y_s|x_s, w_s)$, and $(x_s, w_s) \mapsto \phi^t(y_s|x_s+1, w_s+1) - \phi^t(y_s|x_s, w_s)$ and confirms that they are non-negative and non-increasing. An alternative way of checking concavity is to solve problem (4.29)-(4.30). Next, we explain this alternative method and discuss the construction of the two dimensional piece-wise linear concave function.

We define m to denote the total number of states in our dynamic programming model in (4.28). Let $u_i \in \mathbb{Z}_+^2$, $i = 1, ..., m$ be the coordinates corresponding to these states. For ease of notation, we define $\mathcal{U} = \{1, ..., m\}$ to denote the set of states in our model. Suppose the corresponding objective function values are given by $\lambda_i := \phi(u_i)$, $i \in \mathcal{U}$. Since the objective function is concave over $\mathbb{R}_+^2$, we have for all i and j pairs, the subgradient inequality

$$\lambda_j \leq \lambda_i + \gamma_i^\intercal (u_j - u_i).$$

If we introduce auxiliary variables $\hat{\lambda}_i$, $i \in \mathcal{U}$, then we can write an optimization problem as

$$\begin{aligned} &\text{minimize} && \sum_{i \in \mathcal{U}} |\lambda_i - \hat{\lambda}_i| && && (4.29) \\ &\text{subject to} && \hat{\lambda}_j \leq \hat{\lambda}_i + \gamma_i^\intercal (u_j - u_i), && i, j \in \mathcal{U}, && (4.30) \end{aligned}$$

where $\hat{\lambda}_i \in \mathbb{R}$ and $\gamma_i \in \mathbb{R}_+^2$, $i \in \mathcal{U}$ are the decision variables. After loosing the absolute value function in a straightforward manner, we obtain a linear programming problem

$$\begin{aligned} &\text{minimize} && \sum_{i \in \mathcal{U}} z_i \\ &\text{subject to} && \hat{\lambda}_j \leq \hat{\lambda}_i + \gamma_i^\intercal (u_j - u_i), && i, j \in \mathcal{U}, \\ & && - z_i \leq \lambda_i - \hat{\lambda}_i, && i \in \mathcal{U}, \\ & && z_i \geq \lambda_i - \hat{\lambda}_i, && i \in \mathcal{U}, \\ & && z_i \geq 0 && i \in \mathcal{U}. \end{aligned}$$

This problem can be solved very efficiently. Clearly, its optimal objective function value should be zero with the optimal solution $\hat{\lambda}_i^* = \lambda_i$ and γ_i^* for $i \in \mathcal{U}$ if the objective function $(x_s, w_s) \mapsto \phi_s(.|x_s, w_s)$ is concave. Then, we can write

$$\phi_s(y_s|x_s, w_s) = \min_{i \in \mathcal{U}} \left\{ \lambda_i + \gamma_i^\intercal \left(\begin{bmatrix} x_s \\ w_s \end{bmatrix} - u_i \right) \right\}$$

Equivalently,

$$\phi_s(y_s|x_s, w_s) = \min_{i \in \mathcal{U}} \left\{ \gamma_i^\intercal \begin{bmatrix} x_s \\ w_s \end{bmatrix} + b_i \right\}$$

where $b_i = \lambda_i - \gamma_i^\intercal u_i$, $i \in \mathcal{U}$. By using this result, we can rewrite the model (4.24) - (4.27) as a linear programming problem.

4.3 Computational Experiments

In this section, we present computational results that illustrates the performances of numerous solution methods for the hotel revenue management problem described in Section 4.1. We proceed to describe our simulation setup.

We simulate the arrival of product requests over discrete time periods $\mathcal{T} = \{1, ..., \tau\}$. At each time period, we first generate an arrival request and then apply the corresponding policy. The probability that there is a request for product k at time period t is p_{kt}. To test the performances of the booking policies against varying arrival intensities, we use the load factor parameter ρ, which is given by

$$\rho = \frac{\sum_{t \in \mathcal{T}} \sum_{i \in \mathcal{L}} \sum_{k \in \mathcal{K}} a_{ik} p_{kt}}{\sum_{i \in \mathcal{L}} C_i}$$

In all our numerical experiments, we set the capacity of the hotel and the length of the planning horizon to $C_i = 40$ for all $i \in \mathcal{L}$ and $T = 200$, respectively. To compute the product fares, we define daily prices, σ_i. If a product covers only single day, its fare is set equal to the corresponding daily price. In our simulation set-up, we assume that customers take a discount (κ) if they request to stay more than a day. We compute these fares as follows

$$f_k = \kappa \sum_{i \in \mathcal{L}} a_{ik} \sigma_i.$$

4.3.1 Computational Results for Day-Based Decomposition

In this section, we compare the performances of day-based decomposition methods. We begin by describing the benchmark solution methods.

Linear Programming Formulation (LPF): This is the solution method that we proposed in Section 4.2.1. That is, we solve the linear model given in (4.11)-(4.17) to obtain the optimal values of (α, β). Then, we use these values to compute our accept-reject policy.

Alternative Linear Programming Model (ALP): Day-based decomposition model described in Section 4.2.1 computes the fare allocations and penalties for each day, each product and each time period. To shorten the computational time and understand the effect of time on the fare allocation, we redefine fare allocations and penalties by relaxing the time dependency. In other words, we replace the decision variables $\{(\alpha_{ikt}, \beta_{ikt}) : i \in \mathcal{L}, \forall k \in \mathcal{K}, t = 1, ..., \tau\}$ with $\{(\alpha_{ik}, \beta_{ik}) : i \in \mathcal{L}, \forall k \in \mathcal{K}\}$ in problem (4.11)-(4.17). Then, we use these values to compute our accept-reject policy.

Iterative Heuristic Method (IHM): This is the solution method described in Section 4.2.2. In particular, we solve the linear model (4.11)-(4.17) for each time period starting from the last time period τ to obtain the optimal solution of (α, β). It is an iterative method since we use the optimal values computed for time period $t+1$ to obtain the optimal solution for time period t.

Fare Allocation Heuristic (FAH): As described in Section 4.2.2, we use this heuristic method to allocate fares of each product to the days traversed. Then, we compute the accept-reject policy with these allocated fares.

Deterministic Linear Program (DLP): This is the solution method described in Section 4.2.2. We solve the problem (4.18)-(4.21) to obtain the optimal values of the dual variables associated to capacity constraints (4.19). We use these dual variables as the bid prices for our accept-reject policy. To refine bid prices in our implementation, we solve the problem (4.18)-(4.21) five times over the decision horizon. At each solution time, we replace the right side of constraints (4.19) and (4.20) with the current remaining hotel daily capacities and the updated expected demand.

Deterministic Fare Allocation (DFA): By using the deterministic model given in (4.18)-(4.21), we can obtain a fare allocation policy. In order to do this, we decompose the model (4.18)-(4.21) by days in the service period. We begin by defining a fictitious day δ with infinite capacity and the set of days becomes $\mathcal{L} \cup \{\delta\}$. To rewrite the model, we also define x_{ik} as the number of the reservations that we plan to accept for product k on day i. We can rewrite the problem (4.18)-(4.21) as

$$\begin{aligned}
\text{maximize} \quad & \sum_{k \in \mathcal{K}} f_k x_{\delta k} && && (4.31)\\
\text{subject to} \quad & \sum_{k \in \mathcal{K}} a_{ik} x_{ik} \leq C_i && \forall i \in \mathcal{L} && (4.32)\\
& x_{ik} \leq \sum_{t=1}^{\tau} p_{kt} && \forall i \in \mathcal{L}, \forall k \in \mathcal{K} && (4.33)\\
& x_{\delta k} - x_{ik} = 0 && \forall i \in \mathcal{L}, \forall k \in \mathcal{K} && (4.34)\\
& x_{ik} \geq 0 && \forall i \in \mathcal{L}, \forall k \in \mathcal{K} && (4.35)
\end{aligned}$$

It is easy to see the equivalence between problems (4.18)-(4.21) and (4.31)-(4.35). Let $\{\bar{\alpha}_{ik} : i \in \mathcal{L}, \forall k \in \mathcal{K}\}$ be the optimal values of the dual variables associated to constraints (4.34). In the dual of problem (4.31)-(4.35), we have the constraint $\sum_{i \in \mathcal{L}_k} \bar{\alpha}_{ik} = f_k$ associated with the decision variable $x_{\delta k}$. Therefore, we can use the dual variables $\bar{\alpha}_{ik}$ to obtain the fare allocation. Then, we use these allocated fares to compute optimal policy of the dynamic programming model as in fare allocation heuristic; see Topaloglu [85]. As in the DLP policy, we refine bid prices five times during the decision horizon.

Our experimental design is based on various factors of the network size (n), the load factor (ρ), the discount multiplier (κ), and the walk-in ratio (ω). We test our models in two networks with the number of days $n \in \{3, 6\}$. We use load factor values $\rho \in \{1.2, 1.5\}$ corresponding to low and high loads. The discount factor $\kappa \in \{0.5, 0.7\}$ are used to represent low and high fares. The last parameter set comes from the demand ratio of walk-in commitments $\omega \in \{0.1, 0.2\}$. We label our test problems by using all combinations of these parameters.

As we mentioned in Section 4.2.1, these methods provide upper bounds on the maximum total expected revenue obtained by the dynamic model in (4.1). For different test problems, Table 4.1 compares the upper bounds obtained by these five solution methods. The first three columns indicate the characteristics of the test instances. The next five columns give the optimal objective values of the solution methods. The last four column gives the percentage gaps between LPF and the remaining solution methods. The results show that LPF consistently provides the tightest upper bounds which confirms with Proposition 4.2.2 and Corollary 4.2.1. Moreover, Kunnumkal and Topaloglu [53] show that the upper bound obtained by LPF is tighter than the upper bound obtained by DLP. For all test instances, the upper bounds provided by ALP are significantly close to those provided by LPF. ALP uses the time independent fare allocations and penalties and hence, it is computationally more efficient compared to LPF. These results indicate that we can use ALP instead of LPF for small size networks. Iterative heuristic also provides tight upper bounds. For a majority of the test problems, the percentage gap between LPF and IHM is below 1%. The quality of the upper bound provided by IHM is mostly affected by the arrival intensity of walk-in customers. On the other hand, the upper bound provided by FAH is very loose compared to other dynamic policies. Recall that FAH computes the time and capacity depended load factors by assuming that the remaining capacity on each day is the same. Therefore, the percentage gap increases as the load factor and walk-in customer ratio increase.

Table 4.1: Upper bound percentage gaps on the maximum total expected revenue ($n = 3$)

Instances			% Gap with LPF			
ρ	κ	ω	ALP	IHM	FAH	DLP
1.2	0.5	0.1	0.02%	0.58%	4.16%	3.51%
		0.2	0.03%	0.78%	3.77%	3.65%
	0.7	0.1	0.08%	0.44%	4.68%	3.89%
		0.2	0.04%	0.60%	4.38%	4.06%
1.5	0.5	0.1	0.03%	0.24%	2.90%	3.83%
		0.2	0.02%	2.04%	4.08%	3.30%
	0.7	0.1	0.04%	0.49%	3.50%	4.12%
		0.2	0.03%	0.78%	5.08%	3.59%

Our main computational results are summarized in Tables 4.2 and 4.3. In particular, these two tables respectively show the results for the test problems with three and six days. The organization of these tables is the same as that of Table 4.1. Table 4.2 compares performances of different solutions where the revenue obtained by LPF is used as a base approach to report the relative performances of the remaining approaches. The results indicate that LPF outperforms all other solution methods. We also observe that the performances of LPF and ALP can become significantly close. Therefore, we conjecture that relaxing time dependency in the fare and penalty allocations provides a good approximation to the problem (4.11)-(4.17). Moreover, it improves the solution time. Comparing the average percentage gaps, we also observe that performances of ALP, IHM and DFA can become relatively close. They compete for the second, third and fourth places. There are three test instances where IHM performs significantly worse than DFA. These instances correspond to the cases where arrival intensity of walk-in customers is high. As we mentioned before, IHM decomposes the problem (4.11)-(4.17) into time periods and hence, it does not consider all possible combinations of fare and penalty allocations. Therefore, later arrivals of walk-in customers deteriorates the performance of IHM. On the other hand, DLP consistently provides the lowest expected revenues.

Table 4.2: Percentage gaps relative to the expected revenue of LPF ($n = 3$)

Instances			% Gap with LPF				
ρ	κ	ω	ALP	IHM	FAH	DLP	DFA
1.2	0.5	0.1	0.03%	0.70%	3.95%	4.59%	0.34%
		0.2	0.33%	1.59%	3.33%	4.87%	0.40%
	0.7	0.1	0.88%	0.58%	3.45%	5.47%	0.83%
		0.2	0.32%	0.81%	2.96%	3.45%	1.19%
1.5	0.5	0.1	0.80%	0.33%	4.70%	5.56%	0.37%
		0.2	0.08%	2.24%	3.58%	4.46%	0.59%
	0.7	0.1	0.43%	3.13%	4.91%	7.10%	1.07%
		0.2	0.38%	1.56%	4.05%	4.63%	1.60%

Table 4.3 show the performances of the different solution methods for the test problems with 6 days. To observe sufficient amount of customer arrival, we increase the number of time periods to 300. Due to the long computational time, we do not test the performances of LPF and ALP for these test instances. Moreover, to make a fair comparison, fare allocations and bid prices used in DFA and DLP are refined two times during simulation. In this table, we use IHM as a reference when comparing the expected revenues. Comparing the percentage gaps under this setup, we observe that the performances of IHM and DFA are very close, especially for low walk-in ratio. DFA determines the fare allocations at the beginning of the reservation period and updates them only at specific time periods. Therefore, it may fail to capture the actual dynamics of the system. Comparing the percentage gaps for FAH in Table 4.2 with those in Table 4.3, we note that its performance deteriorates as the network size increases. We caution the reader to the percentage gaps of FAH and DLP. For low values of load factor and walk-in customer ratio, DLP performs better than FAH. Therefore, heuristic fare allocation performs poor in large scale networks although its computational time is very efficient.

We conclude this section by reporting average computation times obtained with different strategies. DLP and FAH require on average less than 1.00 seconds. It takes on average 2.50 and 3.40 seconds to solve DFA and IHM, respectively. Hence, iterative heuristic is comparable

Table 4.3: Percentage gaps relative to the expected revenue of IHM ($n = 6$)

Instances			% Gap with IHM		
ρ	κ	ω	FAH	DLP	DFA
1.2	0.5	0.1	5.58%	4.19%	-0.03%
		0.2	5.93%	23.46%	1.70%
	0.7	0.1	5.13%	3.42%	-0.42%
		0.2	7.69%	22.20%	2.68%
1.5	0.5	0.1	6.69%	22.61%	-0.21%
		0.2	7.99%	10.98%	1.97%
	0.7	0.1	6.40%	22.01%	-1.00%
		0.2	10.64%	11.25%	3.82%

to the widely-applied deterministic models in terms of computational efficiency. The most computational effort is invested in LPF and ALP, which take on average 1,287 and 385 seconds, respectively. We want to point out an important property of LPF method. As we mentioned before, problem (4.11)-(4.17) has a block angular structure and it can be solved by Dantzig-Wolfe decomposition. Each block corresponds to independent single-days. Due to the problem structure, we can make use of parallel computing. Therefore, it is possible to improve the CPU times belonging to LPF and ALP.

4.3.2 Computational Results for CICO Pair-Based Decomposition

In this part of the computational study, we test the performances of the models in the presence of stay-over customers. We use four benchmark strategies.

Stay-Over Decomposition (SOD): This is the solution method that we present in Section 4.2.4. We solve the model (4.24)-(4.27) by relaxing the integrality constraints. To obtain an integer feasible solution, we can simply round down the noninteger capacity allocations. By using these allocations, we compute our dynamic policy.

Stay-Over Bid Price (SBP): We can also use model (4.24)-(4.27) to compute bid prices associated to capacity of each day. The summation of the dual variables corresponding to capacity constraints of the days traversed by the product is used as the bid-price for reservation policy.

Deterministic Stay-Over Model (DSM): Deterministic linear programming is a well-known solution method used to compute optimal capacity policy for the hotel revenue management problem. We extend this model to include stay-over customer requests. Let x_{js} be the number of reservations that we plan to accept for product j in pair s. Similarly, let w_s be the number of reservations that we plan to accept for stay-over request of pair s. Then, the deterministic linear program has the following form:

$$
\begin{aligned}
\text{maximize} \quad & \sum_{s\in\mathcal{S}}\sum_{j\in\mathcal{J}_s} r_{js}x_{js} + \sum_{s\in\mathcal{S}} \theta_s w_s \\
\text{subject to} \quad & \sum_{s\in\mathcal{S}}\sum_{j\in\mathcal{J}_s} a_{is}x_{js} + \hat{a}_{is}w_s \le C_i, && \forall i \in \mathcal{L} \\
& x_{js} \le \sum_{t=1}^{\tau} p_{jt}^{s} && \forall s \in \mathcal{S}, j \in \mathcal{J}_s, \\
& w_s - \sum_{j\in\mathcal{J}_s} \pi_s x_{js} \le 0, && \forall s \in \mathcal{S}, \\
& x_{js}, w_s \ge 0 && \forall s \in \mathcal{S}, j \in \mathcal{J}_s,
\end{aligned}
$$

We use bid prices associated to capacity constraints to compute our accept-reject policy.

Check-in Check-out Pair Decomposition (CICO): This is the solution method given in Section 4.2.3. We solve the model (4.23) to obtain an allocation policy for advance bookings and walk-in customers. This model ignores the stay-over customer requests by simply rejecting them.

Our experimental design is based on various factors of the load factor (ρ), the discount multiplier (κ), the stay-over probability (π_s), and the walk-in ratio (ω). We use load factor values $\rho \in \{1.2, 1.5\}$ corresponding to low and high loads. The discount factor $\kappa \in \{0.5, 0.7\}$ are used to represent low and high fares. We give two sets of stay-over probability, $\{\boldsymbol{\pi}^L, \boldsymbol{\pi}^H\}$, to represent possibly low and high stay-over probability. While $\boldsymbol{\pi}^L$ ranges between 0.05 and 0.20, $\boldsymbol{\pi}^H$ is in (0.15,0.30). We also test the models for varying walk-in ratio $\omega \in \{0.1, 0.2\}$ In the following computational experiments, we set the network size to $n = 3$. We label our test problems by using all combinations of these parameters. Under this setup, we have evaluated the reservation policies of all solution methods. To update the bid prices in our implementation, we reoptimize the bid price policies two times over the decision horizon.

Our main results are summarized in Table 4.4. Comparing the percentage gaps, we observe that SBP consistently provides the highest total expected revenues, whereas CICO consistently provides the lowest total expected revenues. We also observe that there is a noticeable performance gap between SBP and SOD. Although SOD has a dynamic policy, it uses fixed capacity allocations which are determined at the beginning of the decision period. Therefore, it is partially static which affects its performance, especially when the load factor is high. As depicted in Table 4.4, even there are instances when total expected revenue obtained by SOD lags behind that of DLP. Birbil et al. [29] points out that the pair-based models using predetermined allocations perform poor as the network becomes large and congested.

When we look into performances of SBP and DLP, we observe that the percentage differences are mostly affected by the load factor and the stay-over probability. Although both of the approaches use static bid prices for accept-reject decisions, DLP performs poor due to disregarding randomness in the arrival process. There are test problems where the gap between total expected revenues obtained by SBP and DLP is as high as 8.51%. When we compare the performances of CICO and SOD, we see that taking into account the stay-over requests brings a significant advantage. As demonstrated in Table 4.4, there exists test instances which provide evidence to the earlier conjecture. For example in one test instance, the gap between SBP and CICO is 13.19% whereas the gap between SBP and SOD is 2.62%.

Table 4.4: Percentage gaps relative to the expected revenue of SBP ($n = 3$)

Instances				% Gap with SBP		
ρ	$\boldsymbol{\pi}^\bullet$	κ	ω	SOD	DLP	CICO
1.2	$\boldsymbol{\pi}_H^s$	0.5	0.1	2.62%	4.85%	13.19%
		0.5	0.2	4.12%	4.86%	11.52%
		0.7	0.1	1.18%	3.50%	8.97%
		0.7	0.2	4.18%	3.98%	9.53%
	$\boldsymbol{\pi}_L^s$	0.5	0.1	2.42%	3.97%	8.10%
		0.5	0.2	4.32%	4.71%	8.26%
		0.7	0.1	1.38%	2.83%	5.53%
		0.7	0.2	3.74%	4.09%	6.50%
1.5	$\boldsymbol{\pi}_H^s$	0.5	0.1	4.94%	8.51%	11.97%
		0.5	0.2	1.66%	3.41%	7.76%
		0.7	0.1	3.33%	4.72%	9.70%
		0.7	0.2	3.78%	4.01%	8.23%
	$\boldsymbol{\pi}_L^s$	0.5	0.1	3.94%	6.87%	8.35%
		0.5	0.2	1.36%	3.07%	4.23%
		0.7	0.1	3.27%	4.06%	6.52%
		0.7	0.2	3.62%	3.54%	5.66%

Chapter 5

CONCLUSION

In this thesis, we present three different capacity allocation policies which arise from emerging applications in revenue management. We address the gaps in the literature and try to develop practical solution methods. In all cases, the structures of the problem create a challenge for analysis and implementation. Therefore, we concentrate on the approximation methods. Each chapter in this thesis presents a different capacity allocation problem in revenue management. While in Chapters 2 and 3 we study the single-leg revenue management problem, in Chapter 4 we discuss the network revenue management applications. To justify the benefits of our proposed models, we conduct a computational study for each problem.

The first problem is the joint capacity allocation and overbooking problem for the single-leg airline revenue management in Chapter 2. We develop new optimization models for static and dynamic problems that involve no-shows, cancellations, and hence, overbooking. In the static case, we discuss two risk-based models both of which allow class-dependent cancellations and no-shows. Our first static model determines the optimal total booking limit under the greedy policy. Finding the optimal total booking limit under such a general setting is useful in practice, since the overbooking limit can be used as an input to some well-known capacity allocation methods like the EMSR heuristics. In the second static model, we determine both the total booking limit and the partitioned booking limits. Arriving at a computationally difficult model, we develop upper and lower bounding problems to obtain approximate solutions. As preferred in practice, we propose to use the partitioned booking limits obtained by our upper and lower bounding models in a nested way. Thus, the resulting method becomes a heuristic one which does not require a predefined overbooking limit like the EMSR heuristics. In the dynamic case, we propose a model based on two independent streams of events;

arrivals of booking requests and cancellations. Our modeling approach allows the number of cancellations in any time period, given the number of accepted requests at the beginning of that time period is a binomially distributed random variable. We show that it is easy to solve the resulting problem with dynamic programming. Following the characterization of the optimal policy, we also present the nested structure of the optimal allocations. We conduct a computational study to compare the performances of the booking policies obtained by our proposed models against those of the some well-known EMSR-based approaches used in the literature. The numerical results demonstrate that the proposed upper bounding model outperforms the EMSR-based heuristics for the generated test problem instances and performs reasonably well compared to the DP model. We also observe that the policies proposed by our upper bounding model are robust even if we switch from low to high show-up probabilities or increase the overbooking cost. On the other hand, the performance of proposed lower bounding model deviates depending on the number of fare classes and the load factor. We also derived bounds on the error introduced by solving the upper bounding problem instead of solving the corresponding original static model. Computational experiments demonstrate that the error bounds are tighter when the load-factor is higher.

In Chapter 3, we introduce the concept of commitment option with single-leg revenue management problem to the literature. By offering commitment option, airline companies aim to attract price sensitive customers in addition to the customers who have uncertainty in travel time. We analyze the consequences of selling this option along with standard bookings of the products. We derive dynamic and static models for the capacity allocation problem. In the dynamic case, finding the optimal policy for the actual problem require solving a dynamic program with a high-dimensional state vector. Hence, we propose an approximate dynamic programming formulation. In the deterministic case, we present a linear programming model leading to an upper bound on the optimal objective value of the actual problem. We analyze structural properties of deterministic model, and show that the upper bound derived from this model is asymptotically tight.

We conduct a computational study to evaluate the impact of offering these options. To assess the effect of commitment decisions, we compare the performance of our model against different policies. Our numerical results confirm the intuitive expectation that offering commitment option is most beneficial when the purchase probability of the committed seat is high and the length of commitment period is short. Furthermore, considering a policy that ignores the contingent option altogether, explicitly modeling the commitment option can bring significant revenue improvements even when customer arrival intensity for this option is low. Also, making the contingent commitment option available up to only a certain time period can be more profitable than making it available during the whole sales horizon, especially when purchase probability of contingent commitment option is low. As the length of the commitment period increases, limiting the availability of the commitment option may be more beneficial to reap the most benefit from this option. Moreover, in our computational study we also evaluate the performance of our approximate dynamic model. When we compare our proposed model with the actual dynamic model, we see that there is no significant difference between their performances for the short commitment period length. This proves that our approximation performs very well.

In Chapter 4, we study the room allocation problem with walk-in and stay-over customers in a hotel network. Although room allocation problem in hotel revenue management resembles to airline capacity allocation problem, there are two important differences. First, demand structure in hotel RM is different. Customers can change the length of their reservation after their arrival or they can show-up without any reservation. Second, long-term stay is very common in hotel systems. Therefore, it is not a good practice to directly apply the models developed for airline problems to hotel context. In this thesis, we work on the dynamic room allocation problem. Due to the complexity of this problem, we concentrate on approximation methods. We analyze structural properties of the problem and present day-based and pair-based decomposition approaches which can handle walk-in and stay-over customers. In the first part of Chapter 4, we utilize the decomposition idea of Kunnumkal and Topaloglu [53] and study its linear programming formulation for the day-based decomposition model. We also work on the alternative solution approaches which are computationally efficient to solve. Since day-based decomposition generates independent subproblems for each day, it cannot store the number of reserved rooms for each product. Hence, incorporating stay-over customers becomes a challenge. In the second part, we work on stay-over extension and propose a pair-based decomposition model. By analyzing the structural properties of the model, we develop a solution method. The objective is to construct a two dimensional piecewise linear concave function. In our numerical results, we observe that our proposed model performs better than the deterministic model which is widely used in practice.

Our experience with several revenue management applications gave us a thorough understanding about the opportunities and the limitations of the models proposed in this thesis. There are multitude of research direction that we may follow in the future.

Dynamic programming model of joint capacity allocation and overbooking problem introduced in Chapter 2 ignores the fare class dependency and assumes that customers in each fare class can cancel with the same probability. In addition, no-show probability of each customer is also class independent. These assumptions can be relaxed for a more realistic model. Nonetheless, the resulting problem would be difficult to solve due to the high dimensional state space. A potential future research direction would be to develop a decomposition approach which would reduce the original dynamic programming formulation into several independent dynamic programming formulations each having a one dimensional state space. In Chapter 4, we present two decomposition approaches which can be used to approximate class dependent dynamic model.

Another potential future research direction might be the extension of proposed overbooking models in the network environment. As we discussed in Chapter 4, network based capacity allocation problems are quite difficult to solve, and hence, in practice, the methods that require solving a series of single-leg problems are frequently applied. However, directly using our proposed models in decomposed network problem may result in high denied boarding cost. In network extension, assignment of overbooking capacity may need to be performed by considering shared flight capacities in the airline network.

The model we propose in Chapter 3 introduces the contingent commitment problem to the revenue management literature. This option is designed to attract customers who are uncertain about their travel time and who are price sensitive. In our modeling approach, we assumed that customers do not anticipate future price movements. An interesting extension would be to investigate the effect of commitment option when the customers are price sensitive. Moreover, we assume that commitment fee is fixed during the reservation period. Explicitly considering the dynamic pricing of commitment fee would be desirable. Another future direction of this work would be to include the overbooking option. Since contingent commitment reservations may leave, the revenue loss due to the resulting empty seats can be filled by overbooking the flight. In this case, the overbooking limit should be determined by considering the cancellation possibility of contingent commitments.

In Chapter 4, we work on the hotel revenue management problem in case of walk-in and stay-over customer requests. To the best of our knowledge, dynamic programming model of stay-over customers have not been proposed in the literature before. This problem is difficult to solve due to the high dimensional state space. In this study, we assume that customers can request for at most one day extension on their reservations. Although, the stay-over probability of staying more than one day is low, relaxation of this assumption would result in a more realistic model. Moreover, hotel customers have the flexibility to leave at any time during their stay. These customers are known as early departures in hotel revenue management literature. Incorporation of early departure request in the check-in and check-out pair-based decomposition method would be another potential topic for future research. We can consider early departure requests like the cancellations in airline revenue management. In Chapter 2, we present several static and dynamic models for cancellations and no-shows. These models can be used to incorporate early departures in pair-based decomposition methods.

Another future research direction would be investigating the error incurred by our fast heuristic approach. This approach divides the overall problem into smaller subproblems and each subproblem corresponds to a different time period in the planning horizon. In our computational experiments, we observe that this method provides a good approximation and it is computationally efficient. To improve the performance of this solution method, we can decompose it into a smaller set of subproblems. In other words, instead of solving the overall problem at each time period in the planning horizon, we can divide the problem into block of time periods and find the optimal solution for each block. It would be interesting to investigate the dependency between the size of the subproblems and the incurred error.

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