MONGE-AMPÈRE MEASURES AND POLETSKY-STESSIN HARDY SPACES ON BOUNDED HYPERCONVEX DOMAINS

by
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Abstract

In this thesis study, we consider Poletsky-Stessin Hardy (PS-Hardy) spaces that are generated by continuous, plurisubharmonic exhaustion functions on hyperconvex domains.

In the first part of this study we examine these spaces on domains in the complex plane that are bounded by an analytic Jordan curve. In this setting we focus on PS-Hardy spaces generated by exhaustion functions that have finite Monge-Ampre mass but are not necessarily maximal outside of a compact set. This choice gives us new Banach spaces strictly contained in classical Hardy spaces. We characterize PS-Hardy spaces through their boundary values and we show factorization results analogous to unit disc case. Using functional analysis techniques we prove that the algebra of holomorphic functions which are continuous on the boundary are dense in PS-Hardy spaces. Moreover, we consider the composition operators with holomorphic symbols acting on PS-Hardy spaces and show that contrary to classical case, not all composition operators are bounded on PS-Hardy spaces.

In the second part, we study PS-Hardy spaces on polydisc, complex ellipsoid and on strongly convex domains. On complex ellipsoid case, we prove the existence of radial boundary values and then by applying a classical method given by Stein we show the existence of boundary values along admissible approach regions. As an application of this method , we also obtain that polynomials are dense in PS-Hardy spaces on complex ellipsoids. Lastly, we examine the boundedness of composition operators on PS-Hardy spaces on hyperconvex domains in several variables.

Monge-Ampère Ölçümleri ve Sınırlı Hiperkonveks Bölgelerde
Poletsky-Stessin Hardy Uzayları

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Özet

Bu tez çalışmasında, hiperkonveks bölgelerde sürekli ve çoklualtharmonik tükeniş fonksiyonlarınca üretilmiş Poletsky-Stessin Hardy (PS-Hardy) uzayları ele alınmıştır.
Çalışmanın ilk kısmında bu uzaylar karmaşık düzlemde analitik bir Jordan eğrisi ile sınırlanmış bölgelerde incelenmiştir. Bu bağlamda sınırlı Monge-Ampère ağırlığı olan ancak bir kompakt küme dışında maksimal olması gerekmeyen tükeniş fonksiyonları tarafından üretilmiş PS-Hardy uzaylarına odaklanılmıştır. Bu seçim klasik Hardy uzaylarının içinde yeni Banach uzayları vermektedir. PS-Hardy uzayları sınır değerleri üzerinden karakterize edilip, birim disk durumuna benzer şekilde çarpanlara ayırma sonuçları elde edilmiştir. Fonksiyonel analiz teknikleri kullanılarak sınırda sürekli holomorf fonksiyonlar cebirinin PS-Hardy uzayları içerisinde yoğun olduğu kanıtlanmıştır. Buna ek olarak PS-Hardy uzayları üzerinde tanımlı holomorf sembollü bileşke operatörleri incelenmiş ve klasik durumdan farklı olarak PS-Hardy uzayları üzerinde tüm bileşke operatörlerinin sınırlı olmadığı görülmüştür.

İkinci kısımda PS-Hardy uzayları polidisk, karmaşık elipsoit ve mutlak konveks bölgelerde çalışılmıştır. Karmaşık elipsoit durumunda ışınsal sınır değerlerinin varlığı kanıtlanmış ve sonrasında Stein'a ait klasik bir metot kullanılarak sınır değerlerinin makbul yaklaşım bölgelerindeki varlığı gösterilmiştir. Bu metodun bir uygulaması olarak polinomların elipsoit üzerindeki PS-Hardy uzaylarında yoğun olduğu gösterilmiştir. Son olarak çok değişkende hiperkonveks bölgelerde PS-Hardy uzayları üzerindeki bileşke operatörleri incelenmiştir.

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Contents

Introduction

The theory of Hardy Spaces has its origins in the works of G.H.Hardy and J.E.Littlewood in 1920's. The theory was improved and widened by the discoveries of I.I.Privalov, F.and M.Riesz, V.Smirnov and G.Szegö. Most of the initial work was on the unit disc of $\mathbb{C}$, and later the theory was generalized to other classes of domains such as simply connected domains in $\mathbb{C}$, Smirnov domains and multiply connected domains in $\mathbb{C}$ ([17]). Further generalizations are then given in the polydisc ([39]), ball of $\mathbb{C}^n$ ([40]) and strictly pseudoconvex domains with $\mathcal{C}^2$ boundaries ([44]). The unit disc $\mathbb{D} \subset \mathbb{C}$ has two natural generalizations, namely polydisc and unit ball, and one can define Hardy classes by means of integral growth or by means of harmonic majorants on these two. However these two definitions do not coincide even for these classical domains because in the case of unit ball having an $\mathscr{M}$-harmonic majorant is sufficient (a function is $\mathscr{M}$-harmonic iff $\widetilde{\triangle} f = 0$ where $\widetilde{\triangle} f(a) = \triangle(f \circ \varphi_a)(0)$ and φ_a is the automorphism of the ball that interchanges the point a with 0, [40],pg:47) for being in Hardy class $H^p(\mathbb{B})$ of holomorphic functions but in the case of polydisc having a harmonic n-majorant is needed ([39],pg:16).

In 2008, Poletsky and Stessin introduced Poletsky-Stessin Hardy spaces, $H^p_u(\Omega)$, on hyperconvex domains ([34]) to get a “meaningful and uniform theory of H^p spaces” which unifies the different definitions made for various domains. In this setting Ω is a hyperconvex domain and u is a continuous, negative, plurisubharmonic exhaustion function for Ω which has finite Monge-

Ampère mass and the growth condition is constructed using integrals with Monge-Ampère measures defined in ([13]). The general framework of ([34]) is based on the examination of the Poletsky-Stessin Hardy spaces when the exhaustion function u belongs to a special class $\mathcal{E}_0$, i.e the measure $(dd^c u)^n$ has compact support. In this thesis we will consider the Poletsky-Stessin Hardy classes in a broader perspective where most of the work will be done using exhaustion functions belong to a wider class, i.e. the exhaustion functions that have finite Monge-Ampère mass but not necessarily maximal outside of a compact set. Now let us mention the structure of this thesis:

In the first part of this study we will examine the Poletsky-Stessin Hardy spaces in a setting where Ω is a domain in $\mathbb{C}$ containing 0, bounded by an analytic Jordan curve and u is a continuous subharmonic exhaustion function for Ω which has finite Monge-Ampère mass but different from Poletsky & Stessin's work the exhaustion u is not necessarily harmonic outside of a compact set. One of the main consequences of this choice of exhaustion function is that the Poletsky-Stessin Hardy spaces and the classical ones do not always coincide so we have new Banach spaces to be explored inside the classical Hardy spaces. First we will characterize Poletsky-Stessin Hardy classes $H^p_u(\Omega)$ through their boundary values and the corresponding Monge-Ampère boundary measure. This boundary value characterization will enable us to prove factorization properties analogous to classical case and next we will show the algebra $A(\Omega)$ is dense in these spaces. Finally we will examine the composition operators induced by holomorphic self maps and we will see that even on the simplest of such domains , namely the unit disc, not all composition operators are bounded contrary to the classical Hardy space case. Then we will explore the necessary conditions needed for composition operators to be bounded on Poletsky-Stessin Hardy classes.

In the second part we will focus on multidimensional case and we will examine the Poletsky-Stessin Hardy spaces on hyperconvex domains in $\mathbb{C}^n, \quad n > 1$. We start this chapter with a complete comparison between different Hardy classes in the most general setting. Then we will continue with the Poletsky-Stessin Hardy spaces on polydisc in $\mathbb{C}^n$ and we will see the immediate transitions of some important characterization results from the unit disc case. Next we will consider the Poletsky-Stessin Hardy spaces on complex ellipsoids which are the basic examples of domains of finite type in $\mathbb{C}^n$. For Poletsky-Stessin Hardy spaces on complex ellipsoids we will show the existence of boundary values and then we will obtain boundary values along admissible approach regions. In order to examine the boundary behavior on the admissible approach regions we will apply a classical method given by Stein in ([44]) however different from the classical approach we will use Cauchy-Fantappie kernel. Next we will use this general method on strongly convex domains which will provide us an alternative approach through Poletsky-Stessin Hardy spaces, to the boundary behavior of classical Hardy spaces on strictly pseudoconvex domains by localizing the procedures. Lastly we will consider the boundedness of composition operators acting on Poletsky-Stessin Hardy spaces on hyperconvex domains in $\mathbb{C}^n$.

Chapter 1

Preliminaries

In this chapter, we will give the preliminary definitions and some important results that we will use throughout this study.

1.1 Differential Forms and Currents

Let Ω be a domain in $\mathbb{C}^n$ and let $C_0^\infty(\Omega)$ be the space of all smooth functions on Ω with compact supports. A sequence $\{\varphi_j\} \subset C_0^\infty(\Omega)$ converges to 0 if the supports of all φ_j belong to a compact $K \subset \Omega$ and the functions φ_j with all derivatives converge uniformly to 0.
We denote by $\mathcal{D}^{p,q}(\Omega)$ the space of all differential forms

$$\omega = \sum_{|I|=p,|J|=q} \omega_{IJ} dz_I d\overline{z}_J$$

of bidegree (p,q) where $I = (i_1, ..., i_p)$ and $J = (j_1, ..., j_q)$ are subsets of $\{1, 2, ...n\}$, $dz_I = dz_{i_1}...dz_{i_p}$, $d\overline{z}_J = d\overline{z}_{j_1}...d\overline{z}_{j_q}$ and $\omega_{IJ} \in C_0^\infty(\Omega)$. Equipped with the topology of uniform convergence on compacta with all derivatives, $\mathcal{D}^{p,q}(\Omega)$ has a structure of a linear topological space.
The space $\mathcal{D}'_{p,q}(\Omega)$ of continuous linear functionals on $\mathcal{D}^{p,q}(\Omega)$ is called the space of currents of bidimension (p,q) or of bidegree $(n-p, n-q)$. If $\phi \in$

$\mathcal{D}'_{p,q}(\Omega)$ then

$$\phi = \sum_{|I|=n-p,|J|=n-q} \phi_{IJ} dz_I d\overline{z}_J$$

where ϕ_{IJ} are distributions and the pairing $\langle \phi, \omega \rangle$ is given by

$$\langle \phi, \omega \rangle = \sum_{|I|=n-p,|J|=n-q} \langle \phi_{IJ}, \omega_{IJ} \rangle$$

A current $\phi \in \mathcal{D}'_{p,q}(\Omega)$is positive if $\langle \phi, \omega \rangle \geq 0$ for every test form $\omega = i\omega_1 \wedge \overline{\omega_1} \wedge ...i\omega_p \wedge \overline{\omega_p}$, $\omega_j \in \mathcal{D}^{1,0}(\Omega)$. In this case the coefficients ϕ_{IJ} are positive measures.
The differential of ω is defined by $d\omega = \partial\omega + \overline{\partial}\omega$ where

$$\partial\omega = \sum \frac{\partial \omega_{IJ}}{\partial z_k} dz_k \wedge dz_I \wedge d\overline{z}_J$$

$$\overline{\partial}\omega = \sum \frac{\partial \omega_{IJ}}{\partial \overline{z}_k} d\overline{z}_k \wedge dz_I \wedge d\overline{z}_J$$

The operator d^c is defined by $d^c = i(\overline{\partial} - \partial)$. For $\phi \in C^2(\Omega)$ we have

$$dd^c\phi = 2i \sum \frac{\partial^2 \phi}{\partial z_i \partial \overline{z}_j} dz_i \wedge d\overline{z}_j$$

Given a current T, we define dT by the formula $\langle dT, \omega \rangle = \langle T, d\omega \rangle$ and dd^cT by the formula $\langle dd^cT, \omega \rangle = \langle T, dd^c\omega \rangle$. A current is closed if $dT = 0$. The following result and a detailed treatment of differential forms and currents can be found in ([15]):

Proposition 1.1.1. *Every plurisubharmonic function generates a closed, positive* $(1,1)$*-current.*

1.2 Hyperconvex Domains and Maximal Plurisubharmonic Functions

Definition 1. A connected open subset Ω of $\mathbb{C}^n$ is called *hyperconvex* if there exists a plurisubharmonic function $g : \Omega \to [-\infty, 0)$ such that $\{z \in \Omega : g(z) < c\}$ is relatively compact for each $c < 0$. Here g is called a *an exhaustion* for Ω.
In $\mathbb{C}$, hyperconvex domains are in fact the regular domains for the Dirichlet problem.([37],pg:88).

Definition 2. A connected open subset Ω of $\mathbb{C}^n$ is called *strictly pseudoconvex* if there exists a smooth, strictly plurisubharmonic defining function for Ω. i.e. There exists a smooth defining function ρ such that

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial z_j \partial \overline{z_k}}(P) w_j \overline{w_k} > 0 \tag{1.2.1}$$

is positive definite for all $P \in \partial\Omega$ and for all $w \in T_P(\partial\Omega)$ where $T_P(\partial\Omega)$ is the tangent space at P. The form given in (1.2.1) is called the Levi form of the domain Ω.

Definition 3. Let Ω be an open subset of $\mathbb{C}^n$ and let $u : \Omega \to \mathbb{R}$ be a plurisubharmonic function. Then u is *maximal* if for every relatively compact open subset G of Ω, and for each upper semicontinuous function v on $\overline{G}$ such that $v \in PSH(G)$ and $v \le u$ on ∂G, we have $v \le u$ in G.

For the maximal plurisubharmonic functions we also have the following characterization by ([25],pg:93):

Theorem 1.2.1. *Let Ω be an open subset of $\mathbb{C}^n$, and let $u \in C^2 \cap PSH(\Omega)$. Then u is maximal if and only if $(dd^c u)^n = 0$ in Ω.*

Remark 1. Since $dd^c u = \frac{1}{4} \triangle u dz d\overline{z}$ by this theorem we see that in $\mathbb{C}$ the maximal subharmonic functions are exactly the harmonic functions.

Definition 4. Pluricomplex Green function of $\Omega \subset \mathbb{C}^n$ is defined as:

$$g_\Omega(z,w) = \sup u(z)$$

where $u \in PSH(\Omega)$ (including $u \equiv -\infty$), u is non-positive and the function $t \to u(t) - \log|t-w|$ is bounded from above in a neighborhood of w. Pluricomplex Green function $g_\Omega(z,w)$ is a negative plurisubharmonic function with a logarithmic pole at w ([25], pg:222).

When Ω is hyperconvex, $g_\Omega(z,w)$ is a continuous function ([14]) and by the previous theorem $g_\Omega(z,w)$ is maximal in $\Omega \setminus \{w\}$.
An important tool that we will use throughout this study is the following comparison principle for the continuous, plurisubharmonic, exhaustion functions on a hyperconvex domain $\Omega \in \mathbb{C}^n$ ([14]):

Theorem 1.2.2. *Let $\varphi, \psi : \Omega \to [-\infty, 0)$ be continuous, plurisubharmonic, exhaustion functions such that $\varphi \leq \psi \leq 0$ and $\int_\Omega (dd^c\varphi)^n < \infty$. Then*

$$\int_\Omega (dd^c\psi)^n \leq \int_\Omega (dd^c\varphi)^n$$

1.3 Classical Hardy and Hardy-Smirnov Spaces

Definition 5. Hardy Spaces on the unit disc are defined for $1 \leq p \leq \infty$ as [38] :

$$H^p(\mathbb{D}) = \{f \in \mathcal{O}(\mathbb{D}) : \sup_{0<r<1} (\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{\frac{1}{p}} < \infty\} \tag{1.3.1}$$

and

$$H^\infty(\mathbb{D}) = \{f \in \mathcal{O}(\mathbb{D}) : \sup_{z\in\mathbb{D}} |f(z)| < \infty\} \tag{1.3.2}$$

For $1 \le p < \infty$ we equip $H^p(\mathbb{D})$ spaces with the following norms :

$$\|f\|_p = \sup_{0<r<1} (\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta)^{\frac{1}{p}}$$

and for $H^\infty(\mathbb{D})$

$$\|f\|_\infty = \sup_{z \in \mathbb{D}} |f(z)|$$

An important tool in the study of $H^p(\mathbb{D})$ spaces and its applications is the factorization of the holomorphic functions in these classes. For the class $H^p(\mathbb{D})$ we have the following canonical factorization theorem ([17],pg:24):

Theorem 1.3.1. *Every function $f \not\equiv 0$ of class $H^p(\mathbb{D})$ has a unique factorization of the form $f = BSF$ where B is a Blaschke product, S is a singular inner function and F is an outer function for the class $H^p(\mathbb{D})$.*

Definition 6. Hardy spaces on the unit ball of $\mathbb{C}^n$ are defined for $1 \le p \le \infty$ as [40] :

$$H^p(\mathbb{B}) = \{f \in \mathcal{O}(\mathbb{B}) : \sup_{0<r<1} \int_{S(r)} |f(z)|^p d\mu < \infty\}$$

where $S(r)$ is the sphere with center 0 and radius r and μ is the usual surface area measure on the sphere. As usual we define

$$H^\infty(\mathbb{B}) = \{f \in \mathcal{O}(\mathbb{B}) : \sup_{z \in \mathbb{B}} |f(z)| < \infty\}$$

Definition 7. Hardy spaces on the unit polydisc of $\mathbb{C}^n$ are defined for $1 \le p \le \infty$ as [39] :

$$H^p(\mathbb{D}^n) = \{f \in \mathcal{O}(\mathbb{D}^n) : \sup_{0<r<1} (\frac{1}{(2\pi)^n} \int_{\mathbb{T}^n} |f(rz)|^p d\mu)^{\frac{1}{p}} < \infty\}$$

where $\mathbb{T}^n$ is torus and μ is the usual product measure on the torus. And

$$H^\infty(\mathbb{D}^n) = \{f \in \mathcal{O}(\mathbb{D}^n) : \sup_{z \in \mathbb{D}^n} |f(z)| < \infty\}$$

Definition 8. Let Ω be a smoothly bounded domain and λ be a characterizing function for Ω which is defined in a neighborhood of $\overline{\Omega}$ i.e. λ is smooth , $\lambda(x) < 0$ iff $x \in \Omega$, $\partial\Omega = \{\lambda(x) = 0\}$ and $|\nabla\lambda(x)| > 0$ if $x \in \partial\Omega$. (The last condition is equivalent to $\frac{\partial\lambda}{\partial\nu_x} > 0$ where ν_x is the outward normal at x.). Let $\Omega_r = \{z : \lambda(z) < r : r < 0\}$ and $\partial\Omega_r = \{z : \lambda(z) = r\}$.
In ([44]), E.M. Stein defines the class H^p_λ as:

$$H^p_\lambda \doteq \{f| \quad f \quad \textit{holomorphic in} \quad \Omega, \quad \sup_{r<0} \int_{\partial\Omega_r} |f|^p d\sigma_r < \infty\}$$

where $d\sigma_r$ is the induced surface area measure on $\partial\Omega_r$. This space is equipped with the norm

$$\|f\|_p^p = \sup_{r<0} \int_{\partial\Omega_r} |f|^p d\sigma_r$$

Remark 2. The space $H^p_\lambda(\Omega)$ does not depend on the characterizing function used to define Ω and one gets equivalent norms for different characterizing functions.

Definition 9. Let Ω be a domain in $\mathbb{C}$ containing 0 and bounded by an analytic Jordan curve. The classical Hardy space $H^p(\Omega)$ is defined as follows:

$$H^p(\Omega) = \{f \in \mathcal{O}(\Omega) \quad | \quad |f|^p \textit{has a harmonic majorant in} \quad \Omega\} \tag{1.3.3}$$

Definition 10. Let Ω be a domain in $\mathbb{C}$ containing 0 and bounded by an analytic Jordan curve. The Hardy-Smirnov class $E^p(\Omega)$ is defined as follows: A holomorphic function f on Ω is said to be of class $E^p(\Omega)$ if there exists a sequence of rectifiable Jordan curves $C_1, C_2, ..$ in Ω tending to boundary in the sense that C_n eventually surrounds each compact subdomain of Ω such that

$$\int_{C_n} |f(z)|^p ds \leq M < \infty$$

Remark 3. The spaces $H^p(\mathbb{D})$, $H^p(\mathbb{D}^n)$, $H^p(\mathbb{B})$, $H^p_\lambda(\Omega)$, $H^p(\Omega)$ and $E^p(\Omega)$ are Banach spaces for $1 \leq p \leq \infty$.

An important point examined in the study of Hardy spaces is the existence of non-tangential limits which determines the boundary behavior of the given classes. About the existence of non-tangential limits we have the following results ([38], [39], [40], [44]) :

Theorem 1.3.2. *Let* $f \in H^p(\mathbb{D})$,$1 \leq p \leq \infty$ *then for almost all* θ *radial limits of* f *exists, i.e.*

$$f^*(\theta) \doteq \lim_{r\to 1} f_r(\theta) = \lim_{r\to 1} f(re^{i\theta}) \tag{1.3.4}$$

exists almost everywhere with respect to the usual Lebesgue measure on the unit circle.

Remark 4. It should also be noted that H^p classes can also be considered as the normed linear spaces where the norm is defined as the L^p norm of the boundary function, i.e. for $f \in H^p(\mathbb{D})$ we have $\|f\|_{H^p} = \|f^*\|_{L^p}$ ([17], pg:23).

Theorem 1.3.3. *Let* f *be in* $H^p(\mathbb{D}^n)$ *then* $f^*(w)$ *exists for almost all* $w \in T^n$ *with respect to the product measure on the torus.*

Theorem 1.3.4. *If* f *is in* $H^p(\mathbb{B})$, *for* $1 \leq p \leq \infty$, *then for almost all* $w \in S$, $f^*(w)$ *exists with respect to the usual surface area measure*

Theorem 1.3.5. *If* f *is in* $H^p_\lambda(\Omega)$, *for* $p \geq 1$ *the non-tangential limits*

$$f^*(y) = \lim_{\substack{x\to y \\ x\in\Gamma_\alpha(y)}} f(x)$$

exists for almost every $y \in \partial\Omega$ *where*

$$\Gamma_\alpha(y) = \{x \in \Omega : |x-y| < (1+\alpha)\delta(x), \alpha > 0\}$$

where δ *is the distance from* $\partial\Omega$

1.4 Monge-Ampère Measures and Poletsky-Stessin Hardy Spaces

Let Ω be a hyperconvex domain in $\mathbb{C}^n$ and $\varphi : \Omega \to [-\infty, 0)$ be a negative, continuous, plurisubharmonic exhaustion for Ω. Define pseudoball:

$$B_\varphi(r) = \{z \in \Omega : \varphi(z) < r\} \quad , \quad r \in [-\infty, 0),$$

and pseudosphere:

$$S_\varphi(r) = \{z \in \Omega : \varphi(z) = r\} \quad , \quad r \in [-\infty, 0),$$

and set

$$\varphi_r = \max\{\varphi, r\} \quad , \quad r \in (-\infty, 0).$$

In 1985, Demailly introduced the Monge-Ampère measures in the sense of currents as :

$$\mu_r = (dd^c \varphi_r)^n - \chi_{\Omega \setminus B_\varphi(r)} (dd^c \varphi)^n \quad r \in (-\infty, 0)$$

which is supported on $S_\varphi(r)$. We can define the mass of an exhaustion function using the mass of Monge-Ampère measure generated by it as follows:

Definition 11. The Monge-Ampère mass of an exhaustion function u on $\Omega \subset \mathbb{C}^n$ is defined as:

$$MA(u) = \int_\Omega (dd^c u)^n$$

The following result by ([34]) gives us the relation between the decay rate of exhaustion functions near the boundary of Ω and the dominating measures:

Theorem 1.4.1. *Let u and v be continuous, plurisubharmonic exhaustion functions for Ω and let F be a compact set in Ω such that $F \subset B_u(r_0)$ for* *some $r_0 < 0$ and $v(z) \le u(z)$ for all $z \in \Omega \setminus F$. Then for any $c > 1$ and any $a < 1 - c^{-1}$ we have*

$$\mu_{u,r}(\psi) \le c^n \mu_{v,ar}(\psi)$$

when $r \ge r_0$ and ψ is a nonnegative plurisubharmonic function on Ω.

Now let us mention so called Lelong-Jensen Formula which will be used in most of the results in this study as a powerful tool ([14]):

Theorem 1.4.2. *Let $r < 0$ and ϕ be a plurisubharmonic function on Ω then for any negative, continuous, plurisubharmonic exhaustion function u*

$$\int_{S_u(r)} \phi d\mu_{u,r} - \int_{B_u(r)} \phi (dd^c u)^n = \int_{B_u(r)} (r - u) dd^c \phi (dd^c u)^{n-1} \qquad (1.4.1)$$

Inspired by Demailly's work, Muhammed Ali Alan defined Hardy Spaces on hyperconvex domains in terms of Monge-Ampère measures associated with $g_\Omega(z, a)$ in his MSc thesis (2003) in the following manner:

$$H^p_a(\Omega) \doteq \{f \in \mathcal{O}(\Omega) : \sup_{r<0} \int_{S(r)} |f(z)|^p d\mu_{r,a} < \infty\} \quad , \quad 1 \le p < \infty$$

and we know that these classes $H^p_a(\Omega)$ are independent of the pole point a ([34],pg:13).

In 2008, Poletsky & Stessin introduced new Hardy type classes of holomorphic functions on hyperconvex domains in $\mathbb{C}^n$ as follows ([34]) :

$H^p_\varphi(\Omega)$, $p > 0$, is the space of all holomorphic functions f on Ω such that

$$\limsup_{r \to 0^-} \int_\Omega |f|^p d\mu_{\varphi,r} < \infty$$

The norm on these spaces is given by:

$$\|f\|_{H^p_\varphi} = \left(\lim_{r \to 0^-} \int_\Omega |f|^p d\mu_{\varphi,r} \right)^{\frac{1}{p}}$$

and with respect to these norm the spaces $H^p_\varphi(\Omega)$ are Banach spaces ([34],pg:16). Moreover on these Banach spaces point evaluations are continuous ([34], Theorem 3.6).

Now let us see the correspondence between the classical Hardy spaces and the Poletsky-Stessin Hardy classes :

In the case of the unit disc in $\mathbb{C}$ using $\varphi_1(z) = \log|z|$ as the exhaustion function we get

$$\mu_{\varphi_1,r,0} = d\theta \tag{1.4.2}$$

where $d\theta$ is the usual Lebesgue measure on the circle with radius r.

For the unit ball of $\mathbb{C}^n$ when we use $\varphi_2(z) = \log\|z\|$ as the defining function we obtain

$$\mu_{\varphi_2,r,0} = \frac{1}{\sigma(S(r))} d\sigma_r \tag{1.4.3}$$

which is the normalized surface area measure on the sphere with radius r.

Now consider the polydisc $\mathbb{D}^n \subset \mathbb{C}^n$ with $\varphi_3(z) = \log(\max|z_j|)$ as the defining function, we have

$$\mu_{\varphi_3,r,0} = \frac{1}{(2\pi)^n} d\theta_1 d\theta_2 \ldots d\theta_n \tag{1.4.4}$$

which is the usual product measure on the torus. Therefore, the classical Hardy spaces $H^p(\mathbb{D})$, $H^p(\mathbb{B})$ and $H^p(\mathbb{D}^n)$ correspond to the classes $H^p_{\varphi_1}(\mathbb{D})$, $H^p_{\varphi_2}(\mathbb{B})$ and $H^p_{\varphi_3}(\mathbb{D}^n)$ respectively.

Lastly let us give an explicit formula for the norms of holomorphic functions in the Poletsky-Stessin Hardy spaces ([34], Theorem 6.2) :

Theorem 1.4.3. *Let Ω be a hyperconvex domain in $\mathbb{C}^n$ with an exhaustion function u such that the set $L(u) = \{z \in \Omega| \quad u(z) = -\infty\}$ is finite. If f is a holomorphic function on Ω then*

$$\|f\|^p_{H^p_u(\Omega)} = \int_\Omega |f|^p (dd^c u)^n + \int_\Omega (-u) dd^c |f|^p \wedge (dd^c u)^{n-1}$$

Chapter 2

Poletsky-Stessin Hardy Spaces on Domains Bounded by An Analytic Jordan Curve in $\mathbb{C}$

2.1 Comparison Between Hardy Type Classes of Holomorphic Functions in $\mathbb{C}$

In this section we will compare the Poletsky-Stessin Hardy spaces with the classical Hardy and Hardy-Smirnov classes over a hyperconvex domain Ω in $\mathbb{C}$. Let us start with the first comparison:

Theorem 2.1.1. *Let Ω be a domain in $\mathbb{C}$ containing 0 and bounded by an analytic Jordan curve. Suppose φ is a continuous, negative, subharmonic exhaustion function for Ω such that φ is harmonic out of a compact set $K \subset \Omega$. Then for a holomorphic function $f \in \mathcal{O}(\Omega)$, $f \in H^p_\varphi(\Omega)$ if and only if $|f|^p$ has a harmonic majorant.*

Proof. Let $|f|^p$ has a harmonic majorant u on Ω. Then

$$\int_{S(r)} |f|^p d\mu_{\varphi,r} \leq \int_{S(r)} u d\mu_{\varphi,r} = \int_{B(r)} u(dd^c\varphi) \tag{2.1.1}$$

by Lelong-Jensen formula and we know that φ is harmonic outside of the compact set K so

$$\int_{B(r)} u(dd^c\varphi) \leq \int_K u(dd^c\varphi) \leq C_K \|u\|_{L^\infty(K)} \tag{2.1.2}$$

for some constant C_K and this bound is independent of r. Hence

$$\sup_{r<0} \int_{S(r)} |f|^p d\mu_{\varphi,r} \leq M < \infty \tag{2.1.3}$$

for some M

$\Rightarrow f \in H^p_\varphi(\Omega)$. For the converse, suppose $f \in H^p_\varphi(\Omega)$ and $|f|^p$ has no harmonic majorant. Then by ([37], Theorem 4.5.4) we have that

$$\frac{1}{2\pi} \int_\Omega (-g_\Omega(z,w))\Delta|f|^p = \infty \tag{2.1.4}$$

where $g_\Omega(z,w)$ is the Green function of the domain Ω ([37]). Then from Lelong-Jensen formula

$$\frac{1}{2\pi} \int_{S(r)} |f|^p d\mu_{\varphi,r} \geq \frac{1}{2\pi} \int_{B(r)} (r-\varphi)\Delta|f|^p$$

Note that left hand side is bounded independent from r since $f \in H^p_\varphi(\Omega)$. Let us take a compact set $F \subset \Omega$ containing the support of $\triangle\varphi$ and $\{w\}$ such that both φ and $g_\Omega(z,w)$ are bounded on ∂F and $bg_\Omega(z,w) \leq \varphi \leq cg_\Omega(z,w)$ holds on ∂F for some numbers $b, c > 0$. By the maximality of both φ and $g_\Omega(z,w)$ on $\Omega \setminus F$ this inequality holds on $\Omega \setminus F$. Hence near boundary we have

$$\varphi \leq c g_\Omega(z,w)$$

Then by ([34], Theorem 3.1) for a positive constant $a > 0$ we have the following

$$\frac{1}{2\pi}\int_{B(ar)}(ar - g_\Omega(z,w))\Delta|f|^p \leq \frac{1}{2\pi}\int_{S(ar)}|f|^p d\mu_{g,r} \leq \frac{1}{2\pi}\int_{S(r)}|f|^p d\mu_{\varphi,r}$$

and as $r \to 0$ by Fatou lemma we have

$$\int_{\mathbb{D}}(-g_\Omega(z,w))\Delta|f|^p \leq \lim_{r\to 0}\frac{1}{2\pi}\int_{B(ar)}(ar - g_\Omega(z,w))\Delta|f|^p$$

$$\leq \lim_{r\to 0}\frac{1}{2\pi}\int_{S(r)}|f|^p d\mu_{\varphi,r}$$

the last limit is bounded since $f \in H^p_\varphi(\Omega)$ but the first integral goes to infinity by (2.1.4). From this contradiction it follows that $|f|^p$ has a harmonic majorant. $\square$

Remark 5. We would like to give a direct proof for the previous result in our study however, alternatively one can deduce this result by combining (Lemma 3.4, [34]) and (Theorem 5.2.2, [2]). The first one gives us that the exhaustion functions which are maximal outside of a compact set generate the same Poletsky-Stessin Hardy space. Hence, all of them generates the same space that is generated by the Green function and the second one gives us that a holomorphic function f belongs to Poletsky-Stessin Hardy space generated by the Green function if and only if $|f|^p$ has a harmonic majorant. Thus, we obtain the previous result.

Remark 6. This result is not true in general for $n > 1$ if we take harmonic functions as solutions to the equation $\triangle u = 0$ in $\mathbb{C}^n$. As we have seen in the cases of the unit polydisc and the unit ball in $\mathbb{C}^n$, if we take our exhaustions to be Green functions then Poletsky-Stessin Hardy classes coincide with the classical Hardy spaces on the ball and polydisc. In the case of unit ball, having an $\mathscr{M}$-harmonic majorant is sufficient (a function is $\mathscr{M}$-harmonic iff $\widetilde{\triangle} f = 0$ where $\widetilde{\triangle} f(a) = \triangle(f \circ \varphi_a)(0)$ and φ_a is the automorphism of the ball that interchanges the point a with 0, [40],pg:47) for being in the defined class of functions, however in the case of polydisc having a harmonic n-majorant is needed ([39],pg:16).

Corollary 2.1.1. *Let Ω be a domain in $\mathbb{C}$ containing 0 and bounded by an analytic Jordan curve. Suppose $g_\Omega(z,w)$ is the Green function of Ω with the logarithmic pole at $w \in \Omega$. Then $H^p(\Omega) = H^p_{g_\Omega}(\Omega)$ for $p \geq 1$.*

Proof. The Green function $g_\Omega(z,w)$ is harmonic outside of the compact set $\{w\}$ by definition so by previous theorem we have $f \in H^p_{g_\Omega}(\Omega)$ is equivalent to the condition that $|f|^p$ has a harmonic majorant which is by definition means $f \in H^p(\Omega)$. $\square$

Now we would like to compare Poletsky-Stessin Hardy classes with the classical Hardy and Hardy-Smirnov spaces when Ω is a domain in $\mathbb{C}$ that is bounded by an analytic Jordan curve. First of all on Ω we have $E^p(\Omega) = H^p(\Omega)$ by (Theorem 10.2,[17]) and using the previous theorem we see that for any exhaustion function φ which is harmonic outside of a compact set we have $E^p(\Omega) = H^p(\Omega) = H^p_\varphi(\Omega)$. However this need **not** be the case when our exhaustion function u has finite Monge-Ampère mass yet not harmonic outside of a compact set. When an exhaustion function u has finite Monge-Ampère mass then by (Theorem 3.1, [34]) we have $H^p_u(\Omega) \subset H^p_{g_\Omega}(\Omega) = H^p(\Omega)$ where the last equality is given in the previous corollary however by explicitly constructing an exhaustion function u on the unit disc $\mathbb{D}$, we will show that $H^p_u(\Omega)$ need not be equal to $H^p(\Omega)$:

Theorem 2.1.2. *There exists an exhaustion function u with finite Monge-Ampère mass such that the Hardy space $H^p_u(\mathbb{D}) \subsetneqq H^p(\mathbb{D})$.*

Proof. (Without loss of generality assume p=1) In order to prove this result we will first construct an exhaustion function u with finite mass:

Let $\mathbb{D}$ be the unit disc in $\mathbb{C}$, and let ρ be the solution of the Dirichlet problem in the unit disc such that $\triangle\rho = 0$ in $\mathbb{D}$ and $\rho = f$ on $\partial\mathbb{D}$ where $f(z) = -(1-x)^{\frac{3}{4}}$. Then define $u = f - \rho$. We will first show that u is a continuous, subharmonic, exhaustion function for $\mathbb{D}$:
u is continuous: $u(z) = f(z) + \int_{\partial\mathbb{D}} P(z, e^{i\theta})(1-\cos\theta)^{\frac{3}{4}} d\theta$ and both parts on the right hand side are in $C^2(\mathbb{D}) \cap C(\overline{\mathbb{D}})$ hence u is a continuous function.
u is subharmonic: u is a $C^2(\mathbb{D})$ function and $\triangle u = \triangle(-(1-x)^{\frac{3}{4}}) = (1-x)^{\frac{-5}{4}} \geq 0$ hence u is subharmonic.
u is an exhaustion: For this we should show that $A_c = \{x \mid u(x) < c\}$ is relatively compact in $\mathbb{D}$ for all $c < 0$. Suppose not then there exists a sequence x_n in A_c such that it has a subsequence x_{n_k} converging to boundary of $\mathbb{D}$. Now suppose that $x_{n_k} \to x$, $|x| = 1$ and since u is continuous so $u(x_{n_k}) \to u(x)$ but $u(x_{n_k}) < c$ so $u(x) < c$. This contradicts the fact that $u = 0$ on the boundary. Hence A_c is relatively compact in $\mathbb{D}$ for all c and u is an exhaustion.
Since we have a negative, continuous, subharmonic exhaustion function in $\mathbb{D}$ we can define the Monge-Ampère measure μ_u associated with u and we will show that total mass ,$\|\mu_u\|$, of μ_u is finite :

$$\|\mu_u\| = \int_{\mathbb{D}} dd^c u = \int_{-1}^{1} \int_{-\sqrt{(1-x)(1+x)}}^{\sqrt{(1-x)(1+x)}} (1-x)^{-\frac{5}{4}} dy dx = \int_{-1}^{1} 2(1+x)^{\frac{1}{2}}(1-x)^{-\frac{3}{4}} dx$$

$$\leq 2\sqrt{2} \int_{-1}^{1} (1-x)^{-\frac{3}{4}} dx$$

say $t = 1 - x$ then

$$= 2\sqrt{2} \int_{0}^{2} \frac{1}{t^{\frac{3}{4}}} dt < \infty$$

Hence μ_u has finite mass.
We know that for any continuous, negative, subharmonic exhaustion function u the Poletsky-Stessin Hardy space $H_u^p(\mathbb{D}) \subset H^1(\mathbb{D})$ ([34],pg:13) but now we will show that the inclusion is strict by using the exhaustion function u that we constructed above. Now consider the holomorphic function $F(z) =$

$\frac{1}{(1-z)^{2q}}$ for $q < \frac{1}{2}$
First of all we want to show that $F(z) \in H^1(\mathbb{D})$, so we will show the following growth condition is satisfied:

$$\sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta}|d\theta < \infty \quad \text{for} \quad 2q < 1$$

$$|F(z)| = \frac{1}{|1-re^{i\theta}|^{2q}} = \frac{1}{(1+r^2-2Re(re^{i\theta}))^q} = \frac{1}{(1+r^2-2r\cos(\theta))^q}$$

Now,

$$\int_0^{2\pi} \frac{1}{(1+r^2-2r\cos(\theta))^q} d\theta = \int_0^{2\pi} \frac{1}{(\sin^2\theta + (\cos\theta - r)^2)^q} d\theta$$

$$\leq \int_0^{2\pi} \frac{1}{(\cos\theta - r)^{2q}} d\theta$$

and as $r \to 1$ this integral converges for $2q < 1$ and

$$\sup_{0<r<1} \int_0^{2\pi} |F(re^{i\theta}|d\theta < \infty \quad \text{for} \quad q < \frac{1}{2}$$

Now we will show that $F(z) \notin H_u^1(\mathbb{D})$:

$$|F(z)| = \frac{1}{|1-z|^{2q}} = \frac{1}{((1-x)^2+y^2)^q} = ((1-x)^2+y^2)^{-q}$$

from the Lelong-Jensen formula,

$$\int_{S(r)} |F(z)| d\mu_{u,r} = \int_{B(r)} |F(z)| \triangle u + \int_{B(r)} (r-u) \triangle |F(z)|$$

second integral on the right hand side is non-negative so

$$\int_{S(r)} |F(z)| d\mu_{u,r} \geq \int_{B(r)} |F(z)| \triangle u$$

now as $r \to 0$

$$\|F(z)\|_{H_u^1} \geq \lim_{r\to 0}\int_{B(r)} |F(z)| \triangle u \geq \int_{\mathbb{D}} |F(z)| \triangle u$$

where the last inequality is due to Fatou Lemma. On $\mathbb{D}$, $y^2 \leq 1-x^2 \leq 2(1-x)$ so

$$\|F(z)\|_{H_u^1} \geq \int_{\mathbb{D}} |F(z)| \triangle u = \int_{-1}^{1}\int_{-\sqrt{(1-x)(1+x)}}^{\sqrt{(1-x)(1+x)}} ((1-x)^2+y^2)^{-q}(1-x)^{-\frac{5}{4}} dydx$$

$$\geq \int_{-1}^{1}\int_{-\sqrt{(1-x)(1+x)}}^{\sqrt{(1-x)(1+x)}} ((1-x)^2+2(1-x))^{-q}(1-x)^{-\frac{5}{4}} dydx$$

$$\geq \int_{-1}^{1} (1-x)^{\frac{1}{2}}(1-x)^{-q}(3-x)^{-q}(1-x)^{-\frac{5}{4}} dx$$

$$\geq \int_{-1}^{1} \frac{1}{(1-x)^{q+\frac{3}{4}}} dx$$

so for $q > \frac{1}{4}$ this integral diverges. Hence for $\frac{1}{4} < q < \frac{1}{2}$ $F(z) \in H^1(\mathbb{D})$ but $F(z) \notin H_u^1(\mathbb{D})$ $\square$

Remark 7. Moreover from this result we also deduce that if u is an arbitrary exhaustion function with finite Monge-Ampère mass $H_u^p(\Omega)$'s are not always closed subspaces of $H^p(\Omega)$ because as Banach spaces, the inclusion $H_u^p(\Omega) \hookrightarrow H^p(\Omega)$ is continuous which can be deduced from Closed Graph Theorem and the fact that point evaluations are continuous ([34]). However the range is not closed because range includes all bounded functions (hence polynomials) and polynomials are dense in $H^p(\Omega)$ but $H^p(\Omega) \neq H_u^p(\Omega)$ in general.

2.2 Boundary Monge-Ampère Measure and Boundary Value Characterization of Poletsky-Stessin Hardy Spaces

2.2.1 Boundary Monge-Ampère Measures

In [14], Demailly gave the following definition for the boundary Monge-Ampère measures :

Definition 12. Let $\varphi : \Omega \to [-\infty, 0)$ be a continuous plurisubharmonic exhaustion function for Ω. Suppose that the total Monge-Ampère mass of φ is finite, i.e.

$$\int_{\Omega} (dd^c \varphi)^n < \infty \tag{2.2.1}$$

Then as r tends to 0, the measures μ_r converge to a positive measure μ weak*-ly on Ω with total mass $\int_{\Omega}(dd^c\varphi)^n$ and supported on $\partial\Omega$. This limit measure μ is called the boundary Monge-Ampère measure associated with the exhaustion φ.

In certain cases we can explicitly calculate the boundary Monge-Ampère measure :

Proposition 2.2.1. *Let Ω be a bounded, simply connected domain bounded by an analytic Jordan curve and $\mu_{g_a,r}$ be the boundary Monge-Ampère measure associated with the Green function with a logarithmic pole at $a \in \Omega$. Then $\mu_{g_a,r}$ converges to the boundary Monge-Ampère measure μ_g which is given by*

$$\mu_g = \frac{1}{4\pi} \|\nabla(|f(z)|^2)\| ds \tag{2.2.2}$$

where $\|.\|$ denotes the Euclidean norm, f is the conformal map given by Riemann mapping with $f(a) = 0$, $f'(a) > 0$ and ds is the arclength measure. Moreover, μ has finite mass and $ds \ll \mu$ (absolutely continuous) on $\partial\Omega$.

Proof. Let a be a point in Ω and f be the conformal mapping given by Riemann mapping theorem which maps Ω conformally onto the unit disk $\mathbb{D}$ such that $f(a) = 0$. Then the Green function of Ω is

$$g_\Omega(z,a) = g(z) = -\log|f(z)|$$

and let $f(x+iy) = u(x,y) + iv(x,y)$. Now Monge-Ampère measure on Ω reduces to $d^c g|_{S(r)}$ and

$$d^c(-\log|f(z)|) = \frac{1}{2\pi}[\frac{uu_y + vv_y}{u^2+v^2}dx - \frac{uu_x + vv_x}{u^2+v^2}dy]$$

but since on $S(r)$, $g(z) = r$ we have $u^2 + v^2 = e^{-r}$ and hence

$$\begin{aligned}
&\frac{1}{2\pi}[\frac{uu_y + vv_y}{u^2+v^2}dx - \frac{uu_x + vv_x}{u^2+v^2}dy] \\
=& \frac{e^{2r}}{2\pi}[\frac{1}{2}(\frac{\partial}{\partial y}(u^2+v^2)dx - \frac{\partial}{\partial x}(u^2+v^2)dy)] \\
&= \frac{e^{2r}}{4\pi}\frac{\partial}{\partial \eta}(|f(z)|^2)ds \\
&= \frac{e^{2r}}{4\pi}\|\nabla(|f(z)|^2)\|ds
\end{aligned}$$

where η is the unit outward pointing normal.So

$$\mu_{g_a,r} = \frac{e^{2r}}{4\pi}\|\nabla(|f(z)|^2)\|ds$$

Now since Ω is bounded by an analytic curve f conformally extends to the boundary ([18]) and hence as $r \to 0$, we have

$$\mu_g = \frac{1}{4\pi}\|\nabla(|f(z)|^2)\|ds$$

In order to see that ds is absolutely continuous to μ_g, it is enough to show that $\nabla|f(z)|^2 \neq 0$ on the boundary. Assume the contrary, then

$$\begin{aligned}\nabla(u^2+v^2)=0 &\Leftrightarrow (uu_x+vv_x, uu_y+vv_y)=(0,0)\\ &\Leftrightarrow uu_x=-vv_x \quad \text{and} \quad uu_y=-vv_y\end{aligned}$$

Now let $z=(x,y)$ be a point on $\partial\Omega$ then since $-\log|f(z)|=0$ on the boundary we have $|f(z)|=1$ so $u(x,y)$ and $v(x,y)$ can not be both zero.
CASE 1:($u(x,y)$ and $v(x,y)$ are both non-zero)
Combining the above equalities with the Cauchy-Riemann equations we get

$$\begin{aligned}uu_x^2=-vv_xu_x=vu_yv_y=-uu_y^2&=-uv_x^2\\ \Rightarrow u_x^2=-v_x^2 \quad \text{since } u(x,y) &\text{ is non-zero}\\ \Rightarrow u_x=v_x&=0\\ \Rightarrow f'(z)=u_x+iv_x&=0\end{aligned}$$

but this contradicts with f being conformal so there is no such case.
CASE 2:(one of $u(x,y)$ and $v(x,y)$ is zero)
Without loss of generality suppose $u(x,y)=0$ then since they cannot be both zero $v(x,y)\neq 0$

$$\begin{aligned}uu_x(x,y)=-vv_x(x,y)\Rightarrow vv_x(x,y)&=0\\ \text{but} \quad v(x,y)\neq 0\Rightarrow v_x(x,y)&=0\\ uu_y(x,y)=-vv_y(x,y)\Rightarrow vv_y(x,y)&=0\\ \text{but} \quad v(x,y)\neq 0\Rightarrow v_y(x,y)=u_x(x,y)&=0\\ \Rightarrow f'(z)=u_x+iv_x&=0\end{aligned}$$

but this contradicts with f being conformal so there is no such case either.
Hence ds is absolutely continuous to μ_g.
In order to show that μ_g has finite mass we will first show that $\|\nabla(|f(z)|^2)\|$

is bounded:

$$
\begin{aligned}
\|\nabla(|f(z)|^2)\| &= 2\|(uu_x + vv_x, uu_y + vv_y)\| \\
&= 2\sqrt{(uu_x + vv_x)^2 + (uu_y + vv_y)^2} \\
&= 2\sqrt{u^2u_x^2 + 2uvu_xv_x + v^2v_x^2 + u^2u_y^2 + 2uvu_yv_y + v^2v_y^2} \\
&= 2\sqrt{u^2u_x^2 + v^2u_x^2 + v^2v_x^2 + u^2v_x^2} \\
&= 2\sqrt{(u^2 + v^2)(u_x^2 + v_x^2)} \\
&\leq 2\sqrt{u_x^2 + v_x^2} = 2|f'(z)| \leq C_1
\end{aligned}
$$

from Cauchy estimate. Now

$$
\|\mu_g\| = \int_\Omega d\mu_g \leq \frac{C_1}{4\pi}\int_\Omega ds \leq M
$$

since Ω is a bounded domain. Hence μ_g has finite mass. □

Corollary 2.2.1. *Suppose Ω is a bounded domain whose boundary is a Jordan curve Γ which is a union of k analytic arcs Γ_j. Let $h \in H^p(\Omega)$ then h has boundary values h^* a.e.$(d\mu_g)$ on Γ in the normal direction and $h^* \in L^p(\Gamma, d\mu_g)$*

Proof. Let f be the Riemann mapping of Ω onto unit disc Δ. Then f extends to be analytic and conformal on a neighborhood of $\Omega \cup \Gamma_j$ for each j since Γ_j's are all analytic. Further $g = h \circ f^{-1}$ lies in $H^p(\Delta)$ and so g has boundary values a.e. $d\theta$ on $\partial\Delta$ and $g^* \in L^p(\partial\Omega, d\theta)$. Then $h = g \circ f$ has boundary values a.e. ds on Γ hence we have that h has boundary values $h^* = g^* \circ f$ μ_g-a.e. from the previous result. Also $h^* = g^* \circ f \in L^p(\Gamma, ds)$ and hence $h^* \in L^p(\Gamma, d\mu_g)$ □

Remark 8. The above result can be generalized to compact bordered Riemann surfaces by means of a procedure called Schottky Doubling. The double of a multiply connected domain of the plane was first introduced by Schottky and it was generalized to general Riemann surfaces by Picard. When

we study the classes of functions given on the border of a compact bordered Riemann surface by means of Schottky Doubling, we can assume that our compact bordered Riemann surface is the closure of a region Ω of a compact Riemann surface S such that $\Gamma = \partial\Omega$ is the union of a finite number (> 0) of disjoint, regular, analytic Jordan curves. The double S of a Riemann surface R may be defined as follows. Two points of S are associated with, or lie over, each interior point of R and one point of S is associated with each boundary point of R. Two disjoint neighborhoods of S lie over each neighborhood of an interior point of R.
To obtain a more complete description of S consider the Riemann surface (R, Φ) where Φ is the complex structure on R and replace each $h \in \Phi$ by $h^* : p \to -\overline{h}(p)$ and consider the conjugate Riemann surface (R, Φ^*). The new structure is conformal since $h_1^* \circ (h_2^*)^{-1}$ is explicitly $z \to -\overline{(h_1 \circ h_2^{-1})}(-\overline{z})$ which is conformal. On boundary, identify the boundary points of (R, Φ) and (R, Φ^*) by identity mapping. Now S is the topological sum of (R, Φ) and (R, Φ^*) i.e. S is obtained topologically from two copies of R by gluing the conjugate parts at the boundary and identifying the boundary points by identity mapping. As well-known examples, the double of a simply connected domain with boundary is the sphere, while the double of a multiply connected domain with m-boundaries is the sphere with $m-1$ handles, the double of the Mobius strip is the torus.

Remark 9. A function $u : R \to \mathbb{R}$ is harmonic if for some chart $h_\alpha : U_\alpha \to V_\alpha$ the functions $u_\alpha = u \circ h_\alpha^{-1}$ are harmonic in the usual sense.
A function is harmonic on a bordered Riemann surface R only if it has a harmonic extension to an open set on the double S.([1])

Now using doubling argument we have the following generalization:

Proposition 2.2.2. *Let R be a compact bordered Riemann surface whose border is denoted by Γ. Let $f \in H^p(R)$ (i.e $|f|^p$ has a harmonic majorant) then f has boundary values f^* a.e. $(d\mu_g)$ on Γ and $f^* \in L^p(d\mu_g)$.*

Proof. First of all due to Schottky doubling we take R as the closure of a region Ω of a compact Riemann surface S such that $\Gamma = \partial\Omega$ is the union of k analytic arcs, Γ_j. Now we can introduce a univalent conformal map η of an annulus $A = \{\rho < |z| < \rho^{-1}\}$ into S, mapping the unit circle onto a component of Γ and mapping points in its domain of modulus less than 1 into points of Ω([21], pg:208) and by this we conclude that a member, f of $H^p(\Omega)$, composed with η restricted to $A_1 = \{\rho < |z| < 1\}$ belongs to $H^p(A_1)$ i.e. $f \circ \eta = g \in H^p(A_1)$. And since $g \in H^p(A_1)$ has Fatou boundary function $g^* \in L^p(A_1, d\theta)$ on ∂A_1, f has Fatou boundary function $f^* = g^* \circ \eta^{-1} \in L^p(\partial\Omega, ds)$ ([23], pg:70). Also we can conformally map Ω into unit disc by a map $\psi(z)$ and then the Green function of Ω, $g_\Omega(z, a) = -\log|\psi(z)|$ and by (2.2.2) we have boundary Monge-Ampère measure on $\partial\Omega$ as $\mu_g = \frac{1}{4\pi}\|\nabla(|\psi(z)|^2)\|ds$ so $f^* \in L^p(\partial\Omega, d\mu_g)$. $\square$

2.2.2 Boundary Value Characterization of Poletsky-Stessin Hardy Spaces

Let Ω be a domain in $\mathbb{C}$ containing 0 and bounded by an analytic Jordan curve and u be a continuous, negative, subharmonic exhaustion function for Ω with finite Monge-Ampère mass. In the classical Hardy space theory on the unit disc $\mathbb{D}$ we can characterize the H^p spaces through their boundary values inside the L^p spaces of the unit circle and since we have $H^p_u(\Omega) \subset H^p(\Omega)$, any holomorphic function $f \in H^p_u(\Omega)$ has the boundary value function f^* from the classical theory (Theorem 10, [44]). In this section we will give an analogous characterization of the Poletsky-Stessin Hardy spaces through these boundary value functions and boundary Monge-Ampère measure. First, we will show the relation between boundary Monge-Ampère measure and Euclidean measure on the boundary $\partial\Omega$.

Proposition 2.2.3. *Let u be a continuous, negative, subharmonic exhaustion function for Ω with finite Monge-Ampère mass. Then the boundary Monge-*

Ampère measure μ_u and the Euclidean measure on $\partial\Omega$ are mutually absolutely continuous.

Proof. Let φ be a continuous function on $\partial\Omega$ and let the Poisson integral of φ be

$$H(z) = \int_{\partial\Omega} P(z,\xi)\varphi(\xi)d\sigma(\xi)$$

then by Lelong-Jensen formula we have

$$\int_{\partial\Omega} \varphi d\mu_u = \int_{\Omega} H(z)dd^c u = \int_{\Omega}\int_{\partial\Omega} P(z,\xi)\varphi(\xi)d\sigma(\xi)dd^c u$$

and since φ is a continuous function on the boundary and μ_u has finite mass we can use Fubini theorem to get

$$\int_{\partial\Omega} \varphi d\mu_u = \int_{\Omega}\int_{\partial\Omega} P(z,\xi)\varphi(\xi)d\sigma(\xi)dd^c u = \int_{\partial\Omega} \varphi(\xi)\left(\int_{\Omega} P(z,\xi)dd^c u(z)\right)d\sigma(\xi)$$

Now define

$$\beta(\xi) = \int_{\Omega} P(z,\xi)dd^c u(z)$$

We will show that $\beta(\xi)$ is $d\sigma$-integrable: First we see that $\beta(\xi) \geq 0$ and

$$\int_{\partial\Omega} |\beta(\xi)|d\sigma(\xi) = \int_{\partial\Omega} \beta(\xi)d\sigma(\xi) = \int_{\partial\Omega}\int_{\Omega} P(z,\xi)dd^c u(z)d\sigma(\xi) = \int_{\Omega}\int_{\partial\Omega} P(z,\xi)d\sigma(\xi)dd^c u(z)$$

and since

$$\int_{\partial\Omega} P(z,\xi)d\sigma(\xi) = 1$$

we have

$$\int_{\partial\Omega} |\beta(\xi)|d\sigma(\xi) = \int_{\Omega} dd^c u = MA(u) < \infty$$

Hence $\beta \in L^1(d\sigma)$ and we have

$$\int_{\partial\Omega} \varphi d\mu_u = \int_{\partial\Omega} \varphi\beta d\sigma$$

$\Rightarrow d\mu_u = \beta d\sigma$. Now we will also show that $1/\beta \in L^1_u(\partial\Omega)$. In fact, near $\partial\Omega$ we have $u \leq C g_\Omega(z, 0)$ for some positive constant $C > 0$, therefore we have $d\sigma \leq C d\mu_u = \beta d\sigma$. Thus, $1/\beta \leq C$ and since bounded functions belong to $L^1_u(\partial\Omega)$ we have $1/\beta \in L^1_u(\partial\Omega)$. Hence, the result follows. $\square$

Remark 10. From the previous proof we see that for a fixed $\xi \in \partial\Omega$, $\beta_{\tilde{r}}(\xi)$ which is defined as

$$\beta_{\tilde{r}}(\xi) = \int_{B_u(\tilde{r})} P(z, \xi) dd^c u = \int_{S_u(\tilde{r})} P(z, \xi) d\mu_{u,\tilde{r}}$$

converges to $\beta(\xi) = \int_\Omega P(z, \xi) dd^c u(z)$ by Monotone Convergence Theorem since $\beta_{\tilde{r}} \geq 0$ for all $\tilde{r}$, and $\beta_{\tilde{r}}$ is increasing with respect to $\tilde{r}$.

Now we will give the characterization of Poletsky-Stessin Hardy spaces $H^p_u(\Omega)$ through boundary value functions:

Theorem 2.2.1. *Let $f \in H^p(\Omega)$ be a holomorphic function and u be a continuous, negative, subharmonic exhaustion function for Ω. Then $f \in H^p_u(\Omega)$ if and only if $f^* \in L^p(d\mu_u)$ for $1 \leq p < \infty$. Moreover $\|f^*\|_{L^p(d\mu_u)} = \|f\|_{H^p_u(\Omega)}$.*

Proof. Let $f \in H^p_u(\Omega) \subset H^p(\Omega)$ we want to show that $f^* \in L^p(d\mu_u)$. First of all let $p > 1$,

$$\int_{\partial\Omega} |f^*(\xi)|^p d\mu_u = \int_{\partial\Omega} |f^*(\xi)|^p \left(\int_\Omega P(z, \xi) dd^c u(z) \right) d\sigma(\xi)$$

and using Fubini-Tonelli theorem we change the order of integration and get

$$\int_\Omega \left(\underbrace{\int_{\partial\Omega} |f^*(\xi)|^p P(z, \xi) d\sigma(\xi)}_{H(z)} \right) dd^c u(z)$$

Then the harmonic function being the Poisson integral of $|f^*|^p$ is the least harmonic majorant of $|f|^p$ so by ([37], Theorem 4.5.4) we have

$$H(z) = |f(z)|^p - \int_{\Omega} g_{\Omega}(w,z) dd^c |f(w)|^p$$

where $g_{\Omega}(w,z)$ is the Green function of Ω with the logarithmic pole at the point z. By ([14],Theorem 4.14), $g_{\Omega}(w,z)$ is continuous on $\overline{\Omega}$ and subharmonic in Ω hence we have

$$\int_{\Omega} H(z) dd^c u = \int_{\Omega} |f(z)|^p dd^c u - \int_{\Omega} \left(\int_{\Omega} g_{\Omega}(w,z) dd^c u \right) dd^c |f(w)|^p$$

Now using the boundary version of Lelong- Jensen formula ([14],Theorem 3.3) we get

$$\int_{\Omega} g_{\Omega}(w,z) dd^c u(z) = \int_{\Omega} u(z) dd^c g_{\Omega}(w,z) = u(w)$$

therefore

$$\int_{\partial\Omega} |f^*(\xi)|^p d\mu_u = \int_{\Omega} H(z) dd^c u = \int_{\Omega} |f(z)|^p dd^c u - \int_{\Omega} u(z) dd^c |f|^p = \|f\|_{H^p_u(\Omega)} < \infty$$

so we have $f^* \in L^p(d\mu_u)$.
For the converse since $f \in H^p(\Omega)$ we have

$$f(z) = \int_{\Omega} P(z,\xi) f^*(\xi) d\sigma(\xi)$$

now

$$\int_{S_u(r)} |f(z)|^p d\mu_{u,r} = \int_{S_u(r)} \left| \int_{\Omega} P(z,\xi) f^*(\xi) d\sigma(\xi) \right|^p d\mu_{u,r}$$

then by using Hölder inequality we have,

$$\leq \int_{\Omega} \left(\int_{S_u(r)} P(z,\xi) d\mu_{u,r} \right) |f^*(\xi)|^p d\sigma(\xi)$$

by Remark 2 we know $\int_{S_u(r)} P(z,\xi)d\mu_{u,r}$ is increasing and using Monotone Convergence Theorem as $r \to 0$ we get

$$\|f\|_{H^p_u(\Omega)} \leq \int_{\partial\Omega} |f^*(\xi)|^p d\mu_u(\xi) < \infty$$

so $f \in H^p_u(\Omega)$. The case $p = 1$ is just a straightforward application of the above procedure. $\square$

2.2.3 Weak and Strong Limit Values

The existence of boundary values for holomorphic functions depends on the geometry of the domain and the growth of the functions. However throughout this section we have made use of the fact that Poletsky-Stessin Hardy classes are inside the classical ones so we already have the non tangential boundary values. In his 2011 paper ([35])in order to look at the boundary value problem without any boundary smoothness condition, Poletsky gave definitions for two types of boundary values namely, strong and weak limit values for sequences of functions in an abstract setting as follows:

Definition 13. Let K be a compact metric space and $M = \mu_j$ be a sequence of regular Borel measures on K converging weak-$*$ in $C^*(K)$ to a finite measure μ. We denote the set $supp\mu_j$ by K_j and $supp\mu$ by K_0. Let $\phi = \{\phi_j\}$ be a sequence of Borel functions ϕ_j on K_j. We let

$$\|\phi\|_{L^p(M)} = \limsup_{j\to\infty} \|\phi_j\|_{L^p(K_j,\mu_j)}$$

If the measures $\{\phi_j\mu_j\}$ converge weak-$*$ to a measure $\phi_*\mu$ then the function ϕ_* will be called the ***weak limit values*** of ϕ. We will denote by $\mathcal{A}(M)$ the space of all sequences ϕ of Borel functions ϕ_j on K_j which have weak limit values and by $\mathcal{A}^p(M)$ those sequences ϕ in $\mathcal{A}(M)$ for $\|\phi\|_{L^p(M)} < \infty$.
We say that a sequence $\phi \in \mathcal{A}(M)$ has ***the strong limit values*** on K_0 with respect to M if there is a μ-measurable function ϕ^* on K_0 such that for

any $b > a$ and any $\varepsilon, \delta > 0$ there is a j_0 and an open set $O \subset K$ containing $G(a,b) = \{x \in K_0 : a \leq \phi^*(x) \leq b\}$ such that

$$\mu_j(\{\phi_j < a - \varepsilon\} \cap O) + \mu_j(\{\phi_j > b + \varepsilon\} \cap O) < \delta$$

when $j \geq j_0$. The function ϕ^* is called the strong limit values of ϕ.

Now we will show the relation of weak and strong limit values with the boundary values of the functions in the Poletsky-Stessin Hardy classes $H^1_u(\mathbb{D})$ where u is a continuous, negative, subharmonic exhaustion function with finite mass:

Theorem 2.2.2. *Let $f \in H^1_u(\mathbb{D})$ and f^* be the boundary value of f. Then the sequence $\{f\mu_{u,r}\}$ has $f^*\mu_u$ as its weak limit value.*

Proof. First define $\mathcal{P}(z,\xi) = \frac{P(z,\xi)}{\int_{\mathbb{D}} P(w,\xi)dd^c u(w)}$ we see that $\int_{\partial\mathbb{D}} \mathcal{P}(z,\xi)d\mu(\xi) = 1$. Now define

$$p_r(\xi) = \int_{S(r)} \mathcal{P}(z,\xi)d\mu_{u,r}$$

Step 1: We will show that p_r's are uniformly bounded and converge weak-$*$ to 1 on $\partial\mathbb{D}$:

$$p_r(\xi) = \int_{S(r)} \mathcal{P}(z,\xi)d\mu_{u,r} = \frac{1}{\int_{\mathbb{D}} P(w,\xi)dd^c u(w)} \int_{S(r)} P(z,\xi)d\mu_{u,r}(z)$$

and using Lelong-Jensen formula we get

$$= \frac{1}{\int_{\mathbb{D}} P(w,\xi)dd^c u(w)} \int_{B(r)} P(z,\xi)dd^c u(z)$$

Hence $\|p_r\| \leq 1$ for all r, ξ.
Let $\varphi \in C(\overline{\mathbb{D}})$ then

$$\lim_{r\to 0} \int_{\partial\mathbb{D}} \varphi(\xi)p_r(\xi)d\mu_u(\xi) = \lim_{r\to 0} \int_{\partial\mathbb{D}} \varphi(\xi) \left(\int_{S(r)} \mathcal{P}(z,\xi)d\mu_{u,r} \right) d\mu_u(\xi)$$

$$\lim_{r\to 0}\int_{S(r)}\left(\int_{\partial\mathbb{D}}\varphi(\xi)P(z,\xi)d\sigma(\xi)\right)d\mu_{u,r}$$

$$\lim_{r\to 0}\int_{S(r)}\varphi(z)d\mu_{u,r}(z)=\int_{\partial\mathbb{D}}\varphi(\xi)d\mu_u(\xi)$$

the last equality is true since $d\mu_{u,r}$ weak-$*$ converges to $d\mu_u$.Hence p_r's weak-$*$ converge to 1 on $\partial\mathbb{D}$.

Step 2: For $z\in S(r)$ define $\psi_r(z)=\int_{\partial\mathbb{D}}\mathcal{P}(z,\xi)f^*(\xi)d\mu_u(\xi)$ then

$$\int_{S(r)}\psi_r(z)d\mu_{u,r}(z)=\int_{\partial\mathbb{D}}f^*(\xi)\left(\int_{S(r)}\mathcal{P}(z,\xi)d\mu_{u,r}(z)\right)d\mu_u(\xi)$$

$$=\int_{\partial\mathbb{D}}f^*(\xi)p_r(\xi)d\mu_u(\xi)$$

and

$$\int_{S(r)}|\psi_r(z)|d\mu_{u,r}(z)\le\int_{\partial\mathbb{D}}|f^*(\xi)|\left(\int_{S(r)}\mathcal{P}(z,\xi)d\mu_{u,r}(z)\right)d\mu_u(\xi)$$

$$=\int_{\partial\mathbb{D}}|f^*(\xi)|p_r(\xi)d\mu_u(\xi)$$

Hence $\|\psi_r\|_{L^1(S(r))}$ are uniformly bounded and we can take a subsequence $\{\psi_{r_j}\mu_{u,r_j}\}$ converging weak-$*$ to a measure ν then

$$\int_{\partial\mathbb{D}}d\nu=\lim_{j\to\infty}\int_{S(r_j)}\psi_{r_j}d\mu_{u,r_j}=\lim_{j\to\infty}\int_{\partial\mathbb{D}}f^*p_{r_j}d\mu_u=\int_{\partial\mathbb{D}}f^*d\mu_u$$

by ([35],Lemma 4.1). Thus $\nu=f^*\mu_u$ and the sequence $\{\psi_r\mu_{u,r}\}$ converges weak-$*$ to $f^*\mu_u$. Hence,

$$\lim_{r\to 0}\int_{\overline{\mathbb{D}}}\psi_r(z)d\mu_{u,r}=\lim_{r\to 0}\int_{\overline{\mathbb{D}}}\int_{\partial\mathbb{D}}\mathcal{P}(z,\xi)f^*(\xi)d\mu_u(\xi)d\mu_{u,r}$$

$$=\lim_{r\to 0}\int_{\overline{\mathbb{D}}}f(z)d\mu_{u,r}=\int_{\overline{\mathbb{D}}}f^*d\mu_u$$

Therefore f^* is the weak limit value. $\square$

Corollary 2.2.2. *Let $f \in H^1_u(\mathbb{D})$ and f^* be the boundary value of f. Then the sequence $\{|f|\mu_{u,r}\}$ has $|f^*|\mu_u$ as its both weak and strong limit value.*

Proof. For $z \in S(r)$ define $\psi_r(z) = \int_{\partial\mathbb{D}} \mathcal{P}(z,\xi)|f^*(\xi)|d\mu_u(\xi)$ then

$$|f(z)| = \left|\int_{\partial\mathbb{D}} f^*(\xi)P(z,\xi)d\sigma(\xi)\right| \leq \int_{\partial\mathbb{D}} |f^*(\xi)|\mathcal{P}(z,\xi)d\mu_u(\xi) = \psi_r(z)$$

Now using the same argument in the previous result we get that the sequence $\psi_r\mu_{u,r}$ converges weak-$*$ to $|f^*|\mu_u$, therefore the non-negative sequence $\psi_r - |f|$ has zero weak-$*$ limit value hence we have $|f|\mu_{u,r} \to |f^*|\mu_u$ weak-$*$. By ([35],Theorem 3.6) we also have $|f^*|$ as strong limit value. $\square$

2.3 Factorization

Let Ω be a domain in $\mathbb{C}$ containing 0 and bounded by an analytic Jordan curve. Suppose $\psi : \mathbb{D} \to \Omega$ is the conformal map such that $\psi(0) = 0$ and $\varphi = \psi^{-1}$. By Carathéodory theorem for a domain like Ω we have $0 < m \leq |\psi'| \leq M < \infty$ for some $m, M > 0$. Following the definition given in ([3]), we have that :

Definition 14. A holomorphic function h on Ω is Ω-inner if $h \circ \psi$ is inner in the classical sense i.e. $|h \circ \psi| = 1$ for almost all $\xi \in \partial\mathbb{D}$ and Ω-outer if $h \circ \psi$ is outer in the classical sense i.e.

$$\log|h \circ \psi(0)| = \int_{\partial\mathbb{D}} \log|h \circ \psi| d\sigma$$

For the classical Hardy space on the unit disc $\mathbb{D}$ we have the following canonical factorization theorem ([17],pg:24):

Theorem 2.3.1. *Every function $f \not\equiv 0$ of class $H^p(\mathbb{D})$ has a unique factorization of the form $f = BSF$ where B is a Blaschke product, S is a singular* *inner function and F is an outer function for the class $H^p(\mathbb{D})$.*

Using this result and the conformal mapping the following is shown in ([3]):

Proposition 2.3.1. *Every $f \in H^p(\Omega)$ can be factored uniquely up to a unimodular constant as $f = IF$ where I is Ω-inner and F is Ω-outer.*

Inspired by this result we have the following corollary

Corollary 2.3.1. *Let $f \in H^p_u(\Omega)$, $1 \leq p < \infty$, where u is a continuous exhaustion function with finite Monge-Ampère mass. Then f can be factored as $f = IF$ where I is Ω-inner and F is Ω-outer. Moreover $I \in H^p_u(\Omega)$ and $F \in H^p_u(\Omega)$.*

Proof. First of all, since $H^p_u(\Omega) \subset H^p(\Omega)$ by the above proposition it is obvious that f can be factored as $f = IF$ where I is Ω-inner and F is Ω-outer. Now define the measure $\hat{\mu}$ on $\partial\mathbb{D}$ as $\hat{\mu}(E) = \mu_u(\psi(E))$ for any measurable set $E \subset \partial\mathbb{D}$ then it is clear from the change of variables formula that $|I \circ \psi| = 1$ $d\hat{\mu}$-a.e. so

$$\int_{\partial\Omega} |I|^p d\mu_u = \int_{\partial\mathbb{D}} |I \circ \psi|^p d\hat{\mu} = \int_{\partial\mathbb{D}} d\hat{\mu} = \int_{\partial\Omega} d\mu_u = MA(u) < \infty$$

since u has finite Monge-Ampère mass and for the outer part

$$\int_{\partial\Omega} |F|^p d\mu_u = \int_{\partial\mathbb{D}} |F \circ \psi|^p d\hat{\mu} = \int_{\partial\mathbb{D}} |f^* \circ \psi|^p d\hat{\mu} = \int_{\partial\Omega} |f^*|^p d\mu_u < \infty$$

since $f \in H^p_u(\Omega)$. Hence we have $I \in H^p_u(\Omega)$ and $F \in H^p_u(\Omega)$. $\square$

In fact in the particular case of unit disc $\mathbb{D}$ we can say more about this factorization :

Theorem 2.3.2. *Let f be analytic function in $\mathbb{D}$ such that $f \in H^p_u(\mathbb{D})$. Then f can be factored into a Blaschke product B, a singular inner function S and an outer function F such that $B, S, F \in H^p_u(\mathbb{D})$ for $1 \leq p < \infty$.*

Proof. Let $f \in H^p_u(\mathbb{D})$ then $f \in H^p(\mathbb{D})$ so f has canonical decomposition $f = BSF$ where B is a Blaschke product, S is a singular inner function and F is an outer function. The exhaustion u has finite mass so bounded functions belong to class $H^p_u(\mathbb{D})$. Since B and S are bounded ([17],pg:24), $B, S \in H^p_u(\mathbb{D})$. As the outer function $F(z)$ is concerned , we know $F \in H^1(\mathbb{D})$ so we have

$$F(z) = \frac{1}{2\pi}\int_0^{2\pi} P(r, \theta - t)F^*(e^{it})dt$$

Now

$$\int_{S(\widehat{r})} |F(z)|^p d\mu_{u,\widehat{r}} = \frac{1}{2\pi}\int_{S(\widehat{r})} \left|\int_0^{2\pi} P(r, \theta - t)F^*(e^{it})dt\right|^p d\mu_{u,\widehat{r}}$$

$$\leq \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{S(\widehat{r})} P(r, \theta - t)d\mu_{u,\widehat{r}}\right)|F^*(e^{it})|^p dt$$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(\int_{S(\widehat{r})} P(r, \theta - t)d\mu_{u,\widehat{r}}\right)|f^*(e^{it})|^p dt$$

and using the previous result, monotone convergence theorem and the fact that $|f^*| = |F^*|$ (dt)- a.e. we get as $\widehat{r} \to 0$:

$$\|F(z)\|_{H^p_u(\mathbb{D})} \leq \int_{\partial\mathbb{D}} |f^*(e^{it})|^p d\mu_u(t) < \infty$$

$\Rightarrow F(z) \in H^p_u(\mathbb{D})$ $\square$

In the classical H^p space theory a useful tool for the proofs is that any holomorphic function $f \in H^1(\mathbb{D})$ can be expressed as a product of two functions , $f = gh$ where both factors g and $h \in H^2(\mathbb{D})$. Now we will show that there is a similar factorization in the spaces $H^p_u(\Omega)$.

Corollary 2.3.2. *Suppose $1 \leq p < \infty$, $f \in H^p_u(\Omega)$, $f \not\equiv 0$. Then there is a zero-free function $h \in H^2_u(\Omega)$ such that $f = Ih^{\frac{2}{p}}$. In particular every $f \in H^1_u(\Omega)$ is a product of $f = gh$ in which both factors are in $H^2_u(\Omega)$.*

Proof. By the previous theorem $f/I \in H_u^p(\Omega)$. Since f/I has no zero in Ω and Ω is simply connected , there exists $\varphi \in \mathcal{O}(\Omega)$ such that $\exp(\varphi) = f/I$. Define $h = \exp(p\varphi/2)$ then $h \in \mathcal{O}(\Omega)$ and $|h|^2 = |f/I|^p$ and $h \in H_u^2(\Omega)$ and $f = Ih^{\frac{2}{p}}$. To obtain $f = gh$ for $f \in H_u^1(\Omega)$ write $f = Ih^2$ in the form $f = (Ih)h$. □

2.4 Approximation

Let $A(\Omega)$ denote the algebra of holomorphic functions on Ω which are continuous on $\partial\Omega$. We know that the algebra of holomorphic functions $A(\Omega)$ is dense in the classical Hardy spaces when Ω is a domain bounded by an analytic Jordan curve and we will show an analogous approximation result on $H_u^p(\Omega)$ where u is a negative, continuous, subharmonic exhaustion function on Ω with finite Monge-Ampère mass but before this result we should first mention some classes of holomorphic functions from the classical theory on the unit disc which will help us in the proof of approximation result:

Definition 15. An analytic function $f \in \mathcal{O}(\mathbb{D})$ is said to be of class N if the integrals

$$\int_0^{2\pi} \log^+ |f(re^{i\theta})| d\theta$$

are bounded for $r < 1$.

Definition 16. An analytic function $f \in \mathcal{O}(\mathbb{D})$ is in class N^+ if it has the form $f = BSF$ where B is a Blaschke product, S is a singular inner function and F is an outer function for the class N

It is clear that $N \supset N^+ \supset H^p(\mathbb{D})$ for all $p > 0$, (For details see [17]). We will also use the following result (Theorem 2.11, [17]), which plays a crucial role in our approximation result:

Theorem 2.4.1. *If a holomorphic function $f \in N^+$ and $f^* \in L^p$ for some $p > 0$, then $f \in H^p(\mathbb{D})$.*

Now we can give the approximation result for the Poletsky-Stessin Hardy classes $H^p_u(\Omega)$:

Theorem 2.4.2. *The algebra $A(\Omega)$ is dense in $H^p_u(\Omega)$, $1 \le p < \infty$.*

Proof. (**Case 1:** $p > 1$)Let L be a linear functional on $H^p_u(\Omega)$ such that L vanishes on $A(\Omega)$. Then $L(f) = \int_{\partial\Omega} f^*\overline{g^*}d\mu_u$ for some non-zero $g \in L^q(d\mu_u)$ hence $\int_{\partial\Omega} \gamma\overline{g^*}d\mu_u = 0$ for all $\gamma \in A(\Omega)$ (*). Now we need the following lemma:

Lemma 2.4.1. *Let μ be a measure on the boundary $\partial\Omega$ of Ω which is orthogonal to $A(\Omega)$. Then μ is absolutely continuous with respect to $d\mu_{g_{\Omega,0}}$ which is the boundary Monge-Ampère measure with respect to the Green function with pole at* 0.

Proof. The homomorphism "evaluation at 0" of $A(\Omega)$ has the representing measure $d\mu_{g_{\Omega,0}}$ on $\partial\Omega$. Then by generalized F. and M. Riesz Theorem (Theorem 7.6, [19]) the singular part μ_s of μ with respect to $d\mu_{g_{\Omega,0}}$ is orthogonal to $A(\Omega)$ that is $\int_{\partial\Omega} f d\mu_s = 0$ for all $f \in A(\Omega)$ but on a domain like Ω, $A(\Omega)$ is dense in $C(\partial\Omega)$ (Theorem 2.7, [19]) so $\int_{\partial\Omega} g d\mu_s = 0$ for all $g \in C(\partial\Omega)$. Hence $d\mu_s = 0$ and $d\mu << d\mu_{g_{\Omega,0}}$. □

Now by the above lemma and (*) we have $\overline{g^*}d\mu_u << d\mu_{g_{\Omega,0}}$ so by Radon-Nikodym Theorem we have $\overline{g^*}d\mu_u = h^* d\mu_{g_{\Omega,0}}$ for some $h^* \in L^1(d\mu_{g_{\Omega,0}})$ and on a domain Ω which is bounded by an analytic Jordan curve, we have $c_1 d\mu_{g_{\Omega,0}} \le d\sigma \le c_2 d\mu_{g_{\Omega,0}}$, so we have $h^* \in L^1(d\sigma)$. Let φ be the conformal mapping between the unit disc and Ω and $\psi = \varphi^{-1}$. Now consider the function $H^* = h^* \circ \psi$ on $\partial\mathbb{D}$ then since $h^* \in L^1(d\sigma)$ we have $h^* \in L^1(\partial\mathbb{D})$, and since $\psi(0) = 0$ and $\mu_{g_{\Omega,0}}$ is in fact the harmonic measure, we have $d\mu_{g_{\mathbb{D},0}}(e^{i\theta}) = d\mu_{g_{\Omega,0}}(\psi(e^{i\theta}))$ and using $\varphi = \psi^{-1}$ we get

$$\int_{\partial\mathbb{D}} e^{in\theta}(h^* \circ \psi(e^{i\theta}))d\theta = \int_{\partial\Omega} (\varphi(z))^n h^*(z) d\mu_{g_{\Omega,0}}(z) = \int_{\partial\Omega} (\varphi(z))^n \overline{g^*} d\mu_u = 0$$

for all n as a consequence of (*). Hence H^* is the boundary value of an $H^1(\mathbb{D})$ function H, then h which is defined as $H = h \circ \psi$ is in the class $E^1(\Omega)$ by the corollary of (Theorem 10.1, [17]). Moreover since $E^1(\Omega) = H^1(\Omega)$ we have $h \in H^1(\Omega)$ and since $\psi(0) = 0$ we have $h(0) = 0$.
Now take $\alpha \in H^p_u(\Omega)$ and consider the analytic function αh

$$\int_{\partial\Omega} |\alpha^* h^*|^{\frac{1}{2}} d\sigma \leq \left(\int_{\partial\Omega} |\alpha^*| d\sigma\right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |h^*| d\sigma\right)^{\frac{1}{2}} \leq \|\alpha\|^{\frac{1}{2}}_{H^1} \|h\|^{\frac{1}{2}}_{H^1}$$

since $h \in H^1(\Omega)$ and $\alpha \in H^p_u(\Omega) \subset H^p(\Omega) \subset H^1(\Omega)$ so $\alpha h \in H^{\frac{1}{2}}(\Omega)$. On the other hand

$$\int_{\partial\Omega} |\alpha^* h^*| d\sigma \leq c_2 \int_{\partial\Omega} |\alpha^*||h^*| d\mu_{g_{\Omega,0}} = c_2 \int_{\partial\Omega} |\alpha^*||\overline{g^*}| d\mu_u$$

$$\leq \left(\int_{\partial\Omega} |\alpha^*|^p d\mu_u\right)^{\frac{1}{p}} \left(\int_{\partial\Omega} |g^*|^q d\mu_u\right)^{\frac{1}{q}} < \infty$$

since $\alpha \in H^p_u(\Omega)$ and $g^* \in L^q(d\mu_u)$. Hence we have $\alpha^* h^* \in L^1(\partial\Omega)$. Now since $\alpha h \in H^{\frac{1}{2}}(\Omega)$ by the corollary of (Theorem 10.1, [17]) the function $AH = \alpha h(\psi(w))[\psi'(w)]^2 \in H^{\frac{1}{2}}(\mathbb{D}) \subset N^+$ but since $\alpha^* h^* \in L^1(\partial\Omega)$, $AH^* \in L^1(\partial\mathbb{D})$ hence again by the corollary of (Theorem 10.1, [17]) we have $\alpha h \in H^1(\Omega)$. Finally

$$0 = \alpha h(0) = \int_{\partial\Omega} \alpha h d\mu_{g_{\Omega,0}} = \int_{\partial\Omega} \alpha \overline{g^*} d\mu_u = L(\alpha)$$

for all $\alpha \in H^p_u(\Omega)$ hence $A(\Omega)$ is dense in $H^p_u(\Omega)$.
(**Case 2:** $p = 1$): By the previous corollary we know that if $f \in H^1_u(\Omega)$ then we can factor it out like $f = gh$ where $g, h \in H^2_u(\Omega)$ and from the first part of the proof we know there exist sequences $\{g_n\}, \{h_n\} \in A(\Omega)$ such that $\{g_n\} \to g$ and $\{h_n\} \to h$ in $H^2_u(\Omega)$. Now for f consider the sequence $\{g_n h_n\}$ then,

$$\int_{\partial\Omega} |f - g_n h_n| d\mu_u = \int_{\partial\Omega} |gh - g_n h_n| d\mu_u = \int_{\partial\Omega} |gh - gh_n + gh_n - g_n h_n| d\mu_u$$

$$\leq \left(\int_{\partial\Omega} |g|^2 d\mu_u \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |h - h_n|^2 d\mu_u \right)^{\frac{1}{2}} + \left(\int_{\partial\Omega} |h_n|^2 d\mu_u \right)^{\frac{1}{2}} \left(\int_{\partial\Omega} |g - g_n|^2 d\mu_u \right)^{\frac{1}{2}}$$

But right hand side goes to 0 since $\{g_n\} \to g$ and $\{h_n\} \to h$ in $H_u^2(\Omega)$ hence $\{g_n h_n\} \in A(\Omega)$ converges to f in $H_u^1(\Omega)$.
Combining these two cases we see that $A(\Omega)$ is dense in $H_u^p(\Omega)$ for $1 \leq p < \infty$.

□

Moreover, we know from Mergelyan's Approximation Theorem ([31]) that the algebra $A(\Omega)$ can be uniformly approximated by polynomials therefore, we have the following corollary:

Corollary 2.4.1. *Polynomials are dense in $H_u^p(\Omega)$, $1 \leq p < \infty$.*

2.5 Composition Operators With Analytic Symbols

Let $\phi : \Omega \to \Omega$ be a holomorphic self map of Ω. The linear composition operator induced by the symbol ϕ is defined by $C_\phi(f) = f \circ \phi$, $f \in \mathcal{O}(\Omega)$. In 2003, Shapiro and Smith ([42]) showed that on a domain Ω which is bounded by an analytic Jordan curve, every holomorphic self map ϕ of Ω induces a bounded composition operator on the classical Hardy space $H^p(\Omega)$. Moreover we know that being in the class $H_v^p(\Omega)$ where v is harmonic outside of a compact set is equivalent to having a harmonic majorant hence any composition operator on a Hardy class generated by this sort of exhaustion function is also bounded. As a consequence of Closed Graph Theorem continuity of a composition operator on $H_u^p(\Omega)$ is in fact determined by whether it takes functions from $H_u^p(\Omega)$ to $H_u^p(\Omega)$ or not. However this does **not** always hold when exhaustion function has finite Monge-Ampère mass but not harmonic outside of a compact set as we see from the following example :

Example 1. Suppose u is the exhaustion function that we constructed in Theorem 2.1.2 then we know again from the proof of Theorem 2.1.2 that the function $\frac{1}{(z-1)^{\frac{3}{4}}} \notin H_u^1(\mathbb{D})$. Now consider the operator with symbol $\phi(z) = ze^{i\frac{\pi}{2}}$, and take the function $f(z) = \frac{1}{(z-i)^{\frac{3}{4}}}$, then

$$\|f\|_{H_u^1(\mathbb{D})} = \int_{\partial\mathbb{D}} \frac{1}{(\xi - i)^{\frac{3}{4}}} \beta(\xi) d\sigma(\xi) < \infty$$

since the singularities of $\beta(\xi)$ and $f^*(\xi)$ do not overlap and they are both integrable functions on the boundary so $f(z) \in H_u^1(\mathbb{D})$ but

$$C_\phi(f) = f \circ \phi = \frac{1}{e^{i\frac{3\pi}{8}}(z-1)^{\frac{3}{4}}}$$

and $C_\phi(f) \notin H_u^1(\mathbb{D})$. Therefore not every composition operator is bounded on Poletsky-Stessin Hardy classes even though the symbol function is a nice and simple one like in our example, namely a rotation.

In the next result we will examine the necessary and sufficient conditions for the composition operator C_φ to be bounded on this rather interesting space $H_u^p(\mathbb{D})$ where u is the exhaustion function constructed in the proof of Theorem 2.1.2 and φ is an automorphism of the unit disc:

Theorem 2.5.1. *Let φ be a Mobius transformation such that $\varphi(z) = e^{i\theta}\frac{z-a}{1-\bar{a}z}$ where $a \in \mathbb{D}$ and u is the exhaustion function constructed in the proof of Theorem 1.3. Then the following are equivalent:*
(i) C_φ is a bounded operator on the space $H_u^p(\mathbb{D})$
(ii) There exists a constant $K > 0$ such that $\int_E \beta(\varphi^{*-1}(\eta))d\sigma(\eta) \leq K \int_E \beta(\eta)d\sigma(\eta)$ for all measurable $E \subset \partial\mathbb{D}$ where $d\mu_u = \beta d\sigma$
(iii) $\varphi(1) = 1$

Proof. (It is sufficient to prove the result for $p = 1$)
($i \Leftrightarrow ii$) Let $f \in H_u^1(\mathbb{D})$ and φ be a Mobius transformation then

$$\|f \circ \varphi\|_{H_u^1(\mathbb{D})} = \int_{\partial\mathbb{D}} |f^* \circ \varphi^*| \beta(\xi) d\sigma(\xi) = \int_{\partial\mathbb{D}} |f^*(\eta)| \beta(\varphi^{*-1}(\eta)) |(\varphi^{*-1})'| d\sigma(\eta)$$

Suppose C_φ is bounded on $H_u^1(\mathbb{D})$ then $\|f\circ\varphi\|_{H_u^1(\mathbb{D})} \le M\|f\|_{H_u^1(\mathbb{D})}$ for all $f \in H_u^1(\mathbb{D})$. Now since bounded functions are in $H_u^p(\mathbb{D})$, we have $f(z) \equiv 1 \in H_u^1(\mathbb{D})$ and we will write the above inequality for $f(z) \equiv 1$. Since $|(\varphi^{*-1})'| < N < \infty$ on $\partial\mathbb{D}$ we get

$$\int_{\partial\mathbb{D}} \beta(\varphi^{*-1}(\eta))|(\varphi^{*-1})'|d\sigma(\eta) \le N\int_{\partial\mathbb{D}} \beta(\varphi^{*-1}(\eta))d\sigma(\eta) \le NM\int_{\partial\mathbb{D}} \beta(\eta)d\sigma(\eta)$$

For the converse direction, suppose that there exists a constant $K > 0$ such that $\int_{\partial\mathbb{D}} \beta(\varphi^{*-1}(\eta))d\sigma(\eta) \le K\int_{\partial\mathbb{D}} \beta(\eta)d\sigma(\eta)$ for all measurable $E \subset \partial\mathbb{D}$. Then for any characteristic function χ_E, $E \subset \partial\mathbb{D}$ we have

$$\int_{\partial\mathbb{D}} \chi_E\beta(\varphi^{*-1}(\eta))|(\varphi^{*-1})'|d\sigma(\eta) = \int_E \beta(\varphi^{*-1}(\eta))d\sigma(\eta) \le K\int_E \beta(\eta)d\sigma(\eta)$$

$$= K\int_{\partial\mathbb{D}} \chi_E\beta(\eta)d\sigma(\eta)$$

Hence by monotone convergence theorem for any positive integrable function g we have

$$\int_{\partial\mathbb{D}} g(\eta)\beta(\varphi^{*-1}(\eta))|(\varphi^{*-1})'|d\sigma(\eta) \le K\int_{\partial\mathbb{D}} g(\eta)\beta(\eta)d\sigma(\eta)$$

so

$$\|f\circ\varphi\|_{H_u^p(\mathbb{D})} = \int_{\partial\mathbb{D}} |f^*|^p\beta(\varphi^{*-1}(\eta))|(\varphi^{*-1})'|d\sigma(\eta) \le N\int_{\partial\mathbb{D}} |f^*|^p\beta(\varphi^{*-1}(\eta))d\sigma(\eta)$$

$$\le C\int_{\partial\mathbb{D}} |f^*|^p\beta(\eta)d\sigma(\eta) = C\|f\|_{H_u^p(\mathbb{D})}$$

Hence C_φ is bounded.

$(i \Leftrightarrow iii)$ Suppose C_φ is bounded and $\varphi(1) \neq 1$ then $\exists\xi \in \partial\mathbb{D}$, $\xi \neq 1$ such that $\varphi(\xi) = 1$ and take the function $f(z) = \frac{1}{(1-z)^{\frac{3}{4}}}$ then we know that

$f(z) \notin H^1_u(\mathbb{D})$. Now consider the function $F(z) = f \circ \varphi^{-1}(z)$ then

$$\|F(z)\|_{H^1_u(\mathbb{D})} = \int_{\partial\mathbb{D}} \frac{1}{|1-\varphi^{-1}(\eta)|^{\frac{3}{4}}} d\mu_u(\eta)$$

$$= \int_{\partial\mathbb{D}\setminus B_\gamma(1)} \frac{1}{|1-\varphi^{-1}(\eta)|^{\frac{3}{4}}} d\mu_u(\eta) + \int_{B_\gamma(1)} \frac{1}{|1-\varphi^{-1}(\eta)|^{\frac{3}{4}}} d\mu_u(\eta) < \infty$$

for some $\gamma > 0$. The first integral in the last line is bounded because on $\partial\mathbb{D} \setminus B_\gamma(1)$, $d\mu_u$ and $d\sigma$ are mutually absolutely continuous and $\frac{1}{|1-\varphi^{-1}(\eta)|^{\frac{3}{4}}}$ is $d\sigma$ integrable and the second integral is bounded because on $B_\gamma(1)$, $\frac{1}{|1-\varphi^{-1}(\eta)|^{\frac{3}{4}}}$ is a bounded function and hence it is $d\mu_u$ integrable.
Hence $F(z) \in H^1_u(\mathbb{D})$ but $F \circ \varphi = f \notin H^1_u(\mathbb{D})$ but this contradicts with the boundedness of C_φ.
Suppose now $\varphi(1) = 1$ from the $(i \Leftrightarrow ii)$ part of the proof we know that if $\frac{\beta(\varphi^{-1}(\eta))}{\beta(\eta)} < M < \infty$ then C_φ is bounded and for the case $\varphi(1) = 1$ we have $\frac{\beta(\varphi^{-1}(\eta))}{\beta(\eta)}$ bounded hence the result follows. □

We can generalize these arguments to a slightly wider class of symbols as follows:

Proposition 2.5.1. *Let $\varphi : \overline{\mathbb{D}} \to \overline{\mathbb{D}}$ be a locally univalent self map of $\mathbb{D}$ such that φ is differentiable in a neighborhood of $\overline{\mathbb{D}}$. Then C_φ is bounded on $H^1_u(\mathbb{D})$ if and only if $\varphi(1) = 1$ and $N^\varphi_\beta(\eta) \le K\beta(\eta)$ for some $K > 0$ and all $\eta \in \partial\mathbb{D}$ where $N^\varphi_\beta(\eta) = \sum_{j\ge1} \beta(\xi_j(\eta))$ and $\{\xi_j(\eta)\}$ are the zeros of $\varphi(z) - \eta$.*

Proof. $(\Rightarrow)$ Suppose C_φ is bounded and $\varphi(1) \ne 1$ then there exists a $\xi \in \partial\mathbb{D}$ such that $\varphi(1) = \xi$, now consider the function $f(z) = \frac{1}{(\xi - z)^{\frac{3}{4}}}$, $f(z) \in H^1_u(\mathbb{D})$ and

$$\|f \circ \varphi\|_{H^1_u(\mathbb{D})} = \int_{\partial\mathbb{D}} |f^* \circ \varphi| d\mu_u = \int_{\partial\mathbb{D}\setminus\overline{B_\gamma(1)}} |f^* \circ \varphi| d\mu_u + \int_{\overline{B_\gamma(1)}} |f^* \circ \varphi| d\mu_u$$

where $\overline{B_\gamma(1)} = \partial\mathbb{D} \cap B_\gamma(1)$ for some small $\gamma > 0$. The first integral in the sum is bounded since over $\partial\mathbb{D} \setminus \overline{B_\gamma(1)}$, $d\mu_u = C d\sigma$ for some $C > 0$

and $f \in H^1_u(\mathbb{D}) \subset H^1(\mathbb{D})$ so boundedness over this region is guaranteed by classical H^p theory but

$$\int_{\overline{B_\gamma(1)}} |f^* \circ \varphi| d\mu_u = \int_{\overline{B_\gamma(1)}} \frac{1}{|\xi - \varphi|^{\frac{3}{4}}} d\mu_u$$

and φ has finite derivative near $\{1\}$ so

$$\int_{\overline{B_\gamma(1)}} \frac{1}{|\xi - \varphi|^{\frac{3}{4}}} d\mu_u \geq M \int_{\overline{B_\gamma(1)}} \frac{1}{|1 - \eta|^{\frac{3}{4}}} d\mu_u \to \infty$$

contradicting C_φ being bounded. Hence $\varphi(1) = 1$.
The inequality $N^\varphi_\beta(\eta) \leq K\beta(\eta)$ is trivially true for $\eta = 1$, so we will consider the case where $\eta \neq 1$ and assume for a contradiction that $N^\varphi_\beta(\eta_0) > K\beta(\eta_0)$ for all K, for some $\eta_0 \neq 1$. Then from the definition of $N^\varphi_\beta(\eta_0)$ we see that $\beta(\varphi(\eta_0)) \to \infty$ which gives $\varphi(\eta_0) = 1$. Then consider the function $f(z) = \frac{1}{(\eta_0 - z)^{\frac{3}{4}}}$ then by the same argument above $f \circ \varphi \notin H^1_u(\mathbb{D})$ contradicting C_φ being bounded hence $N^\varphi_\beta(\eta) \leq K\beta(\eta)$ for all $\eta \in \partial\mathbb{D}$ for some $K > 0$.
($\Leftarrow$) Since φ is locally univalent we can find a countable collection of disjoint open arcs Ω_j with $\sigma(\partial\mathbb{D} \setminus \bigcup \Omega_j) = 0$ and the restriction of φ to each Ω_j is univalent. Write $\psi_j(w)$ for the inverse of φ taking $\varphi(\Omega_j)$ onto Ω_j. Then change of variables formula gives

$$\int_{\Omega_j} |f^* \circ \varphi||\varphi'(\xi)|\beta(\xi) d\sigma(\xi) = \int_{\varphi(\Omega_j)} |f^*(w)|\beta(\psi_j(w)) d\sigma(w)$$

where $\xi = \psi_j(w)$. Now denoting the characteristic function of $\varphi(\Omega_j)$ by χ_j we get

$$\int_{\partial\mathbb{D}} |f^* \circ \varphi||\varphi'(\xi)|\beta(\xi) d\sigma(\xi) = \int_{\varphi(\partial\mathbb{D})} |f^*(w)| \left(\sum_{j \geq 1} \chi_j(w)\beta(\psi_j(w)) \right) d\sigma(w)$$

$$= \int_{\varphi(\partial\mathbb{D})} |f^*(w)| \left(\sum_{j\geq 1} \beta(\xi_j(w)) \right) d\sigma(w)$$

so

$$\int_{\partial\mathbb{D}} |f^* \circ \varphi| d\mu_u \leq M \int_{\partial\mathbb{D}} |f^* \circ \varphi||\varphi'(\xi)|\beta(\xi)d\sigma(\xi)$$

$$= M \int_{\varphi(\partial\mathbb{D})} |f^*(w)| N_\beta^\varphi(w) d\sigma(w) \leq KM \int_{\partial\mathbb{D}} |f^*(w)|\beta(w)d\sigma(w)$$

hence C_φ is a bounded operator. □

Now we will examine the most general case where we will give a sufficiency condition for the boundedness of composition operators with arbitrary holomorphic symbols on the Poletsky-Stessin Hardy Spaces generated by an exhaustion function with finite Monge-Ampère mass:

Notation: Let ψ be a continuous, subharmonic, exhaustion function for $\mathbb{D}$ and $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function then the generalized Nevanlinna function is given as

$$N_\psi^\varphi(w) = \int_{\mathbb{D}} (-\psi) dd^c \log|\varphi - w|$$

Proposition 2.5.2. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a holomorphic function with $\varphi(0) = 0$ and suppose that ψ is a continuous, subharmonic exhaustion function for $\mathbb{D}$. If $\int_{\mathbb{D}} \frac{1}{(1-|\varphi|^2)^{\frac{p}{2}}} dd^c\psi < \infty$ and $\limsup_{|w|\to 1} \frac{N_\psi^\varphi(w)}{-\psi(w)} < \infty$ then C_φ is bounded on $H_\psi^p(\mathbb{D})$, $p \geq 1$.*

Proof. Suppose $f \in H_\psi^p(\mathbb{D})$, then $f \in H^p(\mathbb{D})$ and f can be factored as $f = Bg$ where B is a Blaschke product and g is a non-vanishing function in $H^p(\mathbb{D})$ such that $\|f\|_{H^p(\mathbb{D})} = \|g\|_{H^p(\mathbb{D})}$ and clearly $g^{\frac{p}{2}} \in H^2(\mathbb{D})$. Now suppose $g^{\frac{p}{2}}(z) = \sum a_n z^n$ then using Hölder inequality and Schwarz Lemma we get

$$\int_{\mathbb{D}} |f \circ \varphi|^p dd^c\psi \leq \int_{\mathbb{D}} (|g \circ \varphi|^{\frac{p}{2}})^2 dd^c\psi = \int_{\mathbb{D}} \left(\left| \sum a_n \varphi^n \right|^{\frac{p}{2}} \right)^2 dd^c\psi$$

$$\leq \int_{\mathbb{D}} \left[\left(\sum |a_n|^2 \right) \left(\sum |\varphi^n|^2 \right) \right]^{\frac{p}{2}} dd^c\psi \leq \|g\|^p_{H^p(\mathbb{D})} \left(\int_{\mathbb{D}} \frac{1}{(1-|\varphi|^2)^{\frac{p}{2}}} dd^c\psi \right)$$
$$= \|f\|^p_{H^p(\mathbb{D})} \left(\int_{\mathbb{D}} \frac{1}{(1-|\varphi|^2)^{\frac{p}{2}}} dd^c\psi \right) \leq M\|f\|^p_{H^p_\psi(\mathbb{D})}$$

By ([34], Theorem 9.2) we know that

$$\|f\circ\varphi\|^p_{H^p_\psi(\mathbb{D})} = \int_{\mathbb{D}} |f\circ\varphi|^p dd^c\psi + \int_{\mathbb{D}} \left(\int_{\mathbb{D}} (-\psi) dd^c \log|\varphi - w| \right) dd^c|f|^p$$

now define $N^{\varphi,f}_\psi(w,r) = \int_{T(r)} (-\psi) dd^c \log|f\circ\varphi - w|$ where $T(r) = \{z \in \mathbb{D} | \psi(\varphi(z)) > r\}$. Let $\gamma(r) = \sup \frac{N^{\varphi,f}_\psi(w,r)}{N^f_\psi(w)}$ where supremum is taken over all $f \in H^p_\psi(\mathbb{D})$ and $w \in \mathbb{D}$. Then by ([34], Theorem 8.3), if for some r_0, $\gamma(r_0) < \infty$ then $\int_{\mathbb{D}} \left(\int_{\mathbb{D}} (-\psi) dd^c \log|\varphi - w| \right) dd^c|f|^p \leq N\|f\|_{H^p_\psi(\mathbb{D})}$ so if we show that $\gamma(r)$ is finite then the result follows. Take the set of all points $\{w_i\} \in \mathbb{D}$ such that $f(w_i) = w$ and $\psi(w_i) > r$. Let $\cup_j A_{ij} = \varphi^{-1}(w_i)$ be the decomposition of the preimage of w_i under φ. The multiplicity of $f\circ\varphi$ on A_{ij} is equal to $m_i m_{ij}$ where m_{ij} is the multiplicity of φ on A_{ij} and m_i is the multiplicity of f at w_i, then $N^{\varphi,f}_\psi(w,r) = \sum_i m_i N^\varphi_\psi(w_i)$ and also $N^f_\psi(w) \geq \sum_i m_i(-\psi(w_i))$ hence

$$\frac{N^{\varphi,f}_\psi(w,r)}{N^f_\psi(w)} \leq \frac{\sum_i m_i N^\varphi_\psi(w_i)}{\sum_i m_i(-\psi(w_i))} \leq \max_i \left\{ \frac{N^\varphi_\psi(w_i)}{(-\psi(w_i))} \right\}$$

Thus $\gamma(r) \leq \sup_{r<|w|} \frac{N^\varphi_\psi(w)}{(-\psi(w))}$ on the other hand if $f(w) = w$ then $\frac{N^{\varphi,f}_\psi(w,r)}{N^f_\psi(w)} = \frac{N^\varphi_\psi(w)}{(-\psi(w))}$ Hence $\gamma(r) = \sup_{r<|w|} \frac{N^\varphi_\psi(w)}{(-\psi(w))}$ now since $\limsup_{|w|\to 1} \frac{N^\varphi_\psi(w)}{-\psi(w)} < \infty$ we have $\gamma(r) < \infty$ and the result follows. $\square$

2.6 Duality for $H_u^p(\mathbb{D})$

Let u be a continuous, negative, subharmonic exhaustion function such that $dd^c u$ has finite mass i.e. $\int_{\mathbb{D}} dd^c u < \infty$. In this section we will examine what the dual space of $H_u^p(\mathbb{D})$ will be for $p > 1$. First of all let us remind that the Monge-Ampère boundary measure $d\mu_u$ is given by $d\mu_u = \beta d\sigma$ where $\beta(\xi) = \int_{\mathbb{D}} P(z,\xi) dd^c u(z)$ and $\beta \in L^1(d\sigma)$. Now let us state our result on the dual of the Banach space $H_u^p(\mathbb{D})$ as follows :

Proposition 2.6.1. *For $1 < p < \infty$ the space $(H_u^p(\mathbb{D}))^* = L_u^q / \beta^{-1} H_0^q(\beta^{-q} d\mu_u)$ where $H_0^q(\beta^{-q} d\mu_u)$ is the space of all $H^1(\mathbb{D})$ functions f such that the boundary value function $f^* \in L^q(\beta^{-q} d\mu_u)$ and $\frac{1}{p} + \frac{1}{q} = 1$.*

Proof. In order to describe the dual of $H_u^p(\mathbb{D})$ we should find the annihilator of $H_u^p(\mathbb{D})$ in $(L_u^p)^* = L_u^q$. (Here by annihilator we mean all linear functionals $\phi \in (L_u^p)^*$ such that $\phi(f) = 0$ for all $f \in H_u^p(\mathbb{D})$). Now let ϕ be a bounded linear functional on L_u^p then by Riesz Representation Theorem it has a unique representation

$$\phi(f^*) = \int_{\partial\mathbb{D}} f^* g d\mu_u \quad , \quad g \in L_u^q$$

Now suppose ϕ is an element of annihilator of $H_u^p(\mathbb{D})$ then since $d\mu_u = \beta d\sigma$ we have

$$\phi(f^*) = \int_{\partial\mathbb{D}} f^* g d\mu_u = 0 \quad \forall f \in H_u^p(\mathbb{D})$$

Call $g\beta = \psi$ then since $\beta > 0$, using Hölder inequality we have

$$\int_{\partial\mathbb{D}} |\psi| d\sigma = \int_{\partial\mathbb{D}} |g|\beta d\sigma = \int_{\partial\mathbb{D}} |g| d\mu_u \leq \left(\int_{\partial\mathbb{D}} |g|^q d\mu_u \right)^{\frac{1}{q}} \left(\int_{\partial\mathbb{D}} 1 d\mu_u \right)^{\frac{1}{p}} < \infty$$

since $g \in L_u^q$ and mass of $d\mu_u$ is finite. Hence $\psi \in L^1(d\sigma)$ and since $\int_{\partial\mathbb{D}} f^* \psi d\sigma = 0 \quad \forall f \in H_u^p(\mathbb{D})$, surely $\int_0^{2\pi} e^{in\theta} \psi(e^{i\theta}) d\theta = 0$ for $n = 0, 1, 2....$ Therefore ψ is a holomorphic function with boundary value in $L^1(d\sigma)$ so $\psi \in H^1(\mathbb{D})$ and $\psi(0) = 0$. Now consider the space $\beta^{-1} H_0^q(\beta^{-q} d\mu_u)$, then

$g \in \beta^{-1} H_0^q(\beta^{-q} d\mu_u)$ since $\beta g = \psi \in H^1(\mathbb{D})$, $\psi(0) = 0$ and

$$\int_{\partial\mathbb{D}} |\psi|^q \beta^{-q} d\mu_u = \int_{\partial\mathbb{D}} |g|^q \beta^q \beta^{-q} d\mu_u = \int_{\partial\mathbb{D}} |g|^q d\mu_u < \infty$$

So annihilator of $H_u^p(\mathbb{D}) \subseteq \beta^{-1} H_0^q(\beta^{-q} d\mu_u)$.
For the inverse inclusion take $h \in \beta^{-1} H_0^q(\beta^{-q} d\mu_u)$ then βh is the boundary value of an $H^1(\mathbb{D})$ function f such that $f(0) = 0$, $f^* = \beta h$ and $f \in H_0^q(\beta^{-q} d\mu_u)$, now take an arbitrary $\alpha \in H_u^p(\mathbb{D})$ then

$$\int_{\partial\mathbb{D}} \alpha h d\mu_u = \int_{\partial\mathbb{D}} \alpha h \beta d\sigma = \int_{\partial\mathbb{D}} \alpha f^* d\sigma = 0$$

since $f(0) = 0$. Hence annihilator of $H_u^p(\mathbb{D})$ in L_u^q is the space $\beta^{-1} H_0^q(\beta^{-q} d\mu_u)$, then by ([17], Theorem 7.1) we have $(H_u^p(\mathbb{D}))^* = L_u^q / \beta^{-1} H_0^q(\beta^{-q} d\mu_u)$. $\square$

Chapter 3

Poletsky-Stessin Hardy Spaces in Hyperconvex Domains on $\mathbb{C}^n$, $n > 1$

In this chapter we will examine the Poletsky-Stessin Hardy classes on hyperconvex domains of $\mathbb{C}^n$ for $n > 1$. First we will start with the comparison results between Poletsky-Stessin Hardy spaces and the classical Hardy type spaces defined in various ways by different authors. Next we will look at Poletsky-Stessin Hardy spaces on the unit polydisc in $\mathbb{C}^n$ where most of the results are analogous to unit disc case. Moreover we will consider Poletsky-Stessin Hardy spaces on complex ellipsoids which are well known examples of domains of finite type. Contrary to one dimensional case, we will see that Poletsky-Stessin Hardy classes on complex ellipsoids are not contained in classical Hardy spaces therefore we study the existence of boundary values in detail. In order to understand the boundary behavior and approach regions for Poletsky-Stessin Hardy classes we will revisit Stein's arguments on maximal functions ([44]) in a general setting. Using this method of utilizing maximal functions we will obtain Fatou type theorems concerning the existence of boundary values along some approach regions. Lastly we will apply the methods in ellipsoid case to strictly convex domains in $\mathbb{C}^n$ with smooth boundary. This will enable us to recapture the classical results of Stein on admissible approach regions from a different point of view.

3.1 Comparison Between Classical and Poletsky-Stessin Hardy Spaces on Higher Dimensions

Let Ω be a smoothly bounded hyperconvex domain in $\mathbb{C}^n$ and φ be a continuous, plurisubharmonic, negative exhaustion function on Ω with finite Monge-Ampère mass and let g_z be the Pluricomlex Green Function of Ω with a logarithmic pole at the point $z \in \Omega$. In this section we will compare Poletsky-Stessin Hardy spaces $H^p_\varphi(\Omega)$, $1 \leq p \leq \infty$, with the Hardy type spaces considered by various authors. Before proceeding further let us recall some of the notation which will be used throughout this section :
Let ρ be a real valued function defined in a neighborhood of $\overline{\Omega}$ so that: ρ is of class $\mathcal{C}^2$, $\rho(z) < 0$ when $z \in \Omega$, $\{\rho = 0\} = \partial\Omega$, and $|\nabla\rho(\xi)| > 0$ when $\xi \in \partial\Omega$. Such a function ρ is called a *characterizing function* for the domain Ω. In ([44]) classical Hardy spaces on Ω are defined (without the assumption of hyperconvexity) as follows:

$$H^p \doteq \{f| \quad f \quad \textit{holomorphic in} \quad \Omega, \quad \sup_{r<0} \int_{\partial\Omega_r} |f|^p d\sigma_r < \infty\}$$

where $d\sigma_r$ is the induced surface area measure on $\partial\Omega_r$ and $1 \leq p < \infty$.

Proposition 3.1.1. *Let Ω be a hyperconvex domain and φ be a continuous, negative plurisubharmonic exhaustion function on Ω with finite Monge-Ampère mass. Then $H^p_\varphi(\Omega) \subseteq H^p_{g_z}(\Omega)$ for any $z \in \Omega$.*

Proof. First of all since $g_z(w)$ and $\varphi(w)$ are exhaustion functions for Ω they approach to 0 as w approaches to boundary however g_z is a maximal function on Ω hence near $\partial\Omega$ we have $\varphi \leq cg_z$ for some constant $c > 0$. Hence by ([34], Theorem 3.1) we have $\mu_{g_z,r}(\phi) \leq c^n \mu_{\varphi,r}(\phi)$ for any positive plurisubharmonic function ϕ, hence $H^p_\varphi(\Omega) \subseteq H^p_{g_z}(\Omega)$. $\square$

Theorem 3.1.1. *Suppose that Ω is a smoothly bounded, hyperconvex domain with a plurisubharmonic characterizing function ρ. Then $H^p(\Omega) \subseteq H^p_\rho(\Omega) = H^p_{g_z}(\Omega)$, $1 \leq p < \infty$.*

Proof. First we will show the equality between $H^p_\rho(\Omega)$ and $H^p_{g_z}(\Omega)$:
From the previous result we know that $H^p_\rho(\Omega) \subseteq H^p_{g_z}(\Omega)$. By Hopf lemma ([28], pg:73) there exists a positive constant $c > 0$ such that $g_z \leq -c\mathbf{dist}$ where **dist** is the distance function to boundary. Also since $\rho \in C^2(\overline{\Omega})$, from the mean value theorem we get $|\rho| \leq K\mathbf{dist}$ for some positive constant $K > 0$ and combining these two we get $g_z \leq M\rho$ for some constant $M > 0$ which depends only on c and K. Thus we have $\mu_{\rho,r}(\phi) \leq M\mu_{g_z,ar}(\phi)$, $a > 0$, for any positive plurisubharmonic function ϕ by ([34], Theorem 3.1) so $H^p_{g_z} \subseteq H^p_\rho(\Omega)$. Therefore $H^p_{g_z} = H^p_\rho(\Omega)$.
Now for the first inclusion since ρ is a smooth function we have $d\mu_{\rho,r} = d^c\rho \wedge (dd^c\rho^{n-1})|_{S(r)}$ ([13], Proposition 3.3) and $d\sigma = d^c\rho \wedge (dd^c|z|^2)^{n-1}|_{S(r)}$ ([36], Corollary 3.5) and these are both $(2n-1)$-dim differential forms on the $(2n-1)$-dim manifold so we have $d\mu_{\rho,r} = c(z)d\sigma(z)$. In a neighborhood of $\overline{\Omega}$, ρ is smooth and $\Omega \subset\subset \mathbb{C}^n$ so $c(z)$ is a bounded function. Hence,

$$\int_{S(r)} \phi d\mu_{\rho,r} = \int_{S(r)} \phi(z)c(z)d\sigma(z) \leq K \int_{S(r)} \phi(z)d\sigma(z)$$

$\Rightarrow H^p(\Omega) \subseteq H^p_\rho(\Omega)$. $\square$

Remark 11. In section 3.3 we will show that for an arbitrary exhaustion function φ we can have that $H^p_\varphi(\Omega)$ strictly contains $H^p(\Omega)$. However by ([34],Theorem 3.8) we see that under certain geometric conditions on the domain we can have equality of these two classes.

Theorem 3.1.2. *Let Ω be a strictly pseudoconvex domain then $H^p(\Omega) = H^p_{g_z} = H^p_\rho(\Omega)$.*

3.2 Poletsky-Stessin Hardy Spaces on Polydisc in $\mathbb{C}^n$, $n > 1$

In this section we will examine the characterization of Poletsky-Stessin Hardy spaces on the unit polydisc of $\mathbb{C}^n$ but first we will consider a special type of exhaustion functions in order to see the transfer of some important results concerning boundary value characterization from unit disc to the polydisc, without loss of generality suppose $n = 2$:

Let u_0, u_1 be exhaustion functions defined on the unit disc $\mathbb{D}$ with finite Monge-Ampère mass. Define the following plurisubharmonic function,

$$u_2(z, w) = \max\{v_1(z, w), v_2(z, w)\}$$

where $v_1(z, w) = u_0(z)$ and $v_2(z, w) = u_1(w)$ then we see that u_2 is a plurisubharmonic exhaustion function. Moreover from the facts given below we have $\int_{\mathbb{D}^2}(dd^c u_2)^2 < \infty$ and $supp(\mu_{u_2}) = \partial\mathbb{D} \times \partial\mathbb{D}$ (*) ([9],Cor.4.10).

We will show that in the sense of currents we have $d\mu_{u_2,r} = d\mu_{u_0,r} \wedge d\mu_{u_1,r}$. For this we will use the following facts :

$$(dd^c(\max\{u, v, r\}))^2 = dd^c(\max\{u, r\}) \wedge dd^c(\max\{v, r\})$$

where r is a constant and,

$$(dd^c(\max\{u(z), v(w)\}))^2 = (dd^c u(z)) \wedge (dd^c v(w))$$

(For the first equation see ([9]) and for the second one see ([6])). Hence,

$$d\mu_{u_0,r} \wedge d\mu_{u_1,r} = (dd^c(\max\{v_1, r\}) - \chi_{\mathbb{D}\setminus B_{v_1}(r)} dd^c v_1) \wedge (dd^c(\max\{v_2, r\}) - \chi_{\mathbb{D}\setminus B_{v_2}(r)} dd^c v_2)$$

$$= (dd^c(\max\{v_1, r\}) \wedge dd^c(\max\{v_2, r\})) - \chi_{\mathbb{D}\setminus B_{v_2}(r)} dd^c v_2 dd^c(\max\{v_1, r\})$$

$$-\chi_{\mathbb{D}\setminus B_{v_1}(r)} dd^c v_1 \wedge dd^c(\max\{v_2, r\}) + \chi_{\mathbb{D}\setminus B_{v_1}(r)} \chi_{\mathbb{D}\setminus B_{v_2}(r)} dd^c v_1 dd^c v_2$$

$$= (dd^c(\max\{v_1, r\}) \wedge dd^c(\max\{v_2, r\})) - \chi_{\mathbb{D}^2\setminus B_{u_2}(r)} (dd^c u_2)^2$$

$$= (dd^c(\max\{v_1, v_2, r\}))^2 - \chi_{\mathbb{D}^2\setminus B_{u_2}(r)} (dd^c u_2)^2 = d\mu_{u_2,r}$$

$$\Rightarrow d\mu_{u_2,r} = d\mu_{u_0,r} \wedge d\mu_{u_1,r} \tag{3.2.1}$$

and as $r \to 0$ by (*) and (3.2.1) we have:

$$d\mu_{u_2} = d\mu_{u_0} \wedge d\mu_{u_1}$$

which is in fact the product measure for the measures $d\mu_{u_0}$ and $d\mu_{u_1}$.
In the classical theory of the Hardy spaces of unit disc, the existence of boundary values along admissible approach regions is well-known. When Hardy spaces are generalized to polydisc in $\mathbb{C}^n$, new phenomena emerged since the Poisson kernel and the associated Poisson integral representation of holomorphic functions are carried only on a part of the boundary, namely the distinguished boundary $\mathbb{T}^n$ and as a consequence approach region is also restricted to the product of non-tangential approach regions ([47]). As we have mentioned, in several variables the existence of boundary values is not yet well understood. In ([35]), Poletsky approached the boundary value problem from an abstract point of view where no assumptions about the boundary smoothness are made and consequently no natural definition of approach regions could be given. He introduced the so called weak and strong limit values and when these two limit values are equal he called it *boundary value.* In ([35], pg:22) he introduced a generalization for radial limit value and using this general definition of radial limits he showed that if a Borel function on a hyperconvex domain has radial limits then it has boundary values and these boundary values are exactly the radial limit values ([35], Theorem 5.5). In particular these generalized radial limit values coincide with the usual radial

limit values in the polydisc case ([35], Theorem 7.6).

As a consequence of Theorem 1.4.1 for any exhaustion function u we have $H^p_u(\mathbb{D}^n) \subset H^p_g(\mathbb{D}^n)$ and also we have seen that on the polydisc if we choose our exhaustion function as the Pluricomplex Green function then the Poletsky-Stessin Hardy space coincides with the classical Hardy space of the polydisc. Therefore, for any exhaustion function u on the polydisc, we have $H^p_u(\mathbb{D}^n) \subset H^p_g(\mathbb{D}^n) = H^p(\mathbb{D}^n)$. Functions in $H^p(\mathbb{D}^n)$ have non-tangential limit values over the non-tangential approach region $\Gamma = \Gamma_1 \times \Gamma_2 \times ... \times \Gamma_n$ by ([47], Theorem 4.13) therefore we automatically have boundary values for the Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{D}^n)$ over the polydisc.

We will now characterize the Poletsky-Stessin Hardy classes through their boundary values first with the special choice of exhaustion functions that we mentioned at the beginning of this chapter :

Theorem 3.2.1. *Let $f \in H^p(\mathbb{D}^n)$ be an analytic function. Then $f^* \in L^p(d\mu_{u_n}, \mathbb{T}^n)$ if and only if $f \in H^p_{u_n}(\mathbb{D}^n)$. Moreover the operator which takes $f \in H^p_{u_n}(\mathbb{D}^n)$ to $f^* \in L^p(d\mu_{u_n}, \mathbb{T}^n)$ is an isometry between $H^p_{u_n}(\mathbb{D}^n)$ and a closed subspace of $L^p(d\mu_{u_n}, \mathbb{T}^n)$.*

Proof. Without loss of generality suppose $n = 2$. First suppose that $p > 1$ and let $f \in H^p(\mathbb{D}^2)$ then f^* exists ([39]). Suppose $f^* \in L^p(d\mu_{u_2})$, then by ([39],pg:53) we have that

$$f(z, w) = \int_0^{2\pi} \int_0^{2\pi} P_{r_1}(\theta_1 - t) P_{r_2}(\theta_2 - \theta) f^*(e^{it}, e^{i\theta}) dt d\theta$$

so

$$\int_{S_{u_2}(r)} |f(z, w)|^p d\mu_{u_2,r}$$

$$= \int_{S_{u_2}(r)} \left| \int_0^{2\pi} \int_0^{2\pi} P_{r_1}(\theta_1 - t) P_{r_2}(\theta_2 - \theta) f^*(e^{it}, e^{i\theta}) dt d\theta \right|^p d\mu_{u_2,r}$$

then by Hölder Inequality applied with measure $P_{r_1}(\theta_1 - t) P_{r_2}(\theta_2 - \theta) dt d\theta$

we have

$$\leq \int_0^{2\pi}\int_0^{2\pi}\left(\int_{S_{u_2}(r)} P_{r_1}(\theta_1 - t)P_{r_2}(\theta_2 - \theta)d\mu_{u_2,r}\right)|f^*(e^{it}, e^{i\theta})|^p dtd\theta$$

Now since $d\mu_{u_2,r} = d\mu_{u,r} \wedge d\mu_{u,r}$ we have

$$\int_{S_{u_2}(r)} P_{r_1}(\theta_1 - t)P_{r_2}(\theta_2 - \theta)d\mu_{u_2,r} = \int_{S_u(r)\times S_u(r)} P(z, e^{it})P(w, e^{i\theta})d\mu_{u,r}(z)d\mu_{u,r}(w)$$

$$= \left(\int_{S_u(r)} P(z, e^{it})d\mu_{u,r}(z)\right)\left(\int_{S_u(r)} P(w, e^{i\theta})d\mu_{u,r}(w)\right)$$

then by Lelong-Jensen formula we have

$$= \left(\int_{B_u(r)} P(z, e^{it})dd^c u(z)\right)\left(\int_{B_u(r)} P(w, e^{i\theta})dd^c u(w)\right)$$

and as $r \to 0$, by Monotone Convergence Theorem we get

$$\lim_{r\to 0}\int_{S_{u_2}(r)} P_{r_1}(\theta_1 - t)P_{r_2}(\theta_2 - \theta)d\mu_{u_2,r} = \left(\int_{\mathbb{D}} P(z, e^{it})dd^c u(z)\right)\left(\int_{\mathbb{D}} P(w, e^{i\theta})dd^c u(w)\right)$$

$$= \beta(t)\beta(\theta)$$

(β is as defined in Section 2.1.2)

$$\Rightarrow \|f\|_{H^p_{u_2}(\mathbb{D}^2)} \leq \int_0^{2\pi}\int_0^{2\pi} |f^*(e^{it}, e^{i\theta})|^p \beta(t)\beta(\theta)dtd\theta = \int_{\partial\mathbb{D}\times\partial\mathbb{D}} |f^*|^p d\mu_{u_2} < \infty$$

$\Rightarrow f \in H^p_{u_2}(\mathbb{D}^2)$

For the converse suppose $f \in H^p_{u_2}(\mathbb{D}^2)$,

$$\int_{\partial\mathbb{D}\times\partial\mathbb{D}} |f^*(\xi, \eta)|^p d\mu_{u_2}(\xi, \eta) = \int_{\partial\mathbb{D}\times\partial\mathbb{D}} |f^*(\xi, \eta)|^p \beta(\xi)\beta(\eta)d\sigma(\xi)d\sigma(\eta)$$

$$= \int_{\partial\mathbb{D}\times\partial\mathbb{D}} |f^*(\xi,\eta)|^p \left(\int_{\mathbb{D}\times\mathbb{D}} P(z,\xi)P(w,\eta) dd^c u(z) dd^c u(w) \right) d\sigma(\xi) d\sigma(\eta)$$

$$\int_{\mathbb{D}\times\mathbb{D}} \left(\int_{\partial\mathbb{D}\times\partial\mathbb{D}} |f^*(\xi,\eta)|^p P(z,\xi)P(w,\eta) d\sigma(\xi) d\sigma(\eta) \right) dd^c u(z) dd^c u(w)$$

The integral inside the parenthesis is the Poisson integral of $|f^*(\xi,\eta)|^p$. Call this expression $H(z,w)$. Now consider the function $v(a,b) = \max\left\{\log\left|\frac{a-z}{1-a\bar{z}}\right|, \log\left|\frac{b-w}{1-b\overline{w}}\right|\right\}$ then $d\mu_v(\xi,\eta) = P(z,\xi)P(w,\eta)d\sigma(\xi)d\sigma(\eta)$ hence, by Lelong-Jensen formula we have

$$H(z,w) = \int_{\partial\mathbb{D}\times\partial\mathbb{D}} |f^*(\xi,\eta)|^p d\mu_v(\xi,\eta) = |f(z,w)|^p - \int_{\mathbb{D}\times\mathbb{D}} v(a,b)(dd^c v(a,b))(dd^c|f(a,b)|^p)$$

By ([39], Theorem 3.2.4), $H(z,w)$and $|f(z,w)|^p$ have the same boundary values. Now,

$$\int_{\partial\mathbb{D}\times\partial\mathbb{D}} |f^*(\xi,\eta)|^p d\mu_{u_2} = \int_{\mathbb{D}\times\mathbb{D}} H(z,w) dd^c u(z) dd^c u(w)$$

$$= \int_{\mathbb{D}\times\mathbb{D}} |f(z,w)|^p (dd^c u_2)^2 - \int_{\mathbb{D}\times\mathbb{D}} \left(\int_{\mathbb{D}\times\mathbb{D}} v(a,b)(dd^c v(a,b))(dd^c|f(a,b)|^p) \right) (dd^c u_2)^2$$

call $g(z,w) = \int_{\mathbb{D}\times\mathbb{D}} v(a,b)(dd^c v(a,b))(dd^c|f(a,b)|^p)$ then,

$$\int_{\mathbb{D}\times\mathbb{D}} g(z,w)(dd^c u_2)(dd^c u_2) = \int_{\mathbb{D}\times\mathbb{D}} u_2(dd^c g(z,w))(dd^c u_2) + \underbrace{\int_{\partial(\mathbb{D}\times\mathbb{D})} g(z,w)(dd^c u_2) d^c u_2}_{0}$$

$$- \underbrace{\int_{\partial(\mathbb{D}\times\mathbb{D})} u_2 d^c(g(z,w))(dd^c u_2)}_{0}$$

by ([15],Formula 3.1, pg:144) and the last two integrals are zero since $g(z,w)$ and $u_2(z,w)$ are both zero on the boundary. Hence,

$$\int_{\mathbb{D}\times\mathbb{D}} g(z,w)(dd^c u_2)^2 = \int_{\mathbb{D}\times\mathbb{D}} u_2(dd^c g(z,w))(dd^c u_2)$$

$$= \int_{\mathbb{D}\times\mathbb{D}} u_2 \left(\int_{\mathbb{D}\times\mathbb{D}} (dd^c v(a,b))^2 (dd^c |f(a,b)|^p) \right) (dd^c u_2) = \int_{\mathbb{D}\times\mathbb{D}} u_2 dd^c |f(z,w)|^p dd^c u_2$$

so we get

$$\int_{\partial\mathbb{D}\times\partial\mathbb{D}} |f^*(\xi,\eta)|^p d\mu_{u_2} = \int_{\mathbb{D}\times\mathbb{D}} |f(z,w)|^p (dd^c u_2)^2 - \int_{\mathbb{D}\times\mathbb{D}} u_2 dd^c |f(z,w)|^p dd^c u_2$$

$$= \|f\|_{H^p_{u_2}(\mathbb{D}^2)} < \infty$$

$\Rightarrow f^* \in L^p(d\mu_{u_2}, \partial\mathbb{D}\times\partial\mathbb{D})$. The case where $p = 1$ is a straightforward application of the procedure above.

Since $H^p_{u_n}(\mathbb{D}^n)$ is a closed subspace of $L^p(d\mu_{u_n}, \mathbb{T}^n)$, from the chain of equations above we deduce that the operator which takes $f \in H^p_{u_n}(\mathbb{D}^n)$ to $f^* \in L^p(d\mu_{u_n}, \mathbb{T}^n)$ is an isometry between $H^p_{u_n}(\mathbb{D}^n)$ and a closed subspace of $L^p(d\mu_{u_n}, \mathbb{T}^n)$. □

It is important to note that for this specific choice of exhaustion function, we obtained a product measure on the torus. However if we take an arbitrary exhaustion function u with finite mass, we may not end up with a product measure since the Poisson kernel is not pluriharmonic when $n > 1$. In the next result we generalize the boundary value characterization for the Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{D}^n)$ to the most general case where the exhaustion function u is a continuous, negative, plurisubharmonic function with finite Monge-Ampère mass. Although we have the complete characterization for boundary values and an isomorphism between $H^p_u(\mathbb{D}^n)$ and a closed subspace of $L^p(d\mu_u, \mathbb{T}^n)$, in this case we may lose the isometry since we may not have the Monge-Ampère measure as a product measure.

Theorem 3.2.2. *Let $f \in H^p(\mathbb{D}^n)$, $1 \le p < \infty$, be a holomorphic function. Then $f \in H^p_u(\mathbb{D}^n)$ if and only if the boundary value function $f^* \in L^p(d\mu_u, \mathbb{T}^n)$ where $\mathbb{T}^n$ is torus in $\mathbb{C}^n$.*

Proof. (Without loss of generality assume n=2) Let u be a continuous, negative, plurisubharmonic exhaustion function for $\mathbb{D}^n$ with finite Monge-Ampère mass, then by ([9], Cor:4.10) we know that $supp(d\mu_u) = \mathbb{T}^n$. Let us first give the relation between the Monge-Ampère measure $d\mu_u$ and the Euclidean measure on the torus. Let $\varphi \in C(\mathbb{T}^2)$ be a continuous function and denoting the Poisson integral of this function also as φ we have,

$$\int_{S_u(r)} \varphi d\mu_{u,r} = \int_{S_u(r)} \left(\int_{\mathbb{T}^2} P(z,\xi)P(w,\eta)\varphi(\xi,\eta)d\sigma(\xi)d\sigma(\eta) \right) d\mu_{u,r}$$

$$= \int_{\mathbb{T}^2} \left(\int_{S_u(r)} P(z,\xi)P(w,\eta)d\mu_{u,r} \right) \varphi(\xi,\eta)d\sigma(\xi)d\sigma(\eta)$$

and $d\mu_u$ is the weak-* limit of $d\mu_{u,r}$ and the integral in parenthesis in the last line is increasing in r so by using monotone convergence theorem we have

$$\int_{\mathbb{T}^2} \varphi d\mu_u = \int_{\mathbb{T}^2} \left(\lim_{r\to 0} \int_{S_u(r)} P(z,\xi)P(w,\eta)d\mu_{u,r} \right) \varphi(\xi,\eta)d\sigma(\xi)d\sigma(\eta)$$

and defining

$$\beta(\xi,\eta) = \lim_{r\to 0} \int_{S_u(r)} P(z,\xi)P(w,\eta)d\mu_{u,r}$$

we have $d\mu_u = \beta(\xi,\eta)d\sigma(\xi)d\sigma(\eta)$ and since the exhaustion function u has finite Monge-Ampère mass we have $\beta(\xi,\eta) \in L^1(d\sigma(\xi), d\sigma(\eta))$. Now suppose $f^* \in L^p(d\mu_u, \mathbb{T}^n)$,

$$\int_{S_u(r)} |f|^p d\mu_{u,r} = \int_{S_u(r)} \left| \int_{\mathbb{T}^2} P(z,\xi)P(w,\eta)f^*(\xi,\eta)d\sigma(\xi)d\sigma(\eta) \right|^p d\mu_{u,r}$$

$$\leq \int_{\mathbb{T}^2} \left(\int_{S_u(r)} P(z,\xi)P(w,\eta)d\mu_{u,r} \right) |f^*(\xi,\eta)|^p d\sigma(\xi)d\sigma(\eta)$$

now by monotone convergence theorem we have,

$$\lim_{r\to 0} \int_{S_u(r)} |f|^p d\mu_{u,r} \leq \int_{\mathbb{T}^2} \left(\lim_{r\to 0} \int_{S_u(r)} P(z,\xi)P(w,\eta)d\mu_{u,r} \right) |f^*(\xi,\eta)|^p d\sigma(\xi)d\sigma(\eta)$$

$$= \int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p \beta(\xi,\eta) d\sigma(\xi) d\sigma(\eta) = \|f^*\|^p_{L^p_u} < \infty$$

hence $f \in H^p_u(\mathbb{D}^2)$. Conversely now suppose we have a holomorphic function $f \in H^p_u(\mathbb{D}^2)$.

$$\int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p d\mu_u = \int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p \left(\lim_{r\to 0} \int_{S_u(r)} P(z,\xi)P(w,\eta) d\mu_{u,r} \right) d\sigma(\xi) d\sigma(\eta)$$

by Fatou's Lemma we have then

$$\leq \lim_{r\to 0} \int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p \left(\int_{S_u(r)} P(z,\xi)P(w,\eta) d\mu_{u,r} \right) d\sigma(\xi) d\sigma(\eta)$$

$$= \lim_{r\to 0} \int_{S_u(r)} \left(\int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p P(z,\xi)P(w,\eta) d\sigma(\xi) d\sigma(\eta) \right) d\mu_{u,r}$$

now we will examine the integral in parenthesis

$$\int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p P(z,\xi)P(w,\eta) d\sigma(\xi) d\sigma(\eta)$$

which is equal to the following by ([35],Cor.7.4)

$$= |f(z,w)|^p + \int_{\mathbb{D}^2} (-\upsilon_{(z,w)}(a,b)(dd^c \upsilon_{(z,w)}(a,b))(dd^c|f(a,b)|^p)$$

where $\upsilon_{(z,w)}(a,b) = \max\left\{ \log\left|\frac{a-z}{1-a\bar{z}}\right|, \log\left|\frac{b-w}{1-b\bar{w}}\right| \right\}$. Then

$$= \lim_{r\to 0} \int_{S_u(r)} \left(\int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p P(z,\xi)P(w,\eta) d\sigma(\xi) d\sigma(\eta) \right) d\mu_{u,r}$$

$$= \lim_{r\to 0} \int_{S_u(r)} |f(z,w)|^p d\mu_{u,r}$$

$$= \underbrace{\lim_{r\to 0} \int_{S_u(r)} \left(\int_{\mathbb{D}^2} (-\upsilon_{(z,w)}(a,b)(dd^c \upsilon_{(z,w)}(a,b))(dd^c|f(a,b)|^p) \right) d\mu_{u,r}}_{0}$$

second integral is 0 since $v_{(z,w)}(a,b)$ converges to 0 uniformly on compact subsets as (z,w) converges to boundary ([10]). Therefore we have

$$\int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p d\mu_u \leq = \lim_{r\to 0} \int_{S_u(r)} \left(\int_{\mathbb{T}^2} |f^*(\xi,\eta)|^p P(z,\xi) P(w,\eta) d\sigma(\xi) d\sigma(\eta) \right) d\mu_{u,r}$$

$$\leq = \lim_{r\to 0} \int_{S_u(r)} |f(z,w)|^p d\mu_{u,r} = \|f\|^p_{H^p_u(\mathbb{D}^2)} < \infty$$

so $f^* \in L^p(d\mu_u)$ when $f \in H^p_u(\mathbb{D}^2)$. $\square$

As an immediate consequence of this boundary value characterization we have the following corollary which enables us to see the inclusions between the Poletsky-Stessin Hardy classes in terms of their boundary values:

Corollary 3.2.1. *Let f be a holomorphic function such that $f \in H^t_u(\mathbb{D}^n)$ for some $t \geq 1$. If the boundary value f^* belongs to $L^s_u(\mathbb{T}^n)$ for some $s > t$ then $f \in H^s_u(\mathbb{D}^n)$.*

Remark 12. As in one dimensional case, there exists an exhaustion function u for the polydisc with finite Monge-Ampère mass so that $H^p_u(\mathbb{D}^n) \subsetneq H^p(\mathbb{D}^n)$. We can see this by an example on $\mathbb{D}^n$ that is similar to the disc case. Let $u_n(z_1, z_2, .., z_n) = \max\{u(z_1), u(z_2), .., u(z_n)\}$ where u is the exhaustion function that we constructed in the disc example. Now first of all consider the holomorphic function $f(z_1, z_2, .., z_n) = \dfrac{1}{(1-z_1)^{\frac{3}{4}}(1-z_2)^{\frac{3}{4}}..(1-z_n)^{\frac{3}{4}}}$ on $\mathbb{D}^n$ then from the arguments that were given in the disc example we deduce that $f(z_1, z_2, .., z_n) \in H^1(\mathbb{D}^n)$ but $f(z_1, z_2, .., z_n) \notin H^1_u(\mathbb{D}^n)$ by combining the previous result with the example in the disc case.

3.3 Poletsky-Stessin Hardy Spaces on Complex Ellipsoids

For domains in $\mathbb{C}^n$ we will next consider the complex ellipsoids which are considered as model cases for domains of finite type. It should be noted that although complex ellipsoids are convex domains they are not strictly pseudoconvex since they have Levi flat points at the boundary. The complex ellipsoid $\mathbb{B}^{\mathbf{p}} \in \mathbb{C}^n$ is given as

$$\mathbb{B}^{\mathbf{p}} = \{z \in \mathbb{C}^n, \rho(z) = \sum_{j=1}^{n} |z_j|^{2p_j} - 1 < 0\}$$

where $\mathbf{p} = (p_1, p_2, ..., p_n) \in \mathbb{Z}^n$. One can easily see that $u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + ... + |z_n|^{2p_n})$ is an exhaustion function for $\mathbb{B}^{\mathbf{p}}$ so we can consider the Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{B}^{\mathbf{p}})$ associated with this exhaustion function. In ([22]), Hansson considered Hardy type spaces where the growth condition is determined by the measures which are restrictions of the measure $\partial\rho \wedge (\overline{\partial}\partial\rho)^{n-1}$ on the sublevel sets of the defining function ρ. If we choose the exhaustion function u then these measures are in fact the Monge-Ampère measures $d\mu_{u,r}$ therefore the Poletsky-Stessin Hardy classes $H^p_u(\mathbb{B}^{\mathbf{p}})$ coincide with the Hardy type classes defined by Hansson. His main results contain a generalization of the classical M.Riesz theorem to Cauchy-Fantappie integrals of $L^2_u(\partial\mathbb{B}^{\mathbf{p}})$ functions and boundedness of Cauchy-Fantappie integral operator H on $BMO(\partial\mathbb{B}^{\mathbf{p}})$ (For details see [22]). In this section we will show that unlike the one variable case, for $n > 1$ Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{B}^{\mathbf{p}})$ are not included in the classical Hardy spaces $H^p(\mathbb{B}^{\mathbf{p}})$ on complex ellipsoids. Hence in this case we do not automatically inherit the existence of boundary values from the theory of classical Hardy spaces. Existence and the behavior of boundary values have not been considered in ([22]) so we will start with exhibiting the existence of the radial limits for holomorphic functions in $H^1_u(\mathbb{B}^{\mathbf{p}})$.

Theorem 3.3.1. *Let $f \in H^1_u(\mathbb{B}^{\mathbf{p}})$ be a holomorphic function. Then the radial limit function $f^*(\xi) = \lim_{\tilde{r}\to 1} f(\tilde{r}\xi)$, $\xi \in \partial\mathbb{B}^{\mathbf{p}}$ exists μ_u-almost everywhere and $f^* \in L^1_{\mu_u}(\partial\mathbb{B}^{\mathbf{p}})$.*

Proof. Let $\mathbb{B}^{\mathbf{p}}$ be the complex ellipsoid determined by the exhaustion function $u(z) = \log(|z_1|^{2p_1} + |z_2|^{2p_2} + ... + |z_n|^{2p_n})$ and let $\xi = (\xi_1, \xi_2, .., \xi n) \in \partial\mathbb{B}^{\mathbf{p}}$, $t \in \mathbb{D}$. Suppose that E is the ellipse which is the intersection of the complex line joining 0 to ξ and the ellipsoid $\mathbb{B}^{\mathbf{p}}$. An exhaustion function for E is $g_E(t) = \log(A_1|t|^{2p_1} + A_2|t|^{2p_2} + .. + A_n|t|^{2p_n})$ where $A_i = |\xi_i|^{2p_i}$, $1 \leq i \leq n$. The Monge-Ampère measure associated with the exhaustion function u is $d\mu_{u,r} = d^c u \wedge dd^c u|_{S_u(r)}$ and let A_0 be the $n-1$-dimensional manifold of complex lines passing through the point $0 \in \mathbb{B}^{\mathbf{p}}$ ([45]). Now take $f \in H^1_u(\mathbb{B}^{\mathbf{p}})$ then

$$\int_{S_u(r)} |f| d\mu_{u,r} = \int_{S_u(r)} |f|(d^c u \wedge dd^c u) = \int_{A_0} \left(\int_{l_z \cap S_u(r)} |f| d^c u \right) \omega$$

where we have the pull-back measure $\pi^*\omega = dd^c u$ and $\pi : \bar{\mathbb{B}}^{\mathbf{p}} \to A_0$ is the function given by $\pi(z) = [0, z] = l_z$ with l_z being the line joining 0 and z.
We can use the above generalization of Fubini theorem since π is a submersion and $\pi|_{supp(d^c u)}$ is proper ([15], pg:17).
The measure $d^c u$ on $l_z \cap S_u(r)$ is equal to $d^c g_E(t)$ on $S_g(r)$ and since it is a smoothly bounded domain $d^c g_E(t)$ on $S_g(r) = d\mu_{g,r}$ so

$$\int_{S_u(r)} |f| d\mu_{u,r} = \int_{A_0} \left(\int_{S_g(r)} |f| d\mu_{g,r} \right) \omega$$

and by Fatou's lemma $\int_{A_0} \left(\liminf_{r\to 0} \int_{S_g(r)} |f| d\mu_{g,r} \right) \omega < \infty$ for $f \in H^1_u(\mathbb{B}^{\mathbf{p}})$. This implies that for ω-a.e. line $\lim_{r\to 0} \int_{S_g(r)} |f| d\mu_{g,r} < \infty$ so $f \in H^1_g(E)$ and it has radial boundary values $d\sigma(\simeq d\mu_g)$ almost everywhere ([44]). Since f^* is the pointwise limit of measurable functions it is measurable and consider the set $A = \{\xi \in \partial\mathbb{B}^{\mathbf{p}}, f^*(\xi) \quad does \quad not \quad exist\}$, then

$$\int_{\partial\mathbb{B}^{\mathbf{p}}} \chi_A d\mu_u = \int_{A_0} \left(\int_{\partial E} \chi_A(\eta) d\mu_g(\eta) \right) \omega$$

Since $f \in H^1_g(E)$, it has radial limit values $d\mu_g$-a.e. so the integral inside is 0 and we have $\int_{\partial\mathbb{B}^{\mathbf{p}}} \chi_A d\mu_u = 0$. Therefore $f^*(\xi)$ exists μ_u-a.e. Moreover for an analytic function $f \in H^1_g(E)$ we know that the boundary function $f^* \in L^1(\partial E)$ so we have

$$\int_{\partial\mathbb{B}^{\mathbf{p}}} |f^*| d\mu_u = \int_{A_0} \left(\int_{\partial E} |f^*| d\mu_g \right) \omega < \infty$$

hence $f^* \in L^1_{\mu_u}(\partial\mathbb{B}^{\mathbf{p}})$. □

Now we have two Hardy type spaces on $\mathbb{B}^{\mathbf{p}}$, the first one is the Poletsky-Stessin Hardy space $H^1_u(\mathbb{B}^{\mathbf{p}})$ and the other one is $H^1(\mathbb{B}^{\mathbf{p}})$ which is defined with respect to surface area measure in accordance with Stein's definition. We will now show that these spaces are not equal. In fact in contrast to the one variable case Poletsky-Stessin Hardy class strictly contains the classical Hardy space.

Proposition 3.3.1. *Let $\mathbb{B}^{\mathbf{p}}$ be the complex ellipsoid. Then there exists an exhaustion function u such that $H^1(\mathbb{B}^{\mathbf{p}}) \subsetneq H^1_u(\mathbb{B}^{\mathbf{p}})$.*

Proof. We will explicitly construct the exhaustion function u by taking $n = 2$ and $\mathbf{p} = (1, 2)$. First of all the relation between $d\sigma$ and $d\mu_u$ on $\partial\mathbb{B}^2$ is given by $K_1|\xi_2|^2 d\sigma \leq d\mu_u \leq K_2|\xi_2|^2 d\sigma$ for some $K_1, K_2 > 0$ (depending only on dimension and $p = (1, 2)$), now consider the analytic function $f(z_1, z_2) = \dfrac{1}{(1 - z_1^2)^{2\alpha}}$ where $\frac{2}{16} < \alpha < \frac{4}{16}$. We have

$$\int_{\partial\mathbb{B}^2} |f^*||\xi_2|^2 d\sigma = \int_{|\xi_2|^4 < 1} \left(\int_{|\xi_1| = \sqrt{1 - |\xi_2|^4}} |f^*| d\xi_1 \right) |\xi_2|^2 d\xi_2$$

$$= \int_{|\xi_2|^4<1} \left(\int_0^{2\pi} \frac{1}{|1-(\sqrt{1-|\xi_2|^4}e^{i\theta})^2|^{2\alpha}} d\theta \right) |\xi_2|^2 d\xi_2$$

$$= \int_{|\xi_2|^4<1} \left(\int_0^{2\pi} \frac{1}{|1-e^{2i\theta}+|\xi_2|^4 e^{2i\theta}|^{2\alpha}} d\theta \right) |\xi_2|^2 d\xi_2$$

Now we will consider the behavior of the inside integral near the point $\{1\}$ i.e. as $\theta \to 0$ (this is the only problematic point as $|\xi_2| \to 0$).

$$\lim_{\theta \to 0} \frac{(1-2(1-|\xi_2|^4)\cos 2\theta + (1-|\xi_2|^4)^2)^\alpha}{|\xi_2|^{8\alpha}} = 1$$

so our integral becomes for $t > 0, \delta > 0$

$$= \int_{|\xi_2|^4<1} \left(2\int_t^{\pi-t} \frac{1}{|1-e^{2i\theta}+|\xi_2|^4 e^{2i\theta}|^{2\alpha}} d\theta \right) |\xi_2|^2 d\xi_2 + 2\int_{B_\delta(0)} \frac{2t}{|\xi_2|^{8\alpha}} |\xi_2|^2 d\xi_2$$

$$+2\int_{|\xi_2|^4<1 \setminus B_\delta(0)} \frac{2t}{|\xi_2|^{8\alpha}} |\xi_2|^2 d\xi_2$$

since we are away from the singularity first and third integrals are finite and if we take $\frac{2}{16} < \alpha < \frac{4}{16}$ then second integral is also finite and we have $f \in H^1_u(\mathbb{B}^2)$ but $f \notin H^1(\mathbb{B}^2)$ since for this choice of α

$$\int_{|\xi_2|^4<1} \left(\int_0^{2\pi} \frac{1}{|1-e^{2i\theta}+|\xi_2|^4 e^{2i\theta}|^{2\alpha}} d\theta \right) d\xi_2$$

diverges. □

In the previous results we have shown that for the functions in the Poletsky-Stessin Hardy class $H^p_u(\mathbb{B}^{\mathbf{p}})$ we have the radial limit values and throughout the following arguments we will study the behavior of these boundary values in detail. In the classical Hardy space theory on strictly pseudoconvex domains, Stein showed the existence of boundary values along admissible approach regions that are more general than the radial approach. Throughout the rest of the section we will show that for the functions in the Poletsky-Stessin Hardy class $H^p_u(\mathbb{B}^{\mathbf{p}})$ boundary values along admissible approach regions exist. Although we use the general idea in Stein's classical method, our approach differs in two aspects, respectively the use of Cauchy-Fantappie kernel instead of Poisson kernel and the use of radial limits existence of which is shown before. In the study of the boundary behavior of holomorphic functions having the boundary of the domain as a space of homogenous type seems to be a leitmotif because one of the most commonly used methods in order to understand boundary behavior is to use maximal functions ([44], Theorem 3) and the natural setting for this type of analysis is homogenous spaces. Therefore we will start with recalling the properties of homogenous spaces and then as an application of this classical method we will show that polynomials are dense in the Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{B}^{\mathbf{p}})$ on complex ellipsoids. Before proceeding our arguments in $\mathbb{C}^n$ with maximal functions, let us first mention the spaces of homogenous type in $\mathbb{C}^n$ for which we need the following definitions :

Definition 17. Suppose that we are given a space X which is equipped with a function $\rho : X \times X \to \mathbb{R}^+$ such that

- $\rho(x, y) = 0$ if and only if $x = y$
- $\rho(x, y) = \rho(y, x)$
- There is a constant $C_1 > 0$ such that if $x, y, z \in X$ then $\rho(x, z) \leq C_1[\rho(x, y) + \rho(y, z)]$

ρ is called a quasi-metric for the space X.

We will denote the balls in this quasimetric by

$$B(x, r) = \{y \in X : \quad \rho(x, y) < r\}$$

Definition 18. Assume that the space X is equipped with a quasi-metric ρ and a regular Borel measure μ on X. We say that (X, ρ, μ) is a space of homogenous type if the following conditions are satisfied:

- For each $x \in X$ and $r > 0$, $0 < \mu(B(x,r)) < \infty$

- (**Doubling Condition**) There is a constant $C_2 > 0$ such that for any $x \in X$ and $r > 0$ we have $\mu(B(x,2r)) \leq C_2\mu(B(x,r))$

Let $\Omega \subset\subset \mathbb{C}^n$ be a smoothly bounded domain such that we have a quasi-metric ρ on $\overline{\Omega}$ and a regular Borel measure μ on $\partial\Omega$. Let $K(z,\xi) : \Omega\times\partial\Omega \to \mathbb{C}$ be a kernel such that $K(z,\xi) \in L^1(d\mu)$ for $z \in \Omega$, $\xi \in \partial\Omega$. Let us consider the integral operator determined by $K(z,\xi)$ for an $L^p(d\mu)$ function f^*,

$$Kf^*(z) = \int_{\partial\Omega} f^*(\xi)K(z,\xi)d\mu(\xi)$$

and define the associated maximal function as

$$Mf^*(\xi) = \sup_{\varepsilon>0} \frac{1}{\mu(B(\xi,\varepsilon))} \int_{B(\xi,\varepsilon)} |f^*|d\mu$$

From the corresponding results in literature (see eg. [44], Theorem 2; [47], chapter 14) the fundamental theorem of the theory of singular operators which is adopted to our setting can be stated as:

Theorem 3.3.2. *Suppose $f^* \in L^p(d\mu_u)$ and $1 \leq p \leq \infty$*
(a) $\|Mf^*\|_p \leq A_p\|f^*\|_p$ for $1 < p \leq \infty$
(b) The mapping $f^* \to Mf^*$ is of weak type (1-1) i.e. $\mu_u\{\xi : Mf^*(\xi) > \alpha\} \leq \frac{K}{\alpha}\|f^*\|_1$ if $f^* \in L^1(d\mu_u)$.

Now we further suppose that the following conditions are satisfied:

- ρ is a quasi-metric on $\overline{\Omega}$

- $(\partial\Omega, \rho, \mu)$ is a space of homogenous type

- For all $z \in \Omega$, $\xi \in \partial\Omega$ with $\eta = \rho(z,\xi) > 0$ we have

$$|K(z,\xi)| \leq C\frac{1}{\mu(B(\xi,\eta))}$$

for some C independent of ξ, z and η. Such a kernel is called a standard kernel.

Following the method given in ([44],Theorem 3),which was applied for the Poisson integrals of L^p functions, we can now estimate the integral operator given above in this general setting :

Theorem 3.3.3. *Suppose $Kf^*(z)$ is the $K(z,\xi)$-integral of an $L^p(d\mu)$ function f^* where $K(z,\xi)$ satisfies the conditions given above. Let $Q_\alpha(y) = \{z \in \overline{\Omega}, \rho(y,z) < \alpha\delta_y(z)\}$ for $y \in \partial\Omega$, $z \in \Omega$ with $\delta_y(z) = \min\{\rho(z,\partial\Omega), \rho(z,T_y)\}$ (T_y is the tangent plane at y), $\alpha > 0$, be the admissible approach region. Then*

- *When $\rho(y,z) = \varepsilon$ and $z \in Q_\alpha(y)$ the following inequality holds*

$$|Kf^*(z)| \leq \tilde{A}\sum_{k=1}^{\infty}(\mu(B(y,2^k\varepsilon)))^{-1}\int_{B(y,2^k\varepsilon)}|f^*|d\mu$$

- $\sup_{z\in Q_\alpha(y)}|Kf^*(z)| \leq \tilde{A}Mf^*(y)$.

Proof. Let $Kf^*(z)$ be the $K(z,\xi)$-integral of the $L^p(d\mu)$ function f^*,

$$|Kf^*(z)| \leq \int_{\partial\Omega}|f^*||K(z,\xi)|d\mu(\xi)$$

$$= \int_{\rho(\xi,y)<2\varepsilon}|f^*||K(z,\xi)|d\mu(\xi) + \sum_{k=2}^{\infty}\int_{2^{k-1}\varepsilon\leq\rho(\xi,y)<2^k\varepsilon}|f^*||K(z,\xi)|d\mu(\xi)$$

first,

$$\int_{\rho(\xi,y)<2\varepsilon}|f^*||K(z,\xi)|d\mu(\xi) \leq \frac{C}{\mu(B(y,2\varepsilon))}\int_{B(y,2\varepsilon)}|f^*(\xi)|d\mu(\xi)$$

by the condition on the kernel and similarly since ρ is a pseudometric we have $\rho(z,\xi) \geq \tilde{C}(\rho(\xi,y) - \rho(y,z)) \geq \tilde{C}2^{k-1}\varepsilon - \tilde{C}\varepsilon \geq \tilde{\tilde{C}}2^{k-2}\varepsilon$ if $k \geq 2$ whenever

$2^{k-1}\varepsilon \leq \rho(\xi,y) < 2^k\varepsilon$ and $\rho(z,y)=\varepsilon$, so $|K(z,\xi)| \leq \dfrac{2^{2k}\tilde{\tilde{C}}}{\mu(B(y,2^k\varepsilon))}$. Hence for all k,

$$\int_{2^{k-1}\varepsilon<\rho(\xi,y)<2^k\varepsilon} |f^*||K(z,\xi)|d\mu(\xi) \leq \frac{\acute{A}_{\alpha,n}}{2^k\mu(B(y,2^k\varepsilon))}\int_{B(y,2^k\varepsilon)} |f^*(\xi)|d\mu(\xi)$$

Upon summing in k we get the first assertion and the second inequality is an immediate consequence of the first. $\square$

In ([22]), Hansson considered the boundedness of Cauchy-Fantappie integral operator ,H, from $L^2_u(\partial\mathbb{B}^{\mathbf{p}})$ into $H^2_u(\mathbb{B}^{\mathbf{p}})$. In his work he applied an operator theory result known as $T1$-Theorem and in order to use that result he showed the homogeneity of the boundary of the complex ellipsoid with respect to the quasimetric d and the boundary measure $\partial\rho\wedge(\overline{\partial}\partial\rho)^{n-1}$ where the function ρ is defined as $\rho(z)=\sum_{j=1}^n |z_j|^{2p_j}-1$. In fact an easy calculation shows that this measure is the boundary Monge-Ampère measure associated with the exhaustion function $u(z)=\log(|z_1|^{2p_1}+|z_2|^{2p_2}+ ... +|z_n|^{2p_n})$, $\mathbf{p}=(p_1,p_2,...,p_n)\in\mathbb{Z}^n$ of the complex ellipsoid $\mathbb{B}^{\mathbf{p}}$. Now let $d(\xi,z) \doteq |v(\xi,z)|+|v(z,\xi)|$ be the quasimetric defined on $\overline{\mathbb{B}^{\mathbf{p}}}$ where $v(\xi,z)=\langle\partial\rho(\xi),\xi-z\rangle$. Then explicitly $v(\xi,z)=\sum_{j=1}^n p_j|\xi_j|^{2(p_j-1)}\bar{\xi}_j(\xi_j-z_j)$ and define the boundary balls as $B(z,\varepsilon)=\{\xi\in\partial\mathbb{B}^{\mathbf{p}}, d(\xi,z)<\varepsilon\}$. It is shown that $(\partial\mathbb{B}^{\mathbf{p}},d,d\mu_u)$ is a space of homogenous type ([22],pg:1483) and $\dfrac{1}{(v(\xi,z))^n}$ is a standard kernel i.e. $\left|\dfrac{1}{(v(\xi,z))^n}\right| \leq \dfrac{C}{d\mu_u(B(z,\varepsilon))}$ for $d(\xi,z)=\varepsilon>0$ and for some $C>0$ depending only on the dimension and $\mathbf{p}$. In the following argument we will use his homogeneity result to apply the previous rather general procedure on the complex ellipsoid case with the so called Cauchy-Fantappie kernel:

The Cauchy-Fantappie integral (from now on we will refer as CF-integral) of an $L^p(d\mu_u)$ function f^* is defined as

$$Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial\mathbb{B}^{\mathbf{p}}} \frac{f^*(\xi)d\mu_u(\xi)}{(v(\xi,z))^n}$$

Before proceeding to further results let us briefly discuss the Cauchy-Fantappie kernel. In the theory of holomorphic functions in one variable a fundamental tool is Cauchy integral formula and in the case of several variables one wants a suitable generalization to Cauchy integral. One of the possible choices for the generalization is the so called Szegö kernel however except for a few domains Szegö kernel has no explicit formulation. One other choice is the well known Bochner-Martinelli kernel but the major shortcoming of this kernel is that it is not holomorphic in z variable (For details see ([36])). Contrary to Bochner-Martinelli kernel, Cauchy-Fantappie kernel is holomorphic in z hence it is a natural generalization of Cauchy kernel to multivariable case and it has reproducing property for the functions in the algebra $A(\mathbb{B}^{\mathbf{p}})$ ([36], Theorem 3.4). Hardy spaces which are examined in ([22]) are exactly the Poletsky-Stessin Hardy spaces $H^p_u(\mathbb{B}^{\mathbf{p}})$ that are generated by the exhaustion function u and at the beginning of this section it is shown that for the functions in $H^p_u(\mathbb{B}^{\mathbf{p}})$ the boundary value function $f^* \in L^p(d\mu_u)$ exists so the CF-integral of f^* is well-defined. Now we will show that CF-integral has reproducing property for the functions in $H^p_u(\mathbb{B}^{\mathbf{p}})$:

Proposition 3.3.2. *Let $f \in H^p_u(\mathbb{B}^p)$ be a holomorphic function then*

$$f(z) = Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial\mathbb{B}^p} \frac{f^*(\xi)d\mu_u(\xi)}{(v(\xi,z))^n}$$

Proof. By the Fubini type integral formula that we used in Theorem 3.1.5 we get that

$$Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{A_0} \left(\int_{\partial E} \frac{f^*(\eta)}{(v(\eta,z))^n} d\mu_g(\eta)\right)\omega$$

and on every ellipse E by ([44], 9.7) we have reproducing property as a consequence of one variable Cauchy integral formula. Hence the result follows. $\square$

Now define the maximal function for the functions in $L^p(d\mu_u)$ as follows :

$$Mf^*(\xi) = \sup_{\varepsilon>0} \frac{1}{\mu_u(B(\xi,\varepsilon))} \int_{B(\xi,\varepsilon)} |f^*| d\mu_u$$

The next result is a consequence of the general method given in Theorem 3.3.3 for complex ellipsoid case and it gives the relation between the CF-integral and the maximal function of an $L^p(d\mu_u)$ function f^*:

Corollary 3.3.1. *Suppose $Hf(z)$ is the CF-integral of an $L^p(d\mu_u)$ function f^*. Let $Q_\alpha(y) = \{z \in \overline{\mathbb{B}^{\mathbf{p}}}, |v(y,z)| < \alpha\delta_y(z)\}$ for $y \in \partial\mathbb{B}^{\mathbf{p}}$, $z \in \mathbb{B}^{\mathbf{p}}$ with $\delta_y(z) = \min\{d(z,\partial X), d(z,T_y)\}$ (T_y is the tangent plane at y), $\alpha > 0$, be the admissible approach region. Then*

- *When $d(y,z) = \varepsilon$ and $z \in Q_\alpha(y)$ the following inequality holds*

$$|Hf(z)| \leq \tilde{A} \sum_{k=1}^{\infty} (\mu_u(B(y, 2^k\varepsilon)))^{-1} \int_{B(y,2^k\varepsilon)} |f^*| d\mu_u$$

- $\sup_{z\in Q_\alpha(y)} |Hf(z)| \leq \tilde{A} Mf^*(y)$.

Next using this maximal function tools we will see the existence of boundary values on the admissible approach regions $Q_\alpha(y)$, $y \in \partial\mathbb{B}^{\mathbf{p}}$:

Theorem 3.3.4. *Let $f \in H^p_u(\mathbb{B}^{\mathbf{p}})$ be a holomorphic function and $1 \leq p < \infty$. Suppose that f^* is the radial limit function then*

$$\lim_{Q_\alpha(\xi)\ni z \to \xi} f(z) = f^*(\xi)$$

exists for almost every $\xi \in \partial\mathbb{B}^{\mathbf{p}}$.

Proof. If $\varepsilon > 0$ then choose $g \in C(\partial\mathbb{B}^{\mathbf{p}})$ so that $\|f^* - g\|_{L^p_u(\partial\mathbb{B}^{\mathbf{p}})} < \varepsilon^2$. Then we know that $\lim_{Q_\alpha(\xi)\ni z\to\xi} Hg(z) = g(\xi)$ for all $\xi \in \partial\mathbb{B}^{\mathbf{p}}$. Therefore

$$\mu_u\{\xi : \limsup_{Q_\alpha(\xi)\ni z\to\xi} |f(z) - f^*(\xi)| > \varepsilon\} \leq \mu_u\{\xi : \limsup_{Q_\alpha(\xi)\ni z\to\xi} |f(z) - Hg(z)| > \varepsilon/3\}$$

$$+\mu_u\{\xi : \limsup_{Q_\alpha(\xi)\ni z\to\xi} |Hg(z) - g(\xi)| > \varepsilon/3\} + \mu_u\{\xi : \limsup_{Q_\alpha(\xi)\ni z\to\xi} |g(\xi) - f^*(\xi)| > \varepsilon/3\}$$

$$\leq \mu_u\{\xi : C_\alpha M(f^* - g) > \varepsilon/3\} + (\|f^* - g\|_{L^p_u(\partial\mathbb{B}^{\mathbf{p}})}/(\varepsilon/3))^p \leq \acute{C}_\alpha \varepsilon^p$$

Hence the result follows. □

As another application of this method, we will show an approximation result on the Poletsky-Stessin Hardy spaces:

Theorem 3.3.5. *Polynomials are dense in* $H^p_u(\mathbb{B}^{\mathbf{p}})$*.*

Proof. Let $f \in H^p_u(\mathbb{B}^{\mathbf{p}})$ be a holomorphic function, $1 \leq p < \infty$ and let $f_r(\xi) = f(r\xi)$ for $\xi \in \partial\mathbb{B}^{\mathbf{p}}$. Then we have $f(r\xi) \to f^*(\xi)$ μ_u almost everywhere. By the previous proposition we know that $Hf(z) = f(z)$ when $f \in H^1_u(\mathbb{B}^{\mathbf{p}})$. Using this and the previous results on maximal function we have $|f(r\xi)| \leq Mf^*$, where $Mf^* \in L^p_u(\partial\mathbb{B}^{\mathbf{p}})$ then by the Lebesgue Dominated Convergence Theorem we have that $f_r \to f^*$ in $L^p_u(\partial\mathbb{B}^{\mathbf{p}})$. Furthermore the complex ellipsoid is a complete Reinhardt domain so as a consequence of series expansion we deduce that polynomials are dense in $A(\mathbb{B}^{\mathbf{p}})$ in the topology of uniform convergence on compact subsets. Hence polynomials are dense in $H^p_u(\mathbb{B}^{\mathbf{p}})$. □

3.4 Poletsky-Stessin Hardy Spaces on Strongly Convex Domains

In this section, in order to understand the boundary behavior of Poletsky-Stessin Hardy spaces on strongly convex domains we will examine Stein's procedure that we used in ellipsoid case on strongly convex domains in $\mathbb{C}^n$ which are defined as follows ([28]):

Definition 19. Let $\Omega \subset\subset \mathbb{R}^n$ be a domain with $\mathcal{C}^2$ boundary and ρ a defining function for Ω. Fix a point $P \in \partial\Omega$. We say that $\partial\Omega$ is convex at P if

$$\sum_{j,k=1}^{n} \frac{\partial^2 \rho}{\partial x_j \partial x_k}(P) w_j w_k \geq 0$$

for all w in $T_P(\partial\Omega)$ which is the tangent plane at P. We say that $\partial\Omega$ is strongly convex at P if the inequality is strict. If $\partial\Omega$ is (strongly) convex at each boundary point, then we say that Ω is (strongly) convex.

In fact strongly convex domains are strictly pseudoconvex domains therefore Poletsky-Stessin Hardy spaces generated by the pluricomplex Green function coincide with the classical Hardy spaces and the general theory given in ([44]) is applicable to them. However,in this section we will provide an alternative approach through Poletsky-Stessin Hardy spaces, to the boundary behavior of classical Hardy spaces on strictly pseudoconvex domains by localizing the procedures which is possible because of the fact that on strictly pseudoconvex domains for each boundary point one can find a neighborhood which is strongly convex (For details see ([45])).

Let Ω be a strongly convex domain in $\mathbb{C}^n$ with smooth boundary and let $g(z,a)$ be the pluricomplex Green function of Ω with pole at $a \in \Omega$. Then by ([29]) we know that $g(z,a)$ is in $C^\infty(\overline{\Omega} \setminus \{a\})$. Now define the quasimetric

$$d(\xi, z) = |<\partial g(\xi), \xi - z>| + |<\partial g(z), z - \xi>|$$

and the corresponding balls will be defined by

$$B(z,\varepsilon) = \{\xi \in \partial\Omega | \quad d(\xi, z) < \varepsilon\}$$

Recall that the Monge-Ampère boundary measure associated with the Green function $g(z,a)$ is given by $d\mu_g = \partial g \wedge (\overline{\partial}\partial g)^{n-1}$. We will show that $(\partial\Omega, d, d\mu_g)$ is a homogenous space. First of all combining the results of ([30],[44]) we see that for the quasimetric d, the balls have enveloping property i.e. $B_{\varepsilon_1}(z) \cap B_{\varepsilon_2}(\xi) \neq \varnothing$ and $\varepsilon_1 \geq \varepsilon_2$ implies that $B_{\varepsilon_2}(\xi) \subset B_{c\varepsilon_1}(z)$. Hence in order to show that $(\partial\Omega, d, d\mu_g)$ is a space of homogenous type we need to prove the doubling condition with respect to measure $d\mu_g$.

Lemma 3.4.1. *There is a constant $A > 0$ depending only on the surface $\partial\Omega$ so that for all $P \in \partial\Omega$ and all $\delta > 0$*

$$\mu_g(B_{2\delta}(P)) \leq A\mu_g(B_\delta(P))$$

Proof. First assume $\Omega \subset \mathbb{C}^2$. Since $\partial\Omega$ is a smooth compact surface it is sufficient to obtain the inequality for all P and all sufficiently small δ. Without loss of generality assume that $P = 0 \in \partial\Omega$ and near 0, $\partial\Omega$ is given as the graph of a convex function ψ such that $x_4 = \psi(x_1, x_2, x_3)$ where $\mathbb{C}^2 \simeq \mathbb{R}^4$ in real coordinates (x_1, x_2, x_3, x_4). Then by construction, for points $(x_1, x_2, x_3, \psi(x_1, x_2, x_3)) \in B_{2\delta}(0)$, we have $\{(x_1, x_2, x_3)| \quad \psi(x_1, x_2, x_3) < 2\delta\}$ and by the proof of Theorem 4.2 in ([7]) there is a constant A_1 depending only on $\partial\Omega$ such that

$$\{(x_1, x_2, x_3)| \quad \psi(x_1, x_2, x_3) < 2\delta\} \subset A_1\{(x_1, x_2, x_3)| \quad \psi(x_1, x_2, x_3) < \delta\}$$

Now let $\pi : \mathbb{C}^2 \to \mathbb{C}$ be the natural projection $\pi(z_1, z_2) = z_1$, then from the above inclusion we get $\pi(B_{2\delta}(0)) \subset \widetilde{A_1}\pi(B_\delta(0))$, moreover by ([15],2.19) we have

$$\int_{\pi(B_{2\delta}(0))} \pi_*(d\mu_g) = \int_{B_{2\delta}(0)} d\mu_g \tag{3.4.1}$$

and in $\mathbb{C}$ we have $\pi_*(d\mu_g) \approx d\sigma$ so we have $\mu_g(B_{2\delta}(P)) \leq A\mu_g(B_\delta(P))$. For $n > 2$ process is the same via an inductive argument; at each step first we identify the n-dimensional complex space with the $2n$-dimensional real one and use Theorem 4.2 in ([7]) to have an enveloping property between the balls of radii 2δ and δ then we project the boundary balls to $(n-1)$-dimensional complex space where we know the existence of doubling property and using (3.4.1) we obtain the doubling condition for the boundary balls of dimension n.

Hence $(\partial\Omega, d, d\mu_g)$ is a space of homogenous type. $\square$

From now on suppose $\Omega \subset \mathbb{C}^n$ is a strongly convex domain and without loss of generality assume that Ω contains $0 \in \mathbb{C}^n$. Let $g(z, 0)$ be the Pluricomplex Green function with the logarithmic pole at 0. We will consider the Poletsky-Stessin Hardy spaces $H_g^p(\Omega)$, $p > 1$. In order to obtain a Calderon-Zygmund type maximal function argument we need to show that the Cauchy-Fantappie kernel

$$\upsilon(z, \xi) = \frac{1}{| < \partial g(\xi), \xi - z > |^n}$$

is a standard kernel i.e. we need to show $\dfrac{1}{|(\upsilon(z, \xi))^n|} \leq \dfrac{C}{d\mu_g(B_\varepsilon(\xi))}$ for $d(z, \xi) = \varepsilon$.

Lemma 3.4.2. *The Cauchy-Fantappie kernel satisfies the following inequality*

$$\frac{1}{|\upsilon(z, \xi)|} \leq \frac{C}{\mu_g(B(\xi, \varepsilon))}$$

where $d(z, \xi) = \varepsilon$.

Proof. We need to show that $\mu_g(B_\varepsilon(\xi)) \leq C|\upsilon(z, \xi)|$ for $d(z, \xi) = \varepsilon$. First of all since we have $| < \partial g(\xi), \xi - z > | \sim | < \partial g(z), z - \xi > |$ ([30]), we get $|\upsilon(\xi, z)|^n \sim \varepsilon^n$ from the definition of $d(z, \xi)$. Hence we need to show that $\mu_g(B_\varepsilon(\xi)) \leq C\varepsilon^n$ when $d(z, \xi) = \varepsilon$. Now let $\varphi \in C^\infty(\overline{\Omega})$ so that $\varphi \geq 0$,$\varphi = 1$ on $B_\varepsilon(\xi)$ and vanishes outside $\overline{B_{\frac{3\varepsilon}{2}}(\xi)}$ with $\varepsilon(0) = 0$ and $|dd^c\varphi| \leq \frac{M}{\varepsilon^2}$. Now

$$\int_{B_\varepsilon(\xi)} d\mu_g(\eta) \leq \int_{\partial\Omega} \varphi d\mu_g(\eta) = \int_\Omega (-g) dd^c\varphi \wedge (dd^c g)^{n-1} \tag{3.4.2}$$

and now consider the Taylor expansion of $g(z)$ around ξ

$$g(z) = g(\xi) + 2Re < g'_\xi(\xi), z - \xi > + \frac{1}{2} d^2 g(\xi) \left[\frac{z-\xi}{|z-\xi|} \right] |z-\xi|^2 + o(|z-\xi|^2)$$

where

$$d^2 g = 2Re \sum_{j,k=1}^{n} g''_{\xi_j \xi_k} w_j w_k + 2 \sum_{j,k=1}^{n} g''_{\xi_j \overline{\xi_k}} w_j \overline{w_k}$$

and if ε is small enough then we have $\gamma_1 |z-\xi|^2 \leq |-g| \leq \gamma_2 |z-\xi|$ for some $\gamma_1, \gamma_2 > 0$ that depend only on Ω. From (3.4.2) we have

$$\int_{B_\varepsilon(\xi)} d\mu_g(\eta) \leq \int_\Omega (-g) dd^c \varphi \wedge (dd^c g)^{n-1} \leq \gamma_2 \varepsilon . \frac{M}{\varepsilon^2} . \varepsilon^{n+1} = C\varepsilon^n$$

$\Rightarrow \quad \mu_g(B_\varepsilon(\xi)) \leq C\varepsilon^n$
It follows that $\upsilon(\xi, z)$ is a standard kernel. $\square$

Lastly we will consider the existence and the characterization of the boundary values for the holomorphic functions in Poletsky-Stessin Hardy spaces $H^p_g(\Omega)$, $p > 1$. We will apply the same method that we considered in the ellipsoid case so let us first see that for a holomorphic function $f \in H^p_g(\Omega)$, $f^*(\xi) = \lim_{\varepsilon \to 0} f(\xi - \varepsilon\nu)$ exists where ν is the outward unit normal.

Proposition 3.4.1. *Let Ω be a strongly convex domain containing* 0 *and $g(z, 0)$ is the Green function with the logarithmic pole at* 0*. Then for any $f \in H^p_g(\Omega)$ the boundary value function $f^*(\xi) = \lim_{\varepsilon \to 0} f(\xi - \varepsilon\nu)$ exists μ_g-a.e. where ν is the outward unit normal to the boundary.*

Let $\xi \in \partial\Omega$, and let E be the strongly convex domain in $\mathbb{C}$ which is the intersection of the complex line joining 0 to ξ and Ω. Then $g\mid_E$ is an exhaustion function for E. The Monge-Ampère measure associated with the exhaustion function g is given by $d\mu_{g,r} = d^c g \wedge (dd^c g)^{n-1} \mid_{S_g(r)}$, and let A_0 be the set of all complex lines passing $0 \in \Omega$. Take $f \in H^p_g(\Omega)$ then

$$\int_{S_g(r)} |f|^p d\mu_{g,r} = \int_{S_g(r)} |f|^p (d^c g \wedge (dd^c g)^{n-1}) = \int_{A_0} \left(\int_{l_z \cap S_g(r)} |f|^p d^c g \right) \omega$$

where $\pi^*\omega = dd^c g)^{n-1}$ and $\pi : \Omega \to A_0$ is the function given by $\pi(z) = [0, z] = l_z$ with l_z being the line joining 0 and z.
We can use the above generalization of Fubini Theorem since π is a submersion and $\pi\ |_{supp(d^c(g|_E))}$ is proper.([15],2.15).
The measure $d^c g$ on $l_z \cap S_g(r)$ is equal to $d^c(g\ |_E)$ on $S_{g|_E}(r)$. Since E is a smoothly bounded domain $d^c(g\ |_E)$ on $S_{g|_E}(r)$ is equal to $d\mu_{g|_E,r}$ so

$$\int_{S_g(r)} |f|^p d\mu_{g,r} = \int_{A_0} \left(\int_{S_{g|_E}(r)} |f|^p d\mu_{g|_E,r} \right) \omega$$

and by Fatou's Lemma

$$\int_{A_0} \left(\liminf_{r \to 0} \int_{S_{g|_E}(r)} |f|^p d\mu_{g|_E,r} \right) \omega < \infty$$

since $f \in H^p_g(\Omega)$ and this gives us that for ω-a.e $\liminf_{r\to 0} \int_{S_{g|_E}(r)} |f|^p d\mu_{g|_E,r} < \infty$ so $f \in H^p_{g|_E}(E)$ for ω-a.e and it has admissible boundary values $d\sigma \simeq d\mu_{g|_E}$ almost everywhere ([44]). Since f^* is the pointwise limit of a measurable function it is measurable and consider the set $A = \{\xi \in \partial\Omega, \quad f^*(\xi)\, does\ not\ exist\}$ then

$$\int_{\partial\Omega} \chi_A d\mu_g = \int_{A_0} \left(\int_{\partial E} \chi_A(\eta) d\mu_{g|_E}(\eta) \right) \omega$$

but since $f \in H^p_{g|_E}(E)$ it has admissible limits $d\mu_{g|_E}$-a.e so the integral inside is 0 and we have $\int_{\partial\Omega} \chi_A d\mu_g = 0$ therefore $f^*(\xi)$ exists μ_g-a.e. Moreover for a holomorphic function $f \in H^p_{g|_E}(E)$ we know that $f^* \in L^p(\partial E)$ so we have

$$\int_{\partial\Omega} |f^*|^p d\mu_g = \int_{A_0} \left(\int_{\partial E} |f^*|^p d\mu_{g|_E} \right) \omega < \infty$$

hence $f^* \in L^p_{\mu_g}(\partial\Omega)$.
The Cauchy-Fantappie integral of an $L^p_{\mu_g}(\partial\Omega)$ function f^* is defined as

$$Hf(z) = \left(\frac{1}{2\pi i}\right)^n \int_{\partial\Omega} \frac{f^*(\xi)d\mu_g(\xi)}{(\upsilon(\xi,z))^n}$$

Combining the Fubini type integral formula on each complex line with ([44],9.7) we get $Hf = f$ when $f \in H^p_g(\Omega)$. Now define the maximal function for f^* as follows:

$$Mf^*(\xi) = \sup_{\varepsilon>0} \frac{1}{\mu_g(B(\xi,\varepsilon))} \int_{B(\xi,\varepsilon)} |f^*|^p d\mu_g$$

As a consequence of the general maximal function argument given in Theorem 3.3.3 we get

Theorem 3.4.1. *Let $f \in H^p_g(\Omega)$. Then from the general argument about the maximal function we see that the boundary function $f^* \in L^p_{\mu_g}(\partial\Omega)$ satisfies the following:*

- $\sup_{z\in Q_\alpha(y)} |Hf(z)| \le \widetilde{A}Mf^*(y)$ *when* $z \in Q_\alpha(y) = \{z \in \overline{\Omega}, d(y,z) < \alpha\delta_y(z)\}$ *where* $\delta_y(z) = \min\{d(z,\partial X), d(z,T_y)\}$ *(T_y is the tangent plane at y),* $\alpha > 0$.

- $\|Mf^*\|_p \le A_p\|f^*\|_p$ *for* $1 < p \le \infty$

- *The mapping $f^* \to Mf^*$ is of weak type (1-1) i.e.* $\mu_g\{\xi : Mf^*(\xi) > \alpha\} \le \frac{K}{\alpha}\|f^*\|_1$ *if* $f^* \in L^1_{d\mu_g}(\partial\Omega)$

Finally, the proof of the analogous result in ellipsoid case may be imitated verbatim to establish the following result:

Theorem 3.4.2. *Let $f \in H^p_g(\Omega)$ be a holomorphic function and $1 \le p < \infty$. Suppose that f^* is the limit function given in the normal direction then*

$$\lim_{Q_\alpha(\xi)\ni z\to\xi} f(z) = f^*(\xi), \quad a.e. \quad \xi \in \partial\Omega$$

Remark 13. Let us now give a comparison about the approach regions given in our previous results and the classical approach regions considered by different authors. By ([30], [44]) we know that on a convex domain the approach regions Q_α given by the metric d are the admissible approach regions that are discussed in the classical theory. If we consider the shape of these regions we see that near its vertex the distance of the $Q_1(w)$, $w \in \partial\Omega$, along the normal direction changes in a parabolic way which allows also the tangential approaches. (For details of this calculation see [5]). Hence our approach regions are greater than the classical non-tangential approach regions given in ([24]). The approach regions Q_α are the greatest family of approach regions in the sense that they are built using the biggest embedded polydiscs that fit inside the domain (for details see [33]).

3.5 Composition Operators on Poletsky-Stessin Hardy Spaces on Hyperconvex Domains in $\mathbb{C}^n$, $n > 1$

In this section we will consider the boundedness properties of composition operators acting on Poletsky-Stessin Hardy spaces on hyperconvex domains in $\mathbb{C}^n$ for $n > 1$. Before proceeding further let us first briefly discuss the results given in ([34]) about composition operators on Poletsky-Stessin Hardy spaces that are generated by exhaustion functions which are maximal out of compact sets:

Let $D_1 \subset \mathbb{C}^n$ and $D_2 \subset \mathbb{C}^m$ be two hyperconvex domains and u_1, u_2 be the exhaustion functions for D_1 and D_2 respectively. In ([34]), Poletsky and Stessin considered the necessary and sufficient conditions of boundedness of a composition operator induced by a holomorphic mapping between D_1 and D_2 when the exhaustion functions u_1 and u_2 belong to class $\mathcal{E}_0$ i.e. the Monge-Ampère measures $(dd^c u_1)^n$ and $(dd^c u_2)^m$ have compact support. Before proceeding further let us first discuss some of the results given in the case where exhaustion functions are chosen from $\mathcal{E}_0$:

Definition 20. Let D be a hyperconvex domain with the exhaustion function u and f be a holomorphic function on D. Nevanlinna counting function is defined as:

$$N_{u,f}(w) = \int_D (-u)(dd^c u)^{n-1} \wedge dd^c \log|f - w|$$

Let $F : D_1 \to D_2$ be a holomorphic mapping between the hyperconvex domains D_1 and D_2 with exhaustion functions $u_1 \in \mathcal{E}_0$ and $u_2 \in \mathcal{E}_0$ respectively. If f is a holomorphic function on D_2 then the "tail" part of Nevanlinna counting function is defined as follows:

$$N^*_{u_1,F,f}(w,r) = \int_{T(r)} (-u_1)(dd^c u_1)^{n-1} \wedge dd^c \log|f \circ F - w|$$

where $T(r) = D_1 \setminus B_{u_2 \circ F}(r) = \{z \in D_1 : \quad u_2(F(z)) > r\}$. Then *deficiency* of F is defined as

$$\delta_{u_1,u_2,F}(r) = \sup \frac{N^*_{u_1,F,f}(w,r)}{N_{u_2,f}(w)}$$

where the supremum is taken over all $f \in H^p_{u_2}(D_2)$.
In ([34]) sufficiency condition for boundedness of a composition operator is given as follows:

Theorem 3.5.1. *Let $F : D_1 \to D_2$ be a holomorphic mapping between the hyperconvex domains D_1 and D_2 with exhaustion functions $u_1 \in \mathcal{E}_0$ and $u_2 \in \mathcal{E}_0$ respectively. If there exists $r_0 < 0$ such that $\delta_{u_1,u_2,F}(r_0) < \infty$ then the operator $C_F(f) = f \circ F$ is a bounded operator from $H^p_{u_2}(D_2)$ into $H^p_{u_1}(D_1)$.*

To provide necessary conditions, fix a compact set $K \subset D_1$ and for a holomorphic function $f \in H^p_{u_2}(D_2)$ introduce the function

$$\nu_F(w,f) = \frac{|w|^p N_{u_1,f\circ F}(w)}{\|f\|_{H^p_{u_2}}}$$

and for $a > 1$ set

$$\rho_{u_1,u_2,F}(a) = \sup \nu_F(w, f)$$

where the supremum is taken over all $f \in H^p_{u_2}(D_2)$ and all $w \in \mathbb{C}$, $|w| > a \max_{\xi \in K} |f \circ F(\xi)|$. In this setting the necessity condition given in ([34]) is as follows:

Theorem 3.5.2. *Let $F : D_1 \to D_2$ be a holomorphic mapping between the hyperconvex domains D_1 and D_2 with exhaustion functions $u_1 \in \mathcal{E}_0$ and $u_2 \in \mathcal{E}_0$ respectively. If C_F is a bounded operator from $H^p_{u_2}(D_2)$ into $H^p_{u_1}(D_1)$, then $\rho_{u_1,u_2,F}(a) < \infty$ for all $a > 1$.*

However when the exhaustion function is chosen with finite Monge-Ampère mass but not necessarily maximal out of a compact set we can end up with unbounded composition operators even for the simplest symbols, namely automorphisms. Consider the exhaustion function $u_n(z) = \max\{u(z_1), u(z_2), .., u(z_n)\}$ where u is the exhaustion function for unit disc that we constructed in Theorem 2.1.2. We have seen that similar to the one variable case, if we use the symbol $\varphi(z, w) = (z_1 e^{\frac{i\pi}{2}}, z_2 e^{\frac{i\pi}{2}}, .., z_n e^{\frac{i\pi}{2}})$ we can obtain that φ does not induce a bounded composition operator on $H^1_{u_n}(\mathbb{D}^n)$ although every automorphism of the polydisc induces a bounded composition operator on the classical Hardy space $H^1(\mathbb{D}^2)$([43], Cor.3.2.3).
In the following result we will show that, for a bounded hyperconvex domain Ω under certain regularity conditions on u and φ we can construct an exhaustion function ψ for Ω with finite Monge-Ampère mass such that the composition operator C_φ with the holomorphic symbol φ is Lipschitz continuous between the Poletsky-Stessin Hardy spaces $H^p_u(\Omega)$ and $H^p_\psi(\Omega)$ with Lipschitz constant $K = 1$ but for this we need to introduce the classes of compliant functions defined by ([8]):
Recall that the Perron-Bremermann envelope for a given function $f : \partial\Omega \to \mathbb{R}$

is given by:

$$PB_f(z) = \sup \left\{ \omega(z) : \quad \omega \in PSH(\Omega) \quad \limsup_{\substack{v \to \xi \\ v \in \Omega}} \omega(v) \le f(\xi), \forall \xi \in \partial\Omega \right\}$$

Definition 21. A continuous function $f : \partial\Omega \to \mathbb{R}$ which satisfies the following two conditions is called a compliant function:

- $\lim_{\substack{z \to \xi \\ z \in \Omega}} (PB_f + PB_{-f})(z) = 0$ for every $\xi \in \Omega$
- $\int_\Omega (dd^c(PB_f + PB_{-f}))^n < \infty$
  The set of all compliant functions is denoted by $\mathcal{CP}(\partial\Omega)$ and the set of functions for which $PB_{-f} = -PB_f$ is denoted by $\mathcal{CP}_0(\partial\Omega)$

Theorem 3.5.3. *Let $\Omega \subset\subset \mathbb{C}^n$ be a bounded hyperconvex domain. Suppose u is a continuous, negative, plurisubharmonic exhaustion function with finite Monge-Ampère mass and $\varphi : \overline{\Omega} \to \Omega$ is a one-to-one holomorphic self map of Ω. If $u \circ \varphi \in \mathcal{CP}_0(\partial\Omega)$ then there exists a continuous exhaustion function ψ with finite mass such that $C_\varphi : H^p_u(\Omega) \to H^p_\psi(\Omega)$ is continuous for $1 \le p < \infty$.*

Proof. First we will construct the exhaustion function ψ. Let ρ be the solution of the Dirichlet problem for pluriharmonic functions i.e. $dd^c\rho = 0$ on Ω, $\rho \in PSH(\Omega) \cap C(\overline{\Omega})$ and $\rho = u \circ \varphi$ on $\partial\Omega$. We know that this problem is solvable on Ω since $u \circ \varphi \in \mathcal{CP}_0(\partial\Omega)$ by ([12], Theorem 3.5). Now consider the function $\psi = (u \circ \varphi) - \rho$. We see that $\psi = 0$ on the boundary and $(dd^c\psi)^n = (dd^c(u \circ \varphi))^n \ge 0$, since ρ is pluriharmonic and $(u \circ \varphi)$ is a plurisubharmonic function. Moreover ψ is continuous on $\overline{\Omega}$, and since it is continuous and equals to 0 on the boundary it is an exhaustion. Therefore the only part we need is to show that ψ has finite Monge-Ampère mass. First of all by ([15], pg:10 (1.13)) and ([32], Theorem 4.9) we have

$$\int_\Omega (dd^c\psi)^n = \int_\Omega (dd^c(u \circ \varphi))^n = \int_\Omega \varphi^*((dd^c u)^n) \tag{3.5.1}$$

where the form $\varphi^*((dd^cu)^n)$ is the pull-back of $(dd^cu)^n$. Then by ([15],pg:11 (1.17)) we have

$$\int_\Omega \varphi^*((dd^cu)^n) = \int_{\varphi(\Omega)} (dd^cu)^n \tag{3.5.2}$$

then combining the equations (3.5.1) and (3.5.2) we have

$$\int_\Omega (dd^c\psi)^n = \int_{\varphi(\Omega)} (dd^cu)^n \leq \int_\Omega (dd^cu)^n < \infty$$

the last inequality follows from the fact that u has finite Monge-Ampère measure. Thus ψ has finite Monge-Ampère mass.
We see that if $u \circ \varphi$ is continuous on $\partial\Omega$ then ψ is a negative, continuous, plurisubharmonic exhaustion function for Ω with finite mass.
Next we will consider the action of the composition operator C_φ on $H^p_u(\Omega)$:
Let $f \in H^p_u(\Omega)$ then,

$$\|f \circ \varphi\|^p_{H^p_\psi(\Omega)} = \int_\Omega |f \circ \varphi|^p (dd^c\psi)^n + \int_\Omega (-\psi) dd^c |f \circ \varphi|^p \wedge (dd^c\psi)^{n-1}$$

$$= \int_\Omega |f \circ \varphi(z)|^p (dd^c(u \circ \varphi(z)))^n + \int_\Omega (\rho - u \circ \varphi(z)) dd^c |f \circ \varphi(z)|^p \wedge (dd^cu \circ \varphi)^{n-1}$$

and since ρ is negative on Ω we have

$$\leq \int_\Omega |f \circ \varphi(z)|^p (dd^c(u \circ \varphi(z)))^n + \int_\Omega (-u \circ \varphi(z)) dd^c |f \circ \varphi(z)|^p \wedge (dd^cu \circ \varphi)^{n-1}$$

then combining ([32], Theorem 4.9) and ([27], pg:9) we have

$$\int_\Omega |f \circ \varphi(z)|^p (dd^c(u \circ \varphi(z)))^n + \int_\Omega (-u \circ \varphi(z)) dd^c |f \circ \varphi(z)|^p \wedge (dd^cu \circ \varphi)^{n-1}$$

$$= \int_\Omega \varphi^*(|f|^p (dd^cu)^n) + \int_\Omega \varphi^*((-u) dd^c |f|^p \wedge (dd^cu)^{n-1})$$

and since φ is a diffeomorphism to its image we have

$$\int_{\varphi(\Omega)} |f|^p (dd^c u)^n + \int_{\varphi(\Omega)} (-u) dd^c |f|^p \wedge (dd^c u)^{n-1}$$

$$\leq \int_{\Omega} |f|^p (dd^c u)^n + \int_{\Omega} (-u) dd^c |f|^p \wedge (dd^c u)^{n-1} = \|f\|^p_{H^p_u(\Omega)} < \infty$$

Hence C_φ acts continuously from $H^p_u(\Omega)$ to $H^p_\psi(\Omega)$. $\square$

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