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Magic informationally complete POVMs with permutations

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Eigenstates of permutation gates are either stabilizer states (for gates in the Pauli group) or magic states, thus allowing universal quantum computation (M. Planat and Rukhsan-Ul-Haq, Preprint 1701.06443). We show in this paper that a subset of such magic states, when acting on the generalized Pauli group, define (asymmetric) informationally complete POVMs. Such IC-POVMs, investigated in dimensions 2–12, exhibit simple finite geometries in their projector products and, for dimensions 4 and 8 and 9, relate to two-qubit, three-qubit and two-qutrit contextuality.

1. Introduction

Sometimes a field of knowledge gets enriched just by looking at it on a different perspective. Here we are interested in informationally complete (IC) measurements on an unknown density matrix ρ with the perspective of universal quantum computing. In the former subfield, one knows how to build group covariant symmetric measurements (SIC-POVMs) that follow from the action of the generalized Pauli group $\mathcal{P}_d$ on a well-chosen 'fiducial' state [1–3]. In the latter subfield, the group $\mathcal{P}_d$ needs to be extended by a well-chosen 'magic' state of the corresponding dimension to allow universal quantum computation [4,5]. Bravyi & Kitaev [4] introduced the principle of 'magic state distillation': universal quantum computation may be realized thanks to the stabilizer formalism (Clifford group unitaries, preparations and measurements) and the ability to prepare an ancilla in an appropriate single qubit mixed state. Following [6, §IIC], in this paper, a non-stabilizer pure state will be called a magic state. When is such a 'magic' state 'fiducial' for an IC-POVM? To address this question, we restrict our choice to eigenstates of permutation gates not living in $\mathcal{P}_d$ (the stabilizer subgroup of unitaries) as in the recent paper [7]. We recover the Hesse SIC for $d = 3$ and discover asymmetric IC-POVMs for $d > 3$.

In this paper, we first remind in §1 a few necessary concepts for our purpose: POVM concepts and the generalized Pauli group. In §2, we apply the methodology to the derivation of IC-POVM's in dimensions 2–12, then we establish the link to some finite geometries and to two-qubit, three-qubit and two-qutrit contextuality. Section 3 summarizes the results.

A POVM is a collection of positive semi-definite operators $\{E_1, \ldots, E_m\}$ that sum to the identity. In the measurement of a state ρ, the ith outcome is obtained with a probability given by the Born rule $p(i) = \mathrm{tr}(\rho E_i)$. For a minimal IC-POVM, one needs d^2 one-dimensional projectors $\Pi_i = |\psi_i\rangle\langle\psi_i|$, with $\Pi_i = dE_i$, such that the rank of the Gram matrix with elements $\mathrm{tr}(\Pi_i \Pi_j)$, is precisely d^2.

A SIC-POVM obeys the remarkable relation [1]

$$|\langle \psi_i \mid \psi_j \rangle|^2 = \mathrm{tr}(\Pi_i \Pi_j) = \frac{d\delta_{ij} + 1}{d + 1}$$

that allows the recovery of the density matrix as [8]

$$\rho = \sum_{i=1}^{d^2} \left[(d+1)p(i) - \frac{1}{d} \right] \Pi_i.$$

This type of quantum tomography is often known as quantum-Bayesian, where the $p(i)$'s represent agent's Bayesian degrees of belief, because the measurement depends on the filtering of ρ by the selected SIC (for an unknown classical signal, this looks similar to the frequency spectrum).

In this paper, we discover new IC-POVMs (i.e. whose rank of the Gram matrix is d^2) and with Hermitian angles $|\langle \psi_i \mid \psi_j \rangle||_{i \neq j} \in A = \{a_1, \ldots, a_l\}$, a discrete set of values of small cardinality l. A SIC is equiangular with $|A| = 1$ and $a_1 = 1/\sqrt{d+1}$.

The states encountered below are considered to live in a cyclotomic field $\mathbb{F} = \mathbb{Q}[\exp(2i\pi/n)]$, with $n = \mathrm{GCD}(d, r)$, the greatest common divisor of d and r, for some r. The Hermitian angle is defined as $|\langle \psi_i \mid \psi_j \rangle||_{i \neq j} = \|(\psi_i, \psi_j)\|^{1/\deg}$, where $\|.\|$ means the field norm [9, p. 162] of the pair (ψ_i, ψ_j) in $\mathbb{F}$ and deg is the degree of the extension $\mathbb{F}$ over the rational field $\mathbb{Q}$. For the IC-POVMs under consideration below, in dimensions $d = 3, 4, 5, 6$ and 7, one has to choose $n = 3, 12, 20, 6$ and 21, respectively, in order to be able to compute the action of the Pauli group. Calculations are performed with Magma.

1.1. The single qubit SIC-POVM

To introduce our methodology, let us start with the qubit magic state

$$|T\rangle = \cos(\beta)|0\rangle + \exp\left(\frac{i\pi}{4}\right)\sin(\beta)|1\rangle, \quad \cos(2\beta) = \frac{1}{\sqrt{3}},$$

employed for universal quantum computation [4]. It is defined as the $\omega_3 = \exp(2i\pi/3)$-eigenstate of the SH matrix (the product of the Hadamard matrix H and the phase gate $S = \left(\begin{smallmatrix} 1 & 0 \\ 0 & i \end{smallmatrix}\right)$).

Taking the action on $|T\rangle$ of the four Pauli gates I, X, Z and Y, the corresponding (pure) projectors $\Pi_i = |\psi_i\rangle\langle\psi_i|, i = 1\ldots 4$, sum to twice the identity matrix thus building a POVM and the pairwise distinct products satisfy $|\langle\psi_i \mid \psi_j\rangle|^2 = 1/3$. The four elements Π_i form the well known two-dimensional SIC-POVM [1, §2].

By contrast, there is no POVM attached to the magic state $|H\rangle = \cos(\pi/8)|0\rangle + \sin(\pi/8)|1\rangle$.

1.2. The generalized Pauli group

Later, we construct IC-POVMs using the covariance with respect to the generalized Pauli group. Let d be a prime number, the qudit Pauli group is generated by the shift and clock operators as follows:

$$\left.\begin{aligned} X|j\rangle &= |j+1 \quad \mathrm{mod}\ d\rangle \\ Z|j\rangle &= \omega^j|j\rangle, \end{aligned}\right\} \tag{1.1}$$

and

with $\omega = \exp(2i\pi/d)$ a dth root of unity. In dimension $d = 2$, X and Z are the Pauli spin matrices σ_x and σ_z.

A general Pauli (also called Heisenberg–Weyl) operator is of the form

$$T_{(m,j)} = \begin{cases} i^{jm} Z^m X^j & \text{if } d = 2 \\ \omega^{-jm/2} Z^m X^j & \text{if } d \neq 2. \end{cases} \tag{1.2}$$

where $(j, m) \in \mathbb{Z}_d \times \mathbb{Z}_d$. For N particles, one takes the Kronecker product of qudit elements N times.

Stabilizer states are defined as eigenstates of the Pauli group.

(a) Magic qutrit POVM's
(0,1,1) or (0,1,−1)
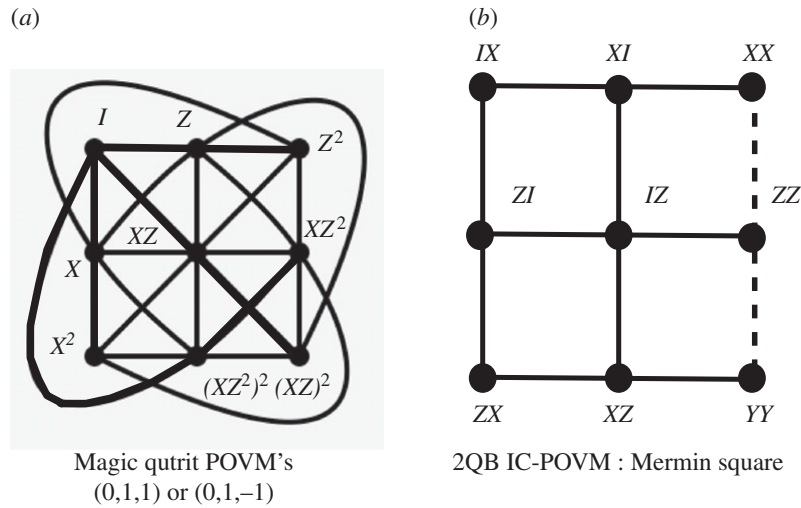
(b) 2QB IC-POVM : Mermin square

Figure 1. (a) The Hesse configuration resulting from the qutrit POVM. The lines of the configuration correspond to traces of triple products of the corresponding projectors equal to 1/8 (for the state (0, 1, −1)) and ±1/8 (for the state (0, 1, 1)). The configuration is labelled in terms of the qutrit operators acting on the magic state. Bold lines feature the lines where all operator pairs are commuting. (b) The triple products of the four dimensional IC-POVM whose trace equal ±1/27 and simultaneously equal plus or minus the identity matrix $\mathcal{I}$ ($-\mathcal{I}$ for the dotted line). This picture identifies to the well-known Mermin square which allows a proof of the Kochen–Specker theorem.

2. Permutation gates, magic states and informationally complete measurements

In the approach of magic states through permutation groups, dimension 2 is trivial as the symmetric group S_2 only contains the identity $I = (1, 2)$ and the shift gate $X = (2, 1) \equiv \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$, that live in the ordinary Pauli group $\mathcal{P}_2$. No magic state may be derived from two-dimensional permutation groups.

The situation changes as soon as $d \geq 3$ with a wealth of magic states [7] having a potential usefulness for our purpose of defining IC-POVMs. From now we focus on magic groups generated by two magic permutation gates.

2.1. In dimension 3

The symmetric group S_3 contains the permutation matrices I, X and X^2 of the Pauli group, where $X = \left(\begin{smallmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{smallmatrix} \right) \equiv (2, 3, 1)$ and three extra permutations $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{smallmatrix} \right) \equiv (2, 3)$, $\left(\begin{smallmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{smallmatrix} \right) \equiv (1, 3)$ and $\left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{smallmatrix} \right) \equiv (1, 2)$, that do not lie in the Pauli group but are parts of the so-called Clifford group (the normalizer of the Pauli group in the unitary group).

Taking the eigensystem of the latter matrices, it is not difficult check that there exists two types of qutrit magic states of the form $(0, 1, \pm 1) \equiv 1/\sqrt{2}(|0\rangle + |1\rangle \pm |2\rangle)$. Then, taking the action of the nine qutrit Pauli matrices, one arrives at the well-known Hesse SIC [10–12].

The Hesse configuration shown in figure 1a is a configuration $[9_4, 12_3]$ with 9 points and 12 lines, 4 lines incident on every point and 3 points on a line. It can also be seen as the three-dimensional affine plane. The reason it occurs in the context of the three-dimensional SIC is as follows. The SIC relations are $\mathrm{tr}(\Pi_i \Pi_j)_{i \neq j} = 1/4$ and, if one takes all projectors satisfying the triple product relation $\mathrm{tr}(\Pi_i \Pi_j \Pi_k)_{i \neq j \neq k} = \pm 1/8$, the corresponding triples (i, j, k) define the Hesse configuration. For the Hesse SIC built from the magic state $(0, 1, -1)$, one only needs the plus sign to recover the Hesse geometry, but for the Hesse SIC built from the magic state $(0, 1, 1)$ both signs are needed (see also [11]).

Observe that the configuration in figure 1a is labelled in terms of qutrit operators acting on the magic states instead of the projector themselves.

2.2. In dimension 4

In dimension 4 and higher, the strategy is to restrict to permutation groups whose two generators are magic gates, gates showing one entry of 1 on their main diagonals. From now, we call such a group a
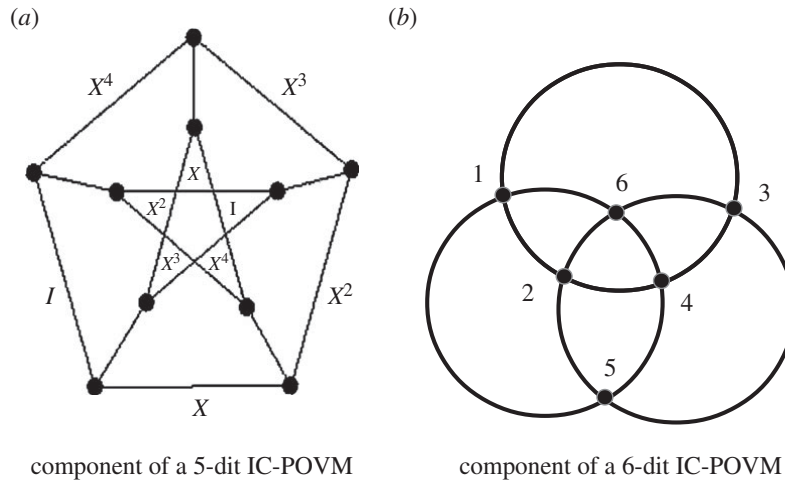
Figure 2. (*a*) A one-point intersection graph for the lines of the 5-dit equiangular IC-POVM defined from the triple products of constant trace $-1/4^3$. (*b*) A component of the 6-dit IC-POVM with magic state $(0, 1, \omega_6 - 1, 0, -\omega_6, 0)$ through the action of Pauli operators 1–6: the lines correspond to 4-tuples products of projectors with constant trace $1/9$ and simultaneously of products equal to $\pm \mathcal{I}$. There are two disjoint copies looking like Borromean rings with points as $[1 \ldots 6] = [\mathcal{I}, ZX^3, Z^2, Z^3X^3, Z^4, Z^5X^3]$ and $[1 \ldots 6] = [X^4, Z, Z^2X^3, Z^3, Z^4X^3, Z^5]$.

magic group. This only happens for a group isomorphic to the alternating group

$$A_4 \cong \left\langle \begin{pmatrix} 1\,0\,0\,0 \\ 0\,0\,0\,1 \\ 0\,1\,0\,0 \\ 0\,0\,1\,0 \end{pmatrix}, \begin{pmatrix} 0\,1\,0\,0 \\ 0\,0\,1\,0 \\ 1\,0\,0\,0 \\ 0\,0\,0\,1 \end{pmatrix} \right\rangle.$$

One finds magic states of type $(0, 1, 1, 1)$ and $(0, 1, -\omega_6, \omega_6 - 1)$, with $\omega_6 = \exp(2i\pi/6)$ [7, §3.3].

Taking the action of the two-qubit Pauli group on the latter type of state, the corresponding pure projectors sum to four times the identity (to form a POVM) and are independent, with the pairwise distinct products satisfying the dichotomic relation $\text{tr}(\Pi_i \Pi_j)_{i \neq j} = |\langle \psi_i \,|\, \psi_j \rangle|^2_{i \neq j} \in \{1/3, 1/3^2\}$. Thus, the 16 projectors Π_i build an asymmetric informationally complete measurement not discovered so far.

The organization of triple products of projectors whose trace is $\pm 1/27$ and simultaneously equal plus or minus the identity matrix $\mathcal{I}$ is shown in figure 1*b*. Instead of labelling coordinates as projectors one may label them with the two-qubit operators acting on the magic state. As a result, the two-qubit (3×3)-grid identifies to the standard Mermin square that is known to allow an operator proof of the Kochen–Specker theorem [13,14].

2.3. In dimension 5

Still restricting to permutation groups generated by two magic gates (magic groups), the smallest group is isomorphic to the semidirect product $\mathbb{Z}_5 \rtimes \mathbb{Z}_4$ of cyclic groups $\mathbb{Z}_4$ and $\mathbb{Z}_5$ [7, §3.4]. One finds magic states of type $(0, 1, 1, 1, 1)$, $(0, 1, -1, -1, 1)$ and $(0, 1, i, -i, -1)$. The latter two types allow to construct IC-POVM's such that the pairwise distinct products satisfy $|\langle \psi_i \,|\, \psi_j \rangle|^2 = 1/4^2$, that is the POVM is equiangular with respect to the field norm defined in the introduction. The first type of magic state is dichotomic with values of the products $1/4^2$ and $(3/4)^2$. The trace of pairwise products of (distinct) projectors is not constant. For example, with the state $(0, 1, -1, -1, 1)$, one gets a field norm equiangular IC-POVM in which the trace is trivalued: it is either $1/16$ or $(7 \pm 3\sqrt{5})/32$. For the state $(0, 1, i, -i, -1)$, there are five values of the trace.

With the symmetric group S_5, one builds magic states of type $(0, 0, 1, 1, 1)$ and IC-POVM's with dichotomic values of the distinct pairwise products equal to $(1/3)^2$ and $(2/3)^2$.

Let us concentrate on the equiangular POVM. Traces of triple products with constant value $-1/4^3$ define lines organized into a geometric configuration of type $(25_{12}, 100_3)$. Lines of the configuration have one or two points in common. The two-point intersection graph consists of 10 disjoint copies of the Petersen graph. One such a Petersen graph is shown in figure 2*a*, the vertices of the graph correspond to the lines and the edges correspond to the one-point intersection of two lines. As before the labelling is in terms of the operators acting on the magic state.

Similar Petersen graphs occur in the organization of triple products for the other five-dimensional IC-POVMs.

2.4. In dimension 6

With the alternating group A_6 generated by two magic gates, one finds an IC-POVM associated to a magic state such as $(0, 1, \omega_6 - 1, 0, -\omega_6, 0)$ with $\text{tr}(\Pi_i \Pi_j)_{i \neq j} = |\langle \psi_i | \psi_j \rangle|^2_{i \neq j} = 1/3$ or $1/3^2$.

Taking the trace of 4-tuple products of projectors whose value is $1/9$ and simultaneously equal $\pm \mathcal{I}$, one gets two copies of a geometry looking like a Borromean ring as shown in figure 2b.

2.5. In dimension 7

Using a magic group isomorphic to $\mathbb{Z}_7 \rtimes \mathbb{Z}_6$ and the magic state $(1, -\omega_3 - 1, -\omega_3, \omega_3, \omega_3 + 1, -1, 0)$, one arrives at an equiangular IC-POVM satisfying $|\langle \psi_i | \psi_j \rangle|^2_{i \neq j} = 1/6^2$. Other magic states are also found that define IC-POVM's with dichotomic products. But no simple structure of the higher order products has been found.

2.6. In dimension 8

In dimension $d = 8$, no IC-POVM was discovered from permutation groups. But it is time to introduce the well-known Hoggar SIC [15,16]. The Hoggar SIC follows from the action of the three-qubit Pauli group on a fiducial state such as $(-1 \pm i, 1, 1, 1, 1, 1, 1, 1)$.

It has been found that triple products are related to combinatorial designs [16]. There are 4032 (resp. 16128) triples of projectors whose products have trace equal to $-1/27$ (resp. $1/27$) [16, (29)]. Within the 4032 triples, those whose product of projectors equal $\pm \mathcal{I}$ (with $\mathcal{I}$ the identity matrix) are organized into a geometric configuration [63$_3$] whose incidence graph is of spectrum $[6^1, 3^{31}, -1^{27}, -3^{14}]$ and automorphism group $G_2(2) = U_3(3) \rtimes \mathbb{Z}_2$ of order 12 096, as in [17]. It is known that there exists two isospectral configurations of this type, one is the so-called generalized hexagon GH(2,2) (also called split Cayley hexagon) and the other one is its dual [18]. These configurations are related to the 12 096 Mermin pentagrams that build a proof of the three-qubit Kochen–Specker theorem [17,19]. From the structure of hyperplanes of our [63$_3$] configuration, one learns that we are concerned with the dual of G_2 as shown in figure 3 (see also [20, Fig. 6a]).

Similarly within the 16 128 triples, set of projectors whose triples equal $\pm \mathcal{I}$ are organized into a configuration [63$_{12}$, 252$_3$] whose incidence graph has spectrum $[33^1, 15^{14}, 9^{21}, 5^{27}, -3^{189}]$ and automorphism group $G_2(2)$ again. The graph shows 63 maximum cliques of size 4 and 72 of size 7. Every maximum clique of size 4 is a Pasch configuration as shown in figure 4.

2.7. In dimension 9

Let us consider a magic group isomorphic to $\mathbb{Z}_3^2 \rtimes \mathbb{Z}_4$ generated by two magic gates. One finds a few magic states such as $(1, 1, 0, 0, 0, 0, -1, 0, -1)$ that, not only can be used to generate a dichotomic IC-POVM with distinct pairwise products $|\langle \psi_i | \psi_j \rangle|^2$ equal to $1/4$ or $1/4^2$, but also show a quite simple organization of triple products. Defining lines as triple of projectors with trace $1/8$, one gets a geometric configuration of type [81$_3$] that split into nine disjoint copies of type [9$_3$]. One of the copies is shown in figure 5.

The configuration [9$_3$] labelled by the operators of figure 5 may be used to provide an operator proof of the Kochen–Specker theorem with two qutrits. The proof is in the same spirit than the one derived for two or three qubits [14] (see also [21] for two-qutrit contextuality). The vertices are projectors instead of just Hermitian operators. On one hand, every operator O can be assigned a value $v(O)$ which is an eigenvalue of O, that is 1, ω_3 or ω_3^2 (with $\omega_3^3 = 1$). Taking the product of eigenvalues over all operators on a line and over all nine lines, one gets 1 since every assigned value occurs three times.

On the other hand, the operators on a line in figure 5 do not necessarily commute but their product is $\mathcal{I} = I \otimes I$, $\omega_3 \mathcal{I}$ or $\omega_3^* \mathcal{I}$, depending on the order of operators in the product. Taking the ordered triples $[1, 6, 9]$, $[9, 7, 8]$, $[2, 4, 8]$, $[1, 3, 2]$, $[8, 5, 1]$, $[3, 5, 7]$, $[3, 4, 9]$, $[4, 5, 6]$ and $[2, 6, 7]$, the triple product of these operators from left to right equals $\mathcal{I}$ except for the dotted line where it is $\omega_3 \mathcal{I}$.

Thus the product law $v(\Pi_{i=1}^9 O_i) = \Pi_{i=1}^9 [v(O_i)]$ is violated. The left-hand side equals ω_3 while the right-hand side equals 1. No non-contextual hidden variable theory is able to reproduce these results. Since the lines are not defined by mutually commuting operators, it is not possible to arrive at a proof of the
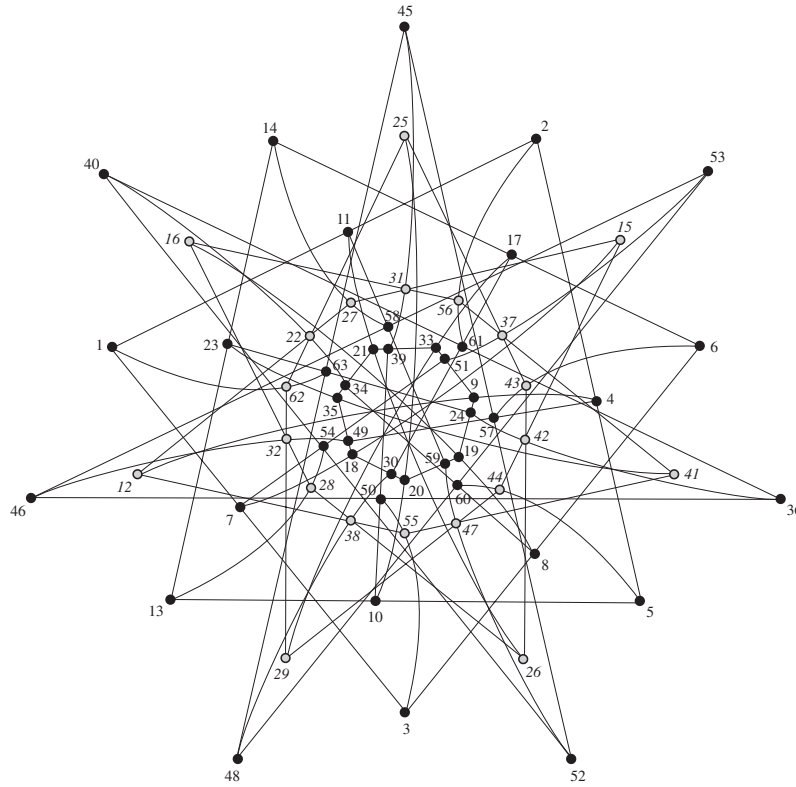
Figure 3. The dual of the generalized hexagon GH(2,2). Grey points have the structure of an embedded generalized hexagon GH(2,1) [18].
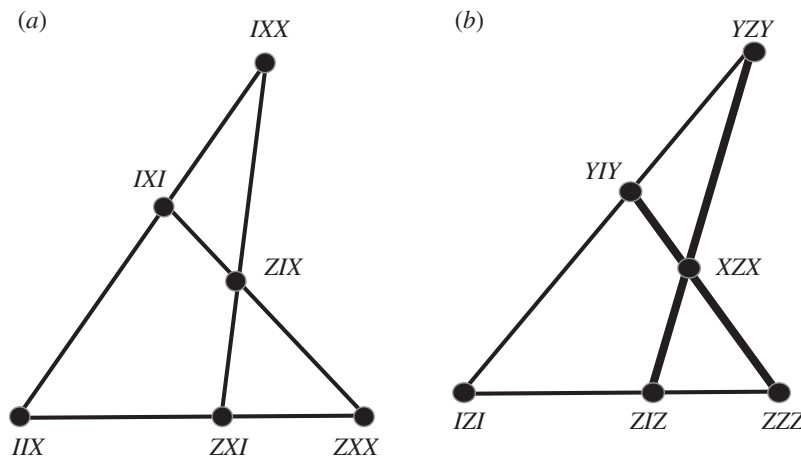

Figure 4. Two types of Pasch blocks in the structure of trace 1/27 triple products of a Hoggar SIC. Thin (resp. thick) lines are for triple products equal to $\mathcal{I}$ (resp. $-\mathcal{I}$).

two-qutrit Kochen–Specker based on vectors instead of operators. In this sense, the proof of contextuality is weaker that the one obtained for two or three qubits.

2.8. Higher dimensions

The same method based on eigenstates of permutation matrices leads to IC-POVMs in dimensions higher than 9. In the next subsection, we provide details about a 12-dimensional IC-POVM covariant under the two-qubit/qutrit Pauli group because the associated triple products contain some geometrical structures as it was the case in lower dimensions.

2.9. In dimension 12

One can build an IC-POVM using a magic group isomorphic to $\mathbb{Z}_2^2 \rtimes (\mathbb{Z}_3^2 \rtimes \mathbb{Z}_2^2)$. A magic state can be taken as $(0, 1, \omega_6 - 1, \omega_6 - 1, 1, 1, \omega_6 - 1, -\omega_6, -\omega_6, 0, -\omega_6, 0)$ or the state obtained from it by permuting
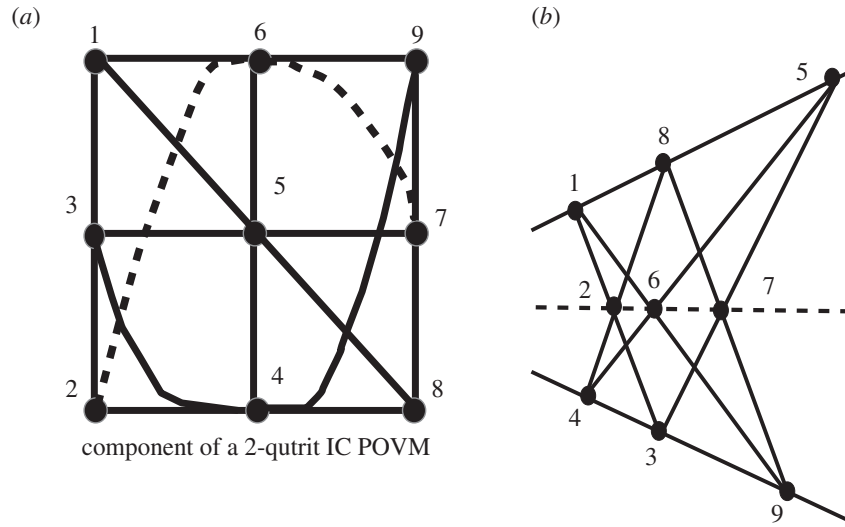
Figure 5. Lines of one component of the two-qutrit IC-POVM built from the magic state $(1, 1, 0, 0, 0, 0, -1, 0, -1)$ alias the Pappus configuration (*b*). The points are labelled in terms of the two-qutrit operators $[1, 2, 3, 4, 5, 6, 7, 8, 9] = [I \otimes Z, I \otimes XZ, I \otimes (XZ^2)^2, Z \otimes I, Z \otimes X, Z \otimes X^2, Z^2 \otimes Z^2, Z^2 \otimes (XZ)^2, Z^2 \otimes XZ^2]$, where X and Z are the qutrit shift and clock operators. The IC-POVM, as labelled, can be used to prove the Kochen–Specker theorem for two qutrits. This is related to the fact that the selected product of operators on a line is the identity matrix $\mathcal{I}$ except for the dotted line where it is $\omega_3 \mathcal{I}$ (see details in the text).
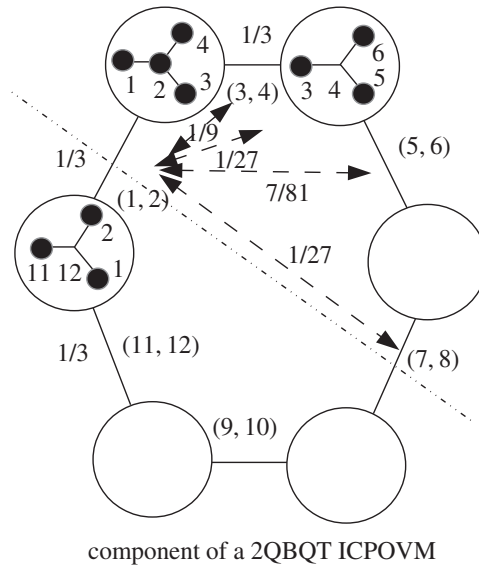

Figure 6. Schematic of a 12-projector component for the IC-POVM built from the magic state given in the text. Each circle contains two triples (e.g. (1, 2, 3) and (1, 2, 4) for the upper left circle). The 12 projectors numbered 1–12 are passed by pairs from one circle to the other (as shown) so that empty circles are easily filled. The four types of pair products $1/3$, $1/9$, $1/27$, $7/81$ occurring are also shown.

the entries $\omega_6 - 1$ and $-\omega_6$ and vice versa. The IC is obtained tanks to the action of the 2QB-QT Pauli group on this state. The distinct trace products are multivalued with eight values. There is an interesting structure of the 144 triple products whose trace is $-1/27$. There are organized into 12 distinct configurations of the type shown in figure 6 (with only four values of pair products occurring).

3. Summary and conclusion

The main contribution of our work is the construction of asymmetric IC-POVMs built thanks to the action of the Pauli group on appropriate permutation generated magic/fiducial states. A summary of the work is in table 1. It is remarkable that the same corpus of ideas may be used simultaneously for permutation groups, universal quantum computing, unambiguous quantum state recovery and also

Table 1. A summary of magic states and the corresponding signatures of IC-POVMs in dimensions 2–12.

dim	magic state	$|\langle\psi_i\,|\,\psi_j\rangle|^2_{i\neq j}$	geometry
2	$|T\rangle$	1/3	tetrahedron [4]
3	$(0,1,\pm1)$	1/4	Hesse SIC [10]
4	$(0,1,-\omega_6,\omega_6-1)$	$\{1/3, 1/3^2\}$	Mermin square[a]
5	$(0,1,-1,-1,1)$	$1/4^2$	Petersen graph
	$(0,1,i,-i,-1)$		
	$(0,1,1,1,1)$	$\{1/3^2, (2/3)^2\}$	
6	$(0,1,\omega_6-1,0,-\omega_6,0)$	$\{1/3, 1/3^2\}$	Borromean rings
7	$(1,-\omega_3-1,-\omega_3,\omega_3,\omega_3+1,-1,0)$	$1/6^2$	unknown
8	$(-1\pm i,1,1,1,1,1,1,1)$	1/9	Hoggar SIC [16], [63$_3$][a]
9	$(1,1,0,0,0,0,-1,0,-1)$	$\{1/4,1/4^2\}$	[9$_3$] configuration[a]
12	$(0,1,\omega_6-1,\omega_6-1,1,1,$ $\omega_6-1,-\omega_6,-\omega_6,0,-\omega_6,0)$	8 values	figure 6

[a] In dimensions 4, 8 and 9, a proof of the two-qubit, two-qutrit and three-qubit Kochen–Specker theorem follows from the IC-POVM. For $d \geq 6$, the magic states leading to an IC (as distinguished) become rare.

quantum contextuality. Further work may focus on extending the range of dimensions where IC-POVM's may be derived, relate the useful magic states to quantum error correction and state distillation [4] and the Bayesian interpretation of quantum mechanics [8].

It is expected that this type of work will clarify the observed efficiency of quantum algorithms based on permutations [22] and the relation between contextuality and quantum computing [23,24].

Q3

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