ELSEVIER
Journal of Mathematical Economics 37 (2002) 215–230
JOURNAL OF Mathematical ECONOMICS
www.elsevier.com/locate/jmateco

Ordinal proportional cost sharing

Yun-Tong Wang [a,*], Daxin Zhu [b]

[a] *Faculty of Arts and Social Sciences, Sabanci University, Orhanli 81474 Tuzla, Istanbul, Turkey*
[b] *Department of Mathematics, Tianjin University, Tianjin 300072, PR China*

Received 29 May 2001; received in revised form 7 April 2002; accepted 8 April 2002

Abstract

We consider cost sharing problems with variable demands of heterogeneous goods. We study the compatibility of two axioms imposed on cost sharing methods: ordinality and average cost pricing for homogeneous (ACPH) goods. We generalize the *ordinal proportional method* (OPM) for the two-agent case, Sprumont [Journal of Economic Theory 81 (1998) 126–162] to arbitrary number of agents.

JEL classification: D63; C71

Keywords: Cost sharing; Ordinality; Average cost pricing

1. Introduction

This paper studies the compatibility of two axioms: ordinality and average cost pricing for homogeneous goods (ACPH), on cost sharing methods. In the two-agent cost sharing problem, Sprumont (1998) has defined a cost sharing method called *ordinal proportional method* (OPM) that satisfies these two axioms. We ask if these two axioms are still compatible by cost sharing methods for cost sharing problems with more than two agents. To answer this question, we generalize the OPM from the two-agent case to the case with any finite number of agents. For this purpose, we study a special *integral equation* system. Our generalization of OPM depends on the existence and the uniqueness of the solution of this equation system.

We consider essentially the same cost sharing model that has been considered in the large literature on the well-known Aumann–Shapley pricing method (A–S) (Billera et al., 1978; Billera and Heath, 1982; Mirman and Tauman, 1982; Samet and Tauman, 1982). In this model, a cost function summarizes the minimum production cost for each demand vector,

* Corresponding author. Tel.: +90-216-483-9268; fax: +90-216-483-9250.
E-mail addresses: wang@sabanciuniv.edu (Y.-T. Wang), dxzhu@tju.edu.cn (D. Zhu).

PII: S0304-4068(02)00016-2

which is a list of quantities representing the demand for each good. Goods are perfectly divisible. Given a demand vector, the total cost must be attributed to these goods. Examples of such cost sharing problem are plenty (see the references in the survey Moulin, 1999).

As is well-known (e.g. Billera and Heath, 1982; Samet and Tauman, 1982), the A–S method satisfies the axioms of additivity (w.r.t. the cost function), dummy (zero price for the dummy good whose marginal cost is always zero), scale invariance (SI) (invariance w.r.t. any re-scaling of the units of the goods), and ACPH (coinciding with the average cost pricing when goods are homogeneous). Recently, Moulin (1995) and other authors (Friedman and Moulin, 1999; Moulin and Shenker, 1992; Sprumont, 1998) criticize the A–S by pointing out that it violates Demand Monotonicity (no agent should pay less when his demand increases), and propose alternative cost sharing methods. Sprumont (1998) further points out that the A–S, although it is scale invariant, is not *ordinal*. The ordinality axiom requires that cost shares be invariant with any increasing transformations of the measurement of the goods. Clearly, it is stronger than the scale invariance axiom.

The ordinality axiom and the ACPH are, in fact, not compatible in the realm of *additive* methods satisfying the dummy axiom (Friedman and Moulin, 1999; Sprumont, 1998). The additivity and dummy axioms, first introduced by Shapley (1953) in the cooperative game theory, have been the two fundamental axioms in the axiomatic cost sharing literature (Billera and Heath, 1982; Friedman and Moulin, 1999; Haimanko, 1998; Moulin, 1999; Wang, 1999). In the meantime, additivity has limited the scope of potential meaningful methods and even become the source of many impossibility results or incompatibilities between compelling axioms (the incompatibility between ordinality and ACPH is an example; for more examples, see Friedman and Moulin, 1999; Moulin, 1999). Recently, there has been a growing interest in dropping the additivity axiom and looking for *nonadditive* methods, which may reconcile the conflicts or recover the compatibilities between some compelling axioms (e.g. Koster et al., 1998; Sprumont, 1998). This paper is in line with the study of *nonadditive* methods. Particularly, we show that ordinality and ACPH are compatible through the (*nonadditive*) OPM for cost sharing problems with any finite number of agents.

2. The model

Let n be a positive integer. Let $N = \{1, \dots, n\}$ be the set of agents (or goods). A demand vector q is a vector in R_+^N. Let $\mathcal{C}_0$ be the set of functions $C : R_+^N \to R_+$ which are nondecreasing ($t \leq t' \Rightarrow C(t) \leq C(t')$ for all $t, t' \in R_+^N$) and satisfy $C(0) = 0$. A cost function is an element in $\mathcal{C}_0$. If the first-order partial derivative of $C \in \mathcal{C}_0$ with respect to its ith argument exists at $t \in R_+^N$, we denote it by $\partial_i C(t)$.[1] Denote $\mathcal{C}_1$ the set of all continuously differentiable functions in $\mathcal{C}_0$ and $\mathcal{C}_2$ those that are twice continuously differentiable. Denote $\mathcal{C}$ a subset of $\mathcal{C}_0$.

A *problem* is a pair $(q; C)$, where q is a demand vector and C is a cost function. Given a problem $(q; C)$, a solution of the problem is a vector $(x_1, \dots, x_n) \in R_+^N$ such that $\sum_1^n x_i = C(q)$. A *method* x is a mapping that associates with each problem $(q; C)$ a solution $x(q; C)$.

[1] If $t_i = 0$, it is understood that $\partial_i C(t)$ stands for the right-hand derivative.

We call a cost function C *homogeneous* if there is a mapping $c : R_+ \to R_+$ such that

$$C(q) = c\left(\sum_{i\in N} q_i\right), \quad q \in R_+^n.$$

Call a problem $(q; C)$ homogeneous if the cost function C is homogeneous.

3. Two axioms

3.1. Average cost pricing for homogeneous goods

We say a method x satisfies the ACPH axiom if

$$x_i(q; C) = \frac{q_i}{\sum_{j=1}^n q_j} C(q), \quad i = 1, \dots, n$$

whenever the problem $(q; C)$ is homogeneous.

We say a method is an *average cost extension* if it satisfies ACPH.

The Aumann–Shapley (A–S) pricing method (Billera et al., 1978; Billera and Heath, 1982; Samet and Tauman, 1982; Tauman, 1988) is an average cost extension:

$$x_i^{\text{AS}}(q, C) = q_i \int_0^1 \partial_i C(tq)\,\mathrm{d}t, \quad i = 1, \dots, n. \tag{1}$$

In fact, the A–S is the unique average cost extension within the family of additive methods. More precisely, the A–S is characterized by the axioms of additivity, dummy, scale invariance (SI), and ACPH (Billera et al., 1978; Billera and Heath, 1982; Samet and Tauman, 1982; Tauman, 1988), where additivity and dummy are the two classical axioms of Shapley (1953), and SI is a property of "measurement invariance" with respect to the "linear transformations" of the measurement units. We restate them as follows.

Additivity: For every $q \in R_+^N$ and $C_1, C_2 \in \mathcal{C}$,

$$x(q; C_1 + C_2) = x(q; C_1) + x(q; C_2).$$

Dummy: Given $(q; C)$. For any $i = 1, \dots, n$, if $\partial_i C(t) = 0,\ \ \forall t \in R_+^N$, then

$$x_i(q; C) = 0.$$

Scale invariance: For any $(q; C)$ and any $r \in R_+^N$, $r \gg 0$,

$$x(q; C) = x((r_1 q_1, \dots, r_n q_n); C^r)$$

where $C^r(t) = C((1/r_1)t_1, \dots, (1/r_n)t_n), \quad t \in R_+^N$.

The following *nonadditive* method, called *proportionally adjusted marginal pricing* (PAMP) method is also an average cost extension:

$$x_i(q; C) = \frac{\partial_i C(q) q_i}{\sum_{j=1}^n \partial_j C(q) q_j} C(q), \quad i = 1, \dots, n$$

and satisfies SI.

However, the example[2] below shows that both the A–S and the PAMP are not independent of the "nonlinear transformations" of the methods of measuring the goods.

Consider the cost function

$$C(t_1, t_2) = t_1 + \sqrt{t_2} + t_1\sqrt{t_2},$$

where t_1, t_2 represent the *distance* from two locations, A and B, to a destination D, and function C represents the cost (e.g. time) of travelling from these two locations to the destination.

Consider the problem $((1, 1); C)$ first.

(1) By the A–S method, A and B's cost shares are

$$x_{\mathrm{A}}^{\mathrm{AS}}((1, 1); C) = \tfrac{5}{3} \quad \text{and} \quad x_{\mathrm{B}}^{\mathrm{AS}}((1, 1); C) = \tfrac{4}{3}.$$

(2) By the PAMP, A and B's cost shares are

$$x_{\mathrm{A}}^{\mathrm{PAMP}}((1, 1); C) = 2 \quad \text{and} \quad x_{\mathrm{B}}^{\mathrm{PAMP}}((1, 1); C) = 1.$$

Now suppose that we use *time* instead of *distance* as the measurement unit of the variables t_1, t_2, and the cost function accordingly changes to

$$\tilde{C}(t_1, t_2) = t_1 + t_2 + t_1 t_2.$$

Re-calculate A and B's cost shares, we then have

(1′) by the A–S method

$$x_{\mathrm{A}}^{\mathrm{AS}}((1, 1); \tilde{C}) = \tfrac{3}{2} \quad \text{and} \quad x_{\mathrm{B}}^{\mathrm{AS}}((1, 1); \tilde{C}) = \tfrac{3}{2}.$$

(2′) by the PAMP

$$x_{\mathrm{A}}^{\mathrm{PAMP}}((1, 1); \tilde{C}) = \tfrac{3}{2} \quad \text{and} \quad x_{\mathrm{B}}^{\mathrm{PAMP}}((1, 1); \tilde{C}) = \tfrac{3}{2}.$$

Therefore, the A–S and the PAMP are not fully independent of the measurement units of the goods, although they both satisfy SI.

To rule out this "measurement dependence", we impose the ordinality axiom given in the following sections.

3.2. *Ordinality*

For completeness, we restate here the definition of ordinality first proposed by Sprumont (1998).

Given the domain $\mathcal{C}$. Let $f = (f_1, \ldots, f_n)$, where $f(t) = (f_1(t_1), \ldots, f_n(t_n)), t = (t_1, \ldots, t_n) \in R_+^N$ and each f_i is a bijection from R_+ onto itself. For each cost function C in $\mathcal{C}$, define $C^f : R_+^n \to R_+$ by

$$C^f(t) = C(f(t)) \quad \text{for all } t \in R_+^n.$$

[2] This example is taken from Sprumont (1998).

Call f an ordinal transformation if $\mathcal{C}$ is closed under it, i.e.

$$C^f \in \mathcal{C} \quad \text{for all} \quad C \in \mathcal{C}.$$

We can easily check that when $\mathcal{C} = \mathcal{C}_1$ a bijection f is an ordinal transformation if and only if it is increasing and continuously differentiable.

Call two problems $(q; C)$ and $(q'; C')$ *ordinally equivalent* if there exists an ordinal transformation f such that

$$C' = C^f \quad \text{and} \quad q = f(q').$$

We say a cost sharing method *ordinal* if it satisfies the following axiom.

Ordinality: If $(q; C)$ and $(q'; C')$ are two ordinally equivalent problems, then $x(q; C) = x(q'; C')$.

Note that if the ordinal transformation is linear, i.e.

$$f(t) = (\lambda_1 t_1, \dots, \lambda_n t_n), (\lambda_1, \dots, \lambda_n) \gg 0,$$

ordinality becomes SI.

As we have shown in the preceding example, both the A–S and the PAMP are not ordinal.

4. Ordinal proportional method

In this section, we assume that all cost functions are twice continuously differentiable, i.e. we consider the domain $\mathcal{C}_2$.

We say that a problem $(q; C)$ is *proportionally normalized* (Sprumont, 1998) if

$$\partial_i C(rq) = 1, 0 \le r \le +\infty, \quad i = 1, \dots, n. \tag{2}$$

If $(q; C)$ is *proportionally normalized*, then we apply the average cost pricing for the solution of the problem, i.e.

$$x_i(q; C) = \frac{q_i}{\sum_{i \in N} q_j} C(q), \quad i = 1, \dots, n.$$

Definition 1. For any given problem $(q; C)$, if $(q^*; C^*)$ is its proportionally normalized problem, then define $x(q; C)$ by

$$x_i(q; C) = \frac{q_i^*}{\sum_{j \in N} q_j^*} C^*(q^*) = \frac{q_i^*}{\sum_{j \in N} q_j^*} C(q), \quad i = 1, \dots, n,$$

and call x ordinal proportional method.
An immediate question is: Can any problem be *proportionally normalized*? The following example says no.

Example 1. Let $N = \{1, 2, 3\}$, $q = (1, 1, 1)$, and

$$C(t_1, t_2, t_3) = t_1 + t_2, \quad (t_1, t_2, t_3) \in R_+^N.$$

Since $\partial_3 C(t_1,t_2,t_3) = 0$, $\forall\,(t_1,t_2,t_3) \in R_+^N$, problem $(q; C)$ cannot be proportionally normalized.

Now we ask under what condition(s) can we guarantee that a problem always has a unique proportional normalization? Our main theorem below provides such "sufficient conditions".

Formally, consider the following question. Given a problem $(q; C)$, under what condition does there exist a unique pair of q^* and f, where $q^* = (q_1^*, \dots, q_n^*)$ and $f(\lambda) = (f_1(\lambda_1), \dots, f_n(\lambda_n))$ such that $(q; C)$ is ordinally equivalent to $(q^*; C^*)$ by the ordinal transformation f, and $(q^*; C^*)$ is *proportionally normalized*, i.e.

$$\begin{cases} \dfrac{\partial C(f(sq^*))}{\partial q_i} {f'}_i(sq_i^*) = 1, \quad s \in (0,+\infty), \;\; i = 1,\dots,n, \\ f(0) = 0, \\ f(q^*) = q. \end{cases}$$

Let $x(s) = (x_1(s),\dots,x_n(s)) := (f_1(sq_1^*),\dots,f_n(sq_n^*))$ and consider the following generalized initial value problem:

$$\begin{cases} \dot{x}_i(s) = \dfrac{q_i^*}{\partial_i C(x(s))}, \quad s \in (0,+\infty), \;\; i = 1,\dots,n, \\ x(0) = 0, \\ x(1) = q. \end{cases} \tag{3}$$

For a given q^*, the initial value problem

$$\begin{cases} \dot{x}_i(s) = \dfrac{q_i^*}{\partial_i C(x(s))}, \quad s \in (0,+\infty), \;\; i = 1,\dots,n \\ x(0) = 0 \end{cases}$$

is equivalent to the following integral equation problem:

$$x_i(s) = q_i^* \int_0^s \frac{1}{\partial_i C(x(t))}\,\mathrm{d}t, \quad s \in (0,+\infty), \;\; i = 1,\dots,n.$$

By the condition $x(1) = q$, the question becomes the existence and uniqueness of the solution to the following integral equation problem:

$$x_i(s) = \frac{q_i}{\int_0^1 [1/\partial_i C(x(t))]\,\mathrm{d}t} \int_0^s \frac{1}{\partial_i C(x(t))}\,\mathrm{d}t, \quad s \in (0,+\infty), \;\; i = 1,\dots,n. \tag{4}$$

Without loss of generality, in the following discussion we always assume that C has been extended on the whole space R^n.

Theorem 1. *Given a problem $(q; C)$. Assume that $q \gg 0$, and the cost function C is twice continuously differentiable, and there exist positive constants $a(C)$, $b(C)$, and $d(C)$, where*

$$d(C) < \frac{1}{2\|q\|} \frac{a^2(C)}{b(C)}$$

(*where* $\|q\| = \max_{i\in N}|q_i|$) *such that*

$$a(C) \le \partial_i C(t) \le b(C), \quad t \in R_+^N, \ \ i = 1, \dots, n,$$

and

$$\sum_{j=1}^{n} |\partial_{ij}^2 C(t)| \le d(C), \quad t \in R_+^N, \ \ i = 1, \dots, n.$$

Then the following equation has a unique solution:

$$x_i(s) = \frac{q_i}{\int_0^1 [1/\partial_i C(x(t))]\,\mathrm{d}t} \int_0^s \frac{1}{\partial_i C(x(t))}\,\mathrm{d}t, \quad s \in (0, +\infty), \ \ i = 1, \dots, n. \tag{5}$$

In other words, the problem $(q; C)$ *can be uniquely proportionally normalized through an ordinal transformation.*

Proof. The proof is divided into three steps.

Step 1. First, we show the existence of a vector function $x(s) = (x_1(s), \dots, x_n(s))$ that satisfies Eq. (5) on $[0, M]$, where $M > 0$.

Let

$$\begin{aligned} X &= C([0, M]; R^n) \\ &= \{x(s) = (x_1(s), \dots, x_n(s)) | x_i(s)(i = 1, \dots, n) : [0, M] \to R \ \text{ continuous}\}. \end{aligned}$$

Define norm $\|x\| = \max_{1\le i\le n} \max_{0\le s\le M} |x_i(s)|$. Then X is a Banach space with respect to this norm.

Define the mapping $T : X \to X$ by

$$\begin{aligned} (Tx)(s) := \Bigg(&\frac{q_i}{\int_0^1 [1/\partial_1 C(x(t))]\,\mathrm{d}t} \int_0^s \frac{1}{\partial_1 C(x(t))}\,\mathrm{d}t, \dots, \frac{q_n}{\int_0^1 [1/\partial_n C(x(t))]\,\mathrm{d}t} \\ &\times \int_0^s \frac{1}{\partial_n C(x(t))}\,\mathrm{d}t\Bigg), \quad s \in [0, M]. \end{aligned}$$

It is obvious that T is continuous and by the Arzelá–Ascoli theorem (Kantorovich and Akilov, 1964), it is also compact (we omit the detail).

Denote

$$r = \max\left\{1, \frac{b(C)}{a(C)} M\|q\|\right\}.$$

Since

$$\|Tx\| = \max_{1\le i\le n} \max_{0\le s\le M} |(Tx)_i(s)| \le \frac{b(C)}{a(C)} M\|q\| \le r,$$

all the solutions of Eq. (5) (on $[0, M]$) satisfy

$$\|x\| \le r.$$

Consider the ball $\bar{B}(0, r) = \{x \in X | \|x\| \leq r\}$ of X. Then $T\bar{B}(0, r) \subseteq \bar{B}(0, r)$. Since T is a continuous compact mapping, by the Schauder fixed-point theorem T has at least one fixed-point x in $\bar{B}(0, r)$, and a fixed-point x is a solution of Eq. (5).

Step 2. Now we show that any solution of Eq. (5) on the finite interval $[0, M]$ ($M \geq 1$) can be uniquely extended on $[0, +\infty)$.

Suppose that $x(s)$ is a solution of Eq. (5) on $[0, 1]$ (its existence is from Step 1), consider the following revised initial value problem.

$$\begin{cases} \dot{\bar{x}}_i(s) = \dfrac{q_i}{\int_0^1 [1/\partial_i C(x(t))]\,\mathrm{d}t} \dfrac{1}{\partial_i C(\bar{x}(s))}, & i = 1, \dots, n \\ \bar{x}(s_0) = \xi, & s_0 \geq 0. \end{cases} \tag{6}$$

It is standard that when C is twice continuously differentiable, for any given ξ, the solution to the above problem is locally unique (see Corduneanu, 1977). On the other hand, by the same argument as in Step 1, we can show that for arbitrary $M \geq 1$, the relatively simpler problem

$$\begin{cases} \dot{\bar{x}}_i(s) = \dfrac{q_i}{\int_0^1 [1/\partial_i C(x(t))]\,\mathrm{d}t} \dfrac{1}{\partial_i C(\bar{x}(s))}, & i = 1, \dots, n \\ \bar{x}(0) = 0 \end{cases} \tag{7}$$

has at least one solution $\bar{x}$ defined on $[0, M]$.

Thus, by combining the above two facts, we can deduce that Eq. (7) has a unique solution $\bar{x}$ defined on $[0, +\infty)$ and it is obvious that

$$\bar{x}(s) = x(s), \quad s \in [0, 1].$$

Step 3. Now we show the uniqueness of the solution of Eq. (5). From Steps 1 and 2, we only need to consider the uniqueness of the solution of Eq. (5) on $[0, 1]$.

Let space X and operator T be the same space and operator as defined in Step 1 ($M = 1$). It is easy to check that now the solution of Eq. (5) satisfies

$$\|x\| \leq \|q\|.$$

For any $g, h \in X$, consider Gâteaux differential of T at g as follows:

$$\begin{aligned} (T'(g)h)(s) &= \frac{\mathrm{d}}{\mathrm{d}\theta} T(g + \theta h)(s)\Big|_{\theta=0} \\ &= \left(\frac{\mathrm{d}}{\mathrm{d}\theta} \left[\frac{q_1}{\int_0^1 [1/\partial_1 C(g + \theta h)]\,\mathrm{d}t} \int_0^s \frac{1}{\partial_1 C(g + \theta h)}\,\mathrm{d}t \right]\Bigg|_{\theta=0}, \dots, \right. \\ &\quad \times \frac{\mathrm{d}}{\mathrm{d}\theta} \left[\frac{q_n}{\int_0^1 [1/\partial_n C(g + \theta h)]\,\mathrm{d}t} \int_0^1 \frac{1}{\partial_n C(g + \theta h)\,\mathrm{d}t} \right. \\ &\quad \left. \left. \times \int_0^s \frac{1}{\partial_n C(g + \theta h)}\,\mathrm{d}t \right]\Bigg|_{\theta=0} \right). \end{aligned}$$

Compute the first component only, i.e.

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}\theta}\left[\frac{q_1}{\int_0^1[1/\partial_1 C(g+\theta h)]\,\mathrm{d}t}\int_0^s \frac{1}{\partial_1 C(g+\theta h)}\,\mathrm{d}t\right]\Bigg|_{\theta=0}\\
&\quad=\left(\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{q_1}{\int_0^1[1/\partial_1 C(g+\theta h)]\,\mathrm{d}t}\right)\right]\int_0^s\left[\frac{1}{\partial_1 C(g+\theta h)}\,\mathrm{d}t\right]\right.\\
&\qquad\left.+\frac{q_1}{\int_0^1[1/\partial_1 C(g+\theta h)]\,\mathrm{d}t}\left[\frac{\mathrm{d}}{\mathrm{d}\theta}\int_0^s\frac{1}{\partial_1 C(g+\theta h)}\,\mathrm{d}t\right]\right)\Bigg|_{\theta=0}\\
&\quad=\frac{q_1\int_0^1[\sum_{j=1}^n \partial_{1j}^2 C(g)h_j/[\partial_1 C(g)]^2]\,\mathrm{d}t}{[\int_0^1(1/\partial_1 C(g)\,\mathrm{d}t]^2}\int_0^s\frac{1}{\partial_1 C(g)}\,\mathrm{d}t=\\
&\qquad-\frac{q_1}{\int_0^1[1/\partial_1 C(g)\,\mathrm{d}t]}\int_0^s\frac{\sum_{j=1}^n\partial_{1j}^2 C(g)h_j}{(\partial_1 C(g))^2}\,\mathrm{d}t.
\end{aligned}
$$

By definition

$$
\|T'(g)h\|=\max_{1\le i\le n}\max_{0\le s\le 1}|(T'(g)h)_i(s)|.
$$

And for each $i=1,\dots,n$

$$
\begin{aligned}
|(T'(g)h)_i(s)|&\le\frac{q_i|\int_0^1[\sum_{j=1}^n\partial_{ij}^2 C(g)h_j/[\partial_i C(g)]^2]\,\mathrm{d}t}{[\int_0^1(1/\partial_i C(g))\,\mathrm{d}t]^2}\int_0^s\frac{1}{\partial_i C(g)}\,\mathrm{d}t\\
&\quad+\frac{q_i}{\int_0^1[1/\partial_i C(g)]\,\mathrm{d}t}\left|\int_0^s\frac{\sum_{j=1}^n\partial_{ij}^2 C(g)h_j}{(\partial_i C(g))^2}\,\mathrm{d}t\right|\\
&\le 2q_i\frac{b(C)}{a^2(C)}\sup_{t\in R^n}\left|\sum_{j=1}^n\partial_{ij}^2 C(t)\right|\|h\|\quad(s\le 1).
\end{aligned}
$$

Since

$$
\sup_{t\in R^n}\left|\sum_{j=1}^n\partial_{ij}^2 C(t)\right|\le d(C),\qquad i=1,\dots,n,
$$

and

$$
d(C)<\frac{1}{2\|q\|}\frac{a^2(C)}{b(C)}.
$$

Therefore,

$$
\gamma:=2\|q\|\frac{b(C)}{a^2(C)}d(C)<1,
$$

then for each $i = 1, \dots, n$

$$2q_i \frac{b(C)}{a^2(C)} \sup_{p \in R^n} \left| \sum_{j=1}^{n} \partial_{ij}^2 C(p) \right| \|h\| \le 2\|q\| \frac{b(C)}{a^2(C)} \sup_{p \in R^n} \left| \sum_{j=1}^{n} \partial_{ij}^2 C(p) \right| \|h\| \le \gamma \|h\|.$$

So,

$$\|T'(g)h\| \le \gamma \|h\|,$$

i.e.

$$\|T'(g)\| \le \gamma, \quad \forall g \in X.$$

Since T is also Fréchet-differentiable, by the mean-value theorem on Banach space we have

$$\|Tg_1 - Tg_2\| \le \sup_{0 \le t \le 1} \|T'(tg_1 + (1-t)g_2)\| \|g_1 - g_2\| \le \gamma \|g_1 - g_2\|,$$

hence, T is a contraction.

Now consider the closed ball $\bar{B}(0, \|q\|) \subset X$. Since $T\bar{B}(0, \|q\|) \subseteq \bar{B}(0, \|q\|)$ and T is a contraction, by the contraction mapping theorem it has a unique fixed-point $x \in \bar{B}(0, \|q\|)$. The unique fixed-point x is the unique solution of Eq. (5) on $[0, 1]$ and by Step 2, x can be uniquely extended to $[0, +\infty)$. Finally, check that each $x_i(s)$, $i = 1, \dots, n$ is a strictly increasing function. This is obvious since $\mathrm{d}x_i(s)/\mathrm{d}s > 0$, $i = 1, \dots, n$. The theorem is proved. □

The next question is: Are these conditions also "necessary"? Unfortunately, the answer is no. The following example demonstrates that a problem may have a proportional normalization but not satisfy the conditions required in the theorem.

Example 2. Let $N = \{1, 2\}$, $q = (1, 1)$.

$$C(t_1, t_2) = t_1^2 + t_2^3, \quad 0 \le t_1, \ t_2 < +\infty.$$

Let

$$t_1 = f_1(t_1') = \sqrt{t_1'}, \qquad t_2 = f_2(t_2') = \sqrt[3]{t_2'}.$$

Then, the proportional normalization is

$$C^*(q_1, q_2) = q_1 + q_2,$$

with

$$q^* = (1, 1), \qquad f = (f_1, f_2).$$

Note that the first-order partial derivatives of C are not bounded away from zero and infinity.

Knowing that not all problems can be proportionally normalized, as shown in Example 1, we ask: Are the problems that can be uniquely proportionally normalized "dense"[3] in the

[3] The word *dense* is referred to the standard topology on the space of cost functions. For simplicity, we fix the demand vector. See Section 6 for the detail.

set of all problems? In other words, for any given problem, is there another problem in the "neighborhood" of the given problem that can be uniquely *proportionally normalized*? The following example suggests a positive answer.

Example 3 (Contrary of Example 1). Let $N = \{1, 2, 3\}$, $q = (1, 1, 1)$, and

$$C(t_1, t_2, t_3) = t_1 + t_2, \quad (t_1, t_2, t_3) \in R_+^N.$$

We have known that the problem $(q; C)$ cannot be proportionally normalized. Now we consider an approximation of $(q; C)$ by $(q'; \tilde{C})$ where $\epsilon > 0$, $q' = (1, 1, 1)$, and

$$\tilde{C}(t_1, t_2, t_3) = t_1 + t_2 + \epsilon t_3, \quad (t_1, t_2, t_3) \in R_+^N.$$

Clearly, $(q'; \tilde{C})$ can be proportionally normalized to $((1, 1, \epsilon); C^f)$, where

$$f_1(t_1) = t_1,\ f_2(t_2) = t_2, \qquad f_3(t_3) = \frac{1}{\epsilon} t_3,$$

and

$$C^f(t_1', t_2', t_3') = t_1' + t_2' + t_3'.$$

In Section 6, we propose a conjecture that any problem has an "approximation" that can be proportionally normalized. We also show that the conjecture is equivalent to the feasibility problem of a system of differential inequalities. However, we do not pursue this question further since it is beyond the scope of this paper.

The next example shows that Theorem 1 indeed identifies a nontrivial family of problems that can be proportionally normalized.

Example 4. For any given $q \gg 0$,[4] consider the cost function

$$C(q) = \sum_{i \in N} \lambda_i q_i,$$

where $\lambda_i > 0$, $i = 1, \ldots, n$.

Let $a(C) = \min_{i \in N} \lambda_i$ and $b(C) = \max_{i \in N} \lambda_i$, and

$$d(C) = \frac{1}{2} \frac{1}{\|q\|} \frac{a^2(C)}{b(C)}.$$

Then,

$$a(C) \le \partial_i C(t) \le b(C), \quad t \in R_+^N, \ \ i = 1, \ldots, n,$$

and

$$\sum_{j \in N} |\partial_{ij}^2 C(t)| \le d(C), \quad t \in R_+^N, \ \ i = 1, \ldots, n.$$

Therefore, $(q; C)$ can be uniquely proportionally normalized.

[4] If $q_i = 0$, $i \in N$, define $x_i(q; C) = 0$ and replace N by $A(q) = \{j \in N | q_j > 0\}$.

Obviously, for the problem given previously, the ordinal transformation that proportionally normalizes it is

$$f_i(t_i) = \frac{1}{\lambda_i} t_i, \quad i = 1, \dots, n,$$

and

$$q^* = (q_1^*, \dots, q_n^*) = (\lambda_1 q_1, \dots, \lambda_n q_n).$$

Therefore,

$$x_i(q; C) = \lambda_i q_i, \quad i = 1, \dots, n.$$

Finally, we check that the OPM given in Definition 1 indeed satisfies ordinality and ACPH.

For ordinality, it is enough to check that any two problems that are ordinally equivalent to each other must have the same proportionally normalized problem (if either has one). This is easily seen from the following diagram.

$$\begin{array}{ccc} (q; C) & \overset{f^* \text{unique}}{\Leftrightarrow} & (q^*; C^*) \\ f \Updownarrow f^{-1} & & f \circ f'^* \Updownarrow f^{-1} \circ f^* \\ (q'; C') & \overset{f'^* \text{unique}}{\Leftrightarrow} & (q'^*; C'^*) \end{array}$$

which implies $(q^*; C^*) = (q'^*; C'^*)$.

Now we check ACPH. Consider a homogeneous problem $(q; C)$ and assume that it is proportionally normalized to problem $(q^*; C^*)$. Since

$$q_i^* = \frac{q_i}{\int_0^1 [1/\partial_i C(x(t)]\, \mathrm{d}t}, \quad i = 1, \dots, n,$$

(see Eq. (5)) and

$$\partial_i C(t) = \partial_j C(t), \quad t \in R_+^N, \quad i, j = 1, \dots, n,$$

therefore,

$$x_i(q; C) = \frac{q_i^*}{\sum_{i \in N} q_j^*} C^*(q^*) = \frac{q_i}{\sum_{i \in N} q_j} C(q), \quad i = 1, \dots, n.$$

Remark. In the two-agent case, Sprumont (1998) does not use the boundary condition on the second-order derivatives in Theorem 1. But he does assume that the first-order derivatives are bounded away from zero and infinity, which is necessary[5] to *guarantee* a proportional normalization. In fact, he provides an entirely different but much simpler proof for the unique existence of a proportional normalization for any given problem satisfying the boundary conditions for the first-order derivatives. However, the technique in Sprumont (1998) is not applicable in the general case here. See Sprumont (1998) for the detail.

[5] But it is not a necessary condition for a problem to have a proportional normalization, as shown in Example 2.

5. Ordinal prices

A price mechanism $p(\cdot, \cdot)$ is a rule that associates with each problem $(q; C)$ a vector of prices:

$$p(q; C) = (p_1(q; C), \dots, p_n(q; C)).$$

The A–S pricing method is a price mechanism, and so is the PAMP (see Section 3). But these two mechanisms are not ordinal, as we have shown in Section 3.

The OPM is an ordinal price mechanism.

In fact, for any given problem $(q; C)$, assume that $(q; C)$ is proportionally normalized to $(q^*; C^*)$ through ordinal transformation f. Let

$$x(s) = (x_1(s), \dots, x_n(s)) = (f_1(sq_1^*), \dots, f_n(sq_n^*)), s \in [0, 1].$$

Define

$$p^*(q; C) = (p_1^*(q; C), \dots, p_n^*(q; C)),$$

where

$$p_i^*(q; C) = \frac{1}{\int_0^1 [1/\partial_i C(x(t))]\,\mathrm{d}t}, \quad i = 1, \dots, n.$$

Then

$$\sum_1^n p_i^*(q; C)q_i = C(q).$$

Actually,

$$\begin{aligned} C(q) &= \int_0^1 \sum_1^n \partial_i C(x(s))\dot{x}_i(s)\,\mathrm{d}s = \sum_1^n q^* \quad \text{(from Eq.(3))} \\ &= \sum_1^n q_i \left[\int_0^1 \frac{1}{\partial_i C(x(t))}\,\mathrm{d}t\right]^{-1} \quad \text{(from Eq.(4))} = \sum_1^n p_i^*(q; C)q_i. \end{aligned}$$

Clearly, the price vector

$$p^*(q; C) = (p_1^*(q; C), \dots, p_n^*(q; C)),$$

is ordinal, namely for each $i = 1, \dots, n$, $p_i^*(q; C)q_i$ is invariant with any increasing transformations of the measurement units of the goods.

6. Discussion

We conjecture that the problems that can be *proportionally normalized*are dense in $\mathcal{C}_1$. Meanwhile, we raise a general question about the existence of solutions or the feasibility of a system of *differential inequalities* that relates to the conjecture.

For convenience, assume that all demand vectors are bounded by $M > 0$, namely $q_i \le M$, $i = 1, \dots, n$. Consider a given problem $(q; C)$ with the constants $0 < a(C) < b(C)$ such that

$$a(C) \le \partial_i C(t) \le b(C), \quad t \in R_+^N, \quad i = 1, \dots, n.$$

If $(q; C)$ can be uniquely proportionally normalized to $(q^*; C^*)$, then it must have

$$|q_i^*| = \frac{q_i}{\int_0^1 [1/\partial_i C(x(t))]\,\mathrm{d}t} \le b(C)M, \quad i = 1, \dots, n$$

where $x(s)$ is the function in Eq. (5) that corresponds to the ordinal transformation f. Therefore, q^* is also bounded (the bound may depend on C).

Note that we do not have the so-called *independence of irrelevant costs* (IIC) property as we do in the case of additive methods, where IIC is a corollary of additivity axiom (see Lemma 1 in Friedman and Moulin, 1999). This implies that for a given problem $(q; C)$, where the demand vector q is bounded by $M > 0$, in its proportionally normalized problem $(q^*; C^*)$ (if there is), the demand vector q^* is also bounded but may be well beyond the previous bound M. That is why we define the proportional normalization condition Eq. (2) on the domain R_+^N.

Conjecture. For any given problem $(q; C)$ and $\epsilon > 0$, where $\|q\| \le M$ and the constants $0 < a(C) < b(C)$ satisfy

$$\epsilon < a(C) \le \partial_i C(t) \le b(C), \quad t \in R_+^N, \quad i = 1, \dots, n,$$

there exists another cost function $\tilde{C}$ on R_+^N such that

1. the function $\tilde{C}$ is in the ϵ-neighborhood of C with respect to the norm:

$$\|C\| = \max_{t \in [0, M\boldsymbol{e}]} C(t) + \max_{t \in [0, M\boldsymbol{e}]} \max_{i \in N} |\partial_i C(t)|,$$

where $\boldsymbol{e} = (1 \dots 1)$, i.e.

$$|\tilde{C}(t) - C(t)| \le \epsilon, \quad t \in [0, M\boldsymbol{e}]$$

$$|\partial_i \tilde{C}(t) - \partial_i C(t)| < \epsilon, \quad t \in [0, M\boldsymbol{e}], \quad i = 1, \dots, n,$$

and outside $[0, M\boldsymbol{e}]$

$$a(C) - \epsilon \le \partial_i \tilde{C}(t) \le b(C) + \epsilon, \quad t \in R_+^N, \quad i = 1, \dots, n,$$

2. the second-order derivatives of $\tilde{C}$ satisfy

$$\sum_{j \in N} |\partial_{ij} \tilde{C}(t)| \le \frac{1}{2} \frac{a^2(\tilde{C})}{2\|M\boldsymbol{e}\| b(\tilde{C})}, \quad t \in R_+^N, \quad i = 1, \dots, n,$$

where $a(\tilde{C}) = a(C) - \epsilon$ and $b(\tilde{C}) = b(C) + \epsilon$.

In other words, the problem $(q; \tilde{C})$ satisfies the conditions in Theorem 1 and thus can be proportionally normalized.

More generally, we ask the following question: Given $\epsilon > 0, \delta > 0$, $M > 0$ and a function $C \in \mathcal{C}_2[0, M\boldsymbol{e}]$, does the following system of differential inequalities always have a feasible solution $\tilde{C}$:

$$-\epsilon < \tilde{C}(t) - C(t) < \epsilon, \quad t \in R_+^N,$$

$$-\epsilon < \partial_i \tilde{C}(t) - \partial_i C(t) < \epsilon, \quad t \in R_+^N, \ i = 1, \ldots, n$$

and

$$\sum_{j \in N} |\partial_{ij} \tilde{C}(t)| \le \delta, \quad t \in R_+^N, \ i = 1, \ldots, n.$$

We do not know the answer yet. Traditionally, to “smoothly” approximate a given function with two variables, one sometimes uses Bezier[6] surfaces and B-spline surfaces (Gerald and Wheatley, 1999). However, the construction of such an approximation is very complex (Gerald and Wheatley, 1999). For the function with three or more variables as in our case, we do not know how to generalize the Bezier (and B-spline) curves or surfaces.

If we can show that the set of all problems that can be proportionally normalized is a dense set in the set of all problems, under certain continuity conditions we may extend the OPM to any problem. Again, this is beyond the scope of this paper.

In conclusion, we show that the ordinality and the ACPH axioms are compatible in the realm of *nonadditive* methods for a fairly rich family of interesting problems with arbitrary finite number of agents.

Acknowledgements

Wang thanks Hervé Moulin, Ahmet Alkan, Albert Erkip, and a referee for comments and suggestions. Wang also gratefully acknowledges the financial support from the Sabanci University Research Fund.

References

Billera, L., Heath, D., Raanan, J., 1978. Internal telephone billing rates: a novel application of nonatomic game theory. Operations Research 26, 956–965.

Billera, L., Heath, D., 1982. Allocation of shared costs: a set of axioms yielding a unique procedure. Mathematics of Operations Research 7, 32–39.

Corduneanu, C., 1977. Principles of Differential and Integral Equations. Chelsea, New York.

Friedman, E., Moulin, H., 1999. Three additive methods to share joint costs or surplus. Journal of Economic Theory 87, 275–312.

[6] Bezier (B-spline) curves and surfaces are named after the French engineer, P. Bezier. These curves or surfaces are smoothly constructed so that they *approximate* and *stay within* the polygon determined by the given points. Note that the Bezier curves (or surfaces) and B-spline curves (or surfaces) may not necessarily pass through the given points.

Gerald, C.F., Wheatley, P.O., 1999. Applied Numerical Analysis. Addison-Wesley, Reading, MA.

Haimanko, O., 1998. Partially symmetric values, mimeo. Hebrew University, Jerusalem.

Kantorovich, L.V., Akilov, G.P. 1964. Functional Analysis in Normed Spaces. Pergamon Press, Oxford.

Koster, M., Tijs, S., Borm, P., 1998. Serial cost sharing methods for multi-commodity situations. Mathematical Social Sciences 36, 229–242.

Mirman, L., Tauman, Y., 1982. Demand compatible equitable cost sharing prices. Mathematics of Operations Research 7, 40–56.

Moulin, H., 1995. On additive methods to share joint costs. Japanese Economic Review 46, 303–332.

Moulin, H., 1999. Axiomatic cost and surplus sharing. In: Arrow, Sen, Suzumura (Eds.), Handbook of Social Choice and Welfare.

Moulin, H., Shenker, S., 1992. Serial cost sharing. Econometrica 60, 1009–1037.

Samet, D., Tauman, Y., 1982. The determination of marginal cost prices under a set of axioms. Econometrica 50, 895–909.

Shapley, L.S., 1953. A value for n-person games. In: Kuhn, H.W., Tucker, A.W. (Eds.), Contributions to the Theory of Games II. Annals of Mathematics Studies, 307–317.

Sprumont, Y., 1998. Ordinal cost sharing. Journal of Economic Theory 81, 126–162.

Tauman, Y., 1988. The Aumann–Shapley prices: a survey. In: Roth, A. (Ed.), The Shapley Value. Cambridge University Press, Cambridge.

Wang, Y.-T., 1999. The additivity and dummy axioms in the discrete cost sharing model. Economics Letters 64, 187–192.