ON THE OPTIMAL CONTROL PROBLEM FOR SINGLE LEG AIRLINE REVENUE MANAGEMENT WITH OVERBOOKING

by

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Kapasite Üstü Reservasyon İçeren Tek Bacakli Uçuslarda Gelir Eniyilemeyi Amaçlayan Kontrol Politikası

Alp Muzaffer Arslan
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Özet

Havayolları endüstrisinde benzer koltukları farklı fiyatlar ile ücretlendirme yaygın bir uygulamadır. Bu politika gözönüne alındıgı zaman, uçak kapasitesinin birden fazla müsteri sınıfı arasında paylaştırılması, havayolu şirketlerinin en temel sorunlarından biridir. Bu tez uçak kapasite paylaşımı problemini incelemekte ve yeni bir model önermektedir. Reservasyon iptalleri ve uçuş anında gelmeyenlerin varlığı nedeniyle boş koltuklarlardan kaynaklanan gelir kaybını engellemek için modelimiz kapasite üstü reservasyona izin vermektedir. Bu çalışmada amacı satışlardan gelen geliri iptal ve fazla reservasyondan kaynaklanan maliyetleri gözönüne alarak eniyilemek olan sürekli zamanlı bir model üzerinde çalıştık. Bu modelde yolcular homojen olmayan Poisson surecine göre gelirken, bir reservasyonun iptal etme süresi ise üssel dağılımını izlemektedir. En iyi politika dinamik program yardımı ile bulunmuştur ve simulasyon yarımı ile literatürde bilinen diğer modellerin ortalamaları ile karşılaştır.

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Abstract

Charging identical seats with different prices is a common practice for airline companies. In that regard one of the main concerns for airline managements is the optimal allocation/partition of the plane capacity between multiple fare classes. This thesis examines the seat allocation problem of airline revenue management and proposes a new model. Due to the occurrence of cancellations and no-shows, we also allow overbooking in order to compensate the revenue loss of empty seats. We study a continuous time model in which the objective is to maximize expected revenue consisting of the fares collected minus the cancellation and overbooking costs. In our model customers arrive according to a nonhomogeneous Poisson process while the time to cancellation of each reservation follows an exponential distribution. An optimal policy is found using dynamic programming and this policy is compared with other policies known in the literature by means of simulation.

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Chapter 1

Introduction

Revenue management plays an important role within applications of Operation Research techniques to real world problems. Its origin started due to airline industry practices towards pricing and overbooking. In 1979 the Airline Deregulation Act became legislation and this resulted in competition in the airline market. Hence it became more urgent for airline companies to reduce cost and/or increase their revenues. To increase market shares some companies also needed to adopt more aggressive pricing strategies to attract customers. As a result of these developments airline companies tried to seek alternative methods of marketing. To realize increasing revenues one needed to develop more sophisticated overbooking and seat allocation policies and this need created the field of Airline Revenue Management (ARM). Nowadays revenue management techniques are not only applied within airline companies but also within other service industries. For an overview on the used techniques and the different fields of applications the reader is referred to Rothstein [27], Talluri, Van Ryzin [32] and Phillips [22].

The main topic in revenue management is to develop techniques which maximize the revenue of a finite number of commodities becoming obsolete at a given time in the future. Due to this general description revenue management is also known as yield management or perishable inventory control. In the airline example considered in this thesis the perishable goods are seats in an airplane and these seats become obsolete after the departure of the plane. Since in our particular example customers with a reservation might cancel their reservation and/or do not show up at departure, customers have different demands airline revenue management uses the combined tools of overbooking, pricing and seat allocation (cf.[21]).

Overbooking is the policy of reserving more seats than the actual capacity of the airplane. Due to actual practice of reservation cancelling or not showing up at the departure applying such a policy can be beneficial for the airline company. However applying overbooking is also risky since more customers than the available capacity might show up at departure. Now the airline company needs to offer alternatives to the overbooked customers and this generates additional costs. Hence there is a tradeoff between obtaining additional revenues due to overbooking and costs of providing alternative ways of transportation.

Pricing is a tool to influence the demand of potential customers for a fixed number of seats. By using pricing techniques one tries to attract demand from different segments of the market. This is done by assigning different prices to more or less similar seats. Associated with these different priced seats are different conditions on its use and these conditions might appeal to specific customers. The most well known example of this segmentation is the differentiation between business and economy class seats. Another example is the possibility to reschedule the flight with or without paying a penalty. Due to this segmentation airline management need to solve the seat allocation problem for these so-called fare classes. This means that airline companies should decide next to the overbooking problem how many seats one needs to reserve for each fare class. The total number of seats reserved for each fare class are called the partitioned booking limits.

To explain this allocation problem between economy and business class seats it is well known that leisure travelers have a tendency to book their economy class tickets at the beginning of the booking period while business class customers prefer the opposite. If total expected demand is estimated to be more than seat capacity, airline management should at the beginning of the booking period decline some of the economy class requests so that they can accept more business class requests at the end of the booking period. If an airline company would accept requests on a first come first served basis this would result in an airplane occupied by economy class customers. However, by not accepting all the early requests of economy class customers yields more revenue because of the higher priced business class seats. At the same time, due to cancellation and no show-ups, it might be profitable to accept more reservations than the actual seat capacity and so one should also consider overbooking. It is clear now that determining the partitioned booking limits for each fare class is also related to the total size of the overbooking limit and we obtain the so-called combined overbooking and seat allocation problem. This problem is the topic of this thesis. In the following section, we first present an overview of the main modeling approaches and techniques available in the literature. At the same time we will discuss in detail the difference between our new model to others available in the literature using a simulation study. In Chapter 2, we construct this model under which we write the net revenue of the airline an our objective function. We study the properties of the value function and we provide an optimal policy.

1.1 Literature Review

In this subsection, we first focus on the difference between so-called static and dynamic model description of the combined overbooking and seat allocation problem. Based on this distinction a more detailed analysis of the different papers considered in the literature will be given in the next subsection. The static pure overbooking problem is actually the first problem (cf.[3]) considered in the revenue management literature. In that paper a single leg model with only one fare class (thereby excluding the seat allocation problem) is proposed and the objective is to reduce the expected loss due to empty seats. After evaluating the univariate expected cost objective for a given total overbooking limit, the optimal total overbooking limit becomes a solution of a minimization problem. In later papers the generalization of this model to multiple fare classes is considered. In this generalization one should now also decide how much capacity will be assigned to each fare class and so next to the overbooking limit the partitioned booking limits are additional decision variables. In this approach one only uses the cumulative distribution of the total demand for each fare class and completely ignores the information given by the evolvement over time of the dynamic arrival process of the fare class requests. Also the dynamic cancellation process of reserved seats is ignored. As input data one also uses in these models the (time average) show up and cancellation probabilities belonging to each fare class. By ignoring the dynamics of the arrival and cancellation process and using only the averaged cancellation and show up statistics and the cdf of the total demand of each fare class one determines before the start of the booking period the expected revenue of a given policy. Such an approach is called static and the determination of an optimal allocation-overbooking policy reduces to solving a maximization problem over a set of so-called feasible policies determined by a finite set of parameters. The objective function in this optimization problem represents the expected revenue or loss for a given policy. In some papers one also considers other service type related objectives. Due to the lack of concavity-type properties for the objective function and the use of integer of decision variables it is computationally hard to find an optimal solution of this mathematical programming problem and one uses heuristics to find a (near optimal) policy. The selected (near optimal) partitioned booking and total booking limit is then used in a nested way (cf.[32]) within a simulation of the time dependent arrival and cancellation process to measure its performance.

It is commonly assumed that in the real world the arrival process of requests for the different fare classes are given by independent (non)homogeneous Poisson processes while the cancellation process of the reservations has a dynamic Markovian structure. Next to the above static approach the literature also discusses an approach to the overbooking-seat allocation problem making use of the sample path information given by the evolvement over time of the arrival process of request and the cancellation process of reservations. Such an approach yields a so-called dynamic model and using technique of dynamic programming one can derive an optimal dynamic decision rule. This decision rule determines whether one should accept or reject an arriving request by considering both the number of reservations (state of the system) at the arrival moment of this specific request and the future expected optimal gain to be earned until departure. As we will show in this thesis such a dynamic policy can be properly analyzed and computed within real time under the assumption of nonhomogeneous Poisson arrival processes of requests, a fare class independent Markovian cancellation process and fare class independent show up probabilities. This generalizes results for similar but less general models considered in the literature. The relation between these models and our new model will be discussed in the next subsection. Also, although easy to formulate as an integer nonlinear programming problem, the static integer programming formulation of the combined overbooking and seat allocation is difficult to solve to optimality. Hence in the literature one cannot find fast algorithms solving this optimization problem. The difficulty in this static formulation is not created by the static cancellation probabilities but created by the show up probabilities assigned to the multiple fare classes. If we consider now only one fare class and hence solve a pure overbooking problem, these difficulties disappear and a simple and fast algorithm can be found for this special static formulation. By this observation it is not surprise that in the literature one can only find heuristic ways to generate a feasible solution based on the following approach. One first determines in the first stage the size of the total booking limit by aggregating the different fare classes into one fare class and adapting in heuristic way the demand characteristic fare classes into parameter for aggregated fare class. Hence one solves in the first stage a static pure overbooking problem. In the second stage the overbooking limit computed by the first aggregate model is taken as the virtual capacity of the plane. One assumes now in the second model that every customer with a reservation will not cancel and always show up. Hence in the second stage one solves a static model with no overbooking. Although it is difficult to solve the nested formulation of this optimization problem to optimality there are fortunately good (nested) heuristics available in the literature. Combining these two stages yields a feasible solution to the overbooking-seat allocation problem. By the above observations it is clear that static pure overbooking models play an important role in solving static overbooking-seat allocation problems. In the next subsection, we will start with reviewing those pure overbooking models. After that, the so-called EMSR heuristics for static models with no overbooking will also be considered. Up to now we only discussed the overbooking-seat allocation problem for one airplane on a direct flight (no stops in between). This is called a single leg problem. However, it is clear from the real world that airline companies have a fleet of airplanes with multiple stops at each flight and therefore mostly operate on a network consisting of different flights. In the literature one also discusses network based models next to the previous single-leg based models. As before these models can be regarded as static or dynamic. The network based model with cancellation and uncertainty of showing up at the departure time of the airplanes is not only difficult to model but also difficult to solve and so only heuristic methods exists in the literature. Therefore in the next subsection we will mainly focus on single leg dynamic and static models discussed in the literature. Due to the importance of pure overbooking models or equivalently overbooking models with no allocation to fare classes (one fare class assumption) we start with these models in the next subsection. After that we consider the important class of static models with no overbooking.

1.1.1 Static and Dynamic Models for Pure Overbooking.

As mentioned previously, the field of revenue management started with the study of pure overbooking models. Most of the studies before Littlewood [20] focused on this problem. Pure static overbooking problems are classified according to their objectives. Cost based approaches emphasize on the opportunity cost of empty seats and additional expected revenue. Service based model are more risk averse and they seek a solution for keeping the number of denied boarding customers at some levels are stated in the work of Lan. [16].

Beckman [3] proposes a model that yields a total booking limit which balances the loss from departing with empty seats with the possible costs due to overbooking. Thompson [34] ignores the probability distribution of the demand of the economy and business class (two fare classes are considered) as well as the revenue generated by those fare classes and focuses on computing the probability of overbooking and the number of denied boardings for given reservations given probability information about cancellation and no shows. Taylor [33] and Rothstein and Stone [28] generalize the Thompson's study by proposing additional treatments for denied boarding customers. As reported in Rothstein [27] the extended model of Rothstein and Stone was implemented at American Airlines. For a more detailed overview of the mostly static overbooking literature before 1985 one should consult Rothstein [27]. After 1985 it became necessary to include different fare classes in the models and so one started in more detail to study the overbooking-seat allocation problem. This will be considered in the next subsection. Finally, in case of overbooking one needs to keep the frustrations of the denied boarding customers low and at the same time quantify the associated additional costs. Simon [30] suggests an auction method which addresses both of the above issues. However, despite its promise, this suggestions seems never to have been implemented in practice [27]. All the above models determine the overbooking limit by means of a static approach. In the paper written by Rothstein [26] a dynamic approach is used. He formulates a dynamic version of the overbooking problem with only only fare class. In his discrete time dynamic model the arrival process is a nonhomogeneous discrete time Markov process and the state of the system is given by the total number of reservations. Also the probabilities of the number of cancellations at a given discrete time depend on the number of reservations at that time and these conditional probabilities are independent of the state of the system before that time. Hence the model has a Markovian structure and one can apply Markov Decision Process techniques to derive an optimal overbooking policy.

1.1.2 Static Models for Overbooking and Seat Allocation.

Seat allocation policies for static models with no overbooking were first considered in the literature after 1970. Littlewood [20] models in a famous paper in 1972 in a nested way the two fare class single leg problem without overbooking. This means that that customers do not cancel and always show up at departure. He defines the concept of marginal seat revenue and then constructs the optimal nested booking limits. Bhatia and Parekh [6], and Richter [23] address the same problem and generalize the results of Littlewood [20]. Brumelle et al. [9] also examine the same problem as Littlewood [20] and generalize Littlewood results to dependent random demands for two fare classes. Belobaba [4] and [5] generalize the model of Littlewood [20] to more than two fare classes and use the exact results of Littlewood [20] derived for two fare classes to obtain a heuristic solution for m fare classes. This is the famous Expected Marginal Seat Revenue (EMSR) heuristic (actually a class of different heuristics bases on similar ideas) and these heuristics select a so-called nested partitioned booking limit for each fare class. To justify the heuristics results of Belobaba [4] and [5], Wollmer [35] and Brumelle and McGill [8] gave a proper formulation of the static nested multiple fare class problem with no overbooking under the assumption that demand is monotonic in fares; i.e., lowest fare is requested first. They prove that under this assumption the heuristic solution proposed by Belobaba is optimal as long as the demand distributions are identical for each fare class. Robinson [24] considers more general demand distributions and shows that applying the EMSR type of heuristics to some of those instances yield very poor results. This means that the EMSR methodology is appealing to use but might give in some instances bad results. Other papers on static models are written by Bodily and Pfeifer [7], Couglan [13] and Aydin et al. [2]. They all use different methods of solving the static overbooking and seat allocation problem by means of heuristics.

1.1.3 Dynamic Models for Overbooking and Seat Allocation

In static single leg formulations, most of the studies assume that, the arrival of the fare classes are ordered due to the desired nested interpretation of the partitioned booking limit. However in the dynamic approach to model a single leg problem, there is no need for such an assumption. In dynamic models one needs to decide upon an arrival of an individual request to accept this or not, and so the order of requests is not of importance.

In the literature all of the dynamic models for single leg seat allocation model with or without overbooking uses mostly the tools of Markov Decision Processes (MDP). One of the earliest dynamic model is given in Alstrup et al. [1] with two fare classes and overbooking. In their model there are cancelations and no-shows. However, the solution approach grows exponentially and becomes burdensome for real-size problems. Lee and Hersh [18] in 1993, proposes a discrete time dynamic model for no-overbooking problem. In their model the multi-fare arrivals are modeled by discrete time independent non-homogenous Poisson processes. Their model allows for bulk arrivals and multi reservation requests. Liang [19] in 1999 formulates the same problem stated in continuous time.

Subramanian et al. [31] extends the dynamic programming approach by also considering cancelations and no-shows. Although nothing mentioned about the arrival pattern in their paper, the problem is solved with Markov decision process in discrete time. They formulate the cancelations fare class dependent, however, then they make it fare class independent for computational reasons. In formulation, Subramanian et al [31], makes some assumptions which do not seem realistic. They divide the booking horizon in small periods. In each period, they formulate that only one event can be occurred: an arrival, a cancelation and no-event. Each event depends on the number of reserved customers at the previous period. Although this assumption seems reasonable for cancelation event, arrival and no-event must be independent from the number of reserved seats. Also they remove constant fare assumption and model the fares with time dependent. Aydın et al. [2] remove the assumption of the dependence of arrivals on number of reserved seats. Also they formulate the cancelations and arrivals are independent processes.

Chatwin [11] formulates similar single leg problem in continuous time. He models the arrival process with (less realistic) homogenous Poisson arrival process. The model allows that refunds and fares to be time dependent. Feng and Xiao [14] uses continuous time approach. The work is closely related with Chatwin's model. Feng and Xiao [14] extends Chatwin's work by removing fare independent no show-ups. Also they model arrival process with time and fare dependent Poisson arrival processes. In addition, Feng and Xiao [14] takes virtual capacity as decision variable whereas the rest of literature dynamic models take it as a parameter. However, they disregard cancelations in the formulation. Brumelle and Walczak [10] extends the Markov Process approach and allow to model the arrivals with more general non-homogenous arrival processses. Although that paper discusses the single leg seat allocation problem with very general Markovian type arrival process, it does not reveal much information about computation of the optimal policy.

1.2 Motivation

In this thesis, we attempt to formulate the single leg airline seat allocation and overbooking problems in continuous time within the Markov Decision Process framework. The booking requests arrive at any time between beginning of the reservation period 0 and departure time T. We do not allow bulk arrivals, therefore all booking requests come one by one. We assume that there are m fare classes and airline sets the constant fares before reservation is opened. We allow that each customer may cancel his/her reservation and cancellations are modeled according to Markovian cancelation process which is empirically showed by Rothstein [27]. If cancelation happens, customer will receive constant refund irrespective of his/her fare classes. Although these assumptions seems not realistic, they are common in the literature in order to reduce to state space and computational burden.

We model this problem different from Subramamanian et al. [31] and Aydin et al. [2] as our formulation is in continuous time based. Although discrete time models seem intuitive, they could not reflect the real world as realistic as continuous time models do. The arrival of booking requests and realization of cancelations can occur any time before departure. Therefore, continuous time models are more natural to formulate problem. In the literature, there exist three continuous time models to deal with similar problems. Chatwin [11] formulates the problem as Markov birth and death process and assumes that arrival processes of booking requests for different fare classes are independent homogeneous Poisson processes. In this thesis, we replace the time homogenous arrivals assumption with a time dependent arrival process. Another continuous time model written by Feng and Xiao [14], ignores the cancelations unlike our formulation. However, they consider the show up probabilities fare class dependent. Brummelle and Wallzcak [10] study on more general arrivals. Although they propose very general model for single leg problem which in bulk arrivals follow Semi-Markov process, their study does not contain any computational results. The model also seems incomputable. Our model is also different regarding setting the overbooking capacity. Most of the papers, overbooking capacity is regarded as a parameter. One exception is study of Feng and Xiao [14]. However, this characterization involves the value function itself and therefore it can be computationally expensive to calculate. Another distinction of our formulation is that does not have an actual virtual capacity.

Although our main contribution is a more natural mathematical analysis of the dynamic optimal policy for a single leg problem with overbooking under general conditions on the nonhomogeneous continuous arrival and homogenous cancellation process, we also perform an extensive computational study comparing the behavior of our optimal policy against the polices generated by several well known EMSR based heuristics. Simulation results show that our proposed policy generates higher revenue comparing to the policies generated by the EMSR based heuristic. Detailed results are presented in the computational section.

The thesis is organized as follows. In Chapter 2 we formulate and analyze by means of the dynamic approach the single leg problem with overbooking under general assumptions on the continuous time arrival and cancellation process. In Chapter 3 we discuss a way of computing the value function of the problem. In Chapter 4 we present by means of a simulation study the results generated by our optimal policy and compare these results with results given by the policies generated by several EMSR based heuristics available in the literature. Our concluding remarks can be found in Chapter 5.

Chapter 2

Analysis of a Continuous Time Single Leg Overbooking Model

In this section we present a continuous time dynamic model for the single leg seat allocation problem with no shows and cancellations. This means that also overbooking is allowed and the objective of our dynamic model is to select an optimal decision policy (if it exists) which maximizes the expected net revenue. Before discussing in detail the existence and construction of such an optimal policy we first introduce our model.

Consider a flight with seat capacity P and for this flight m different fare classes can be reserved. Let $\bar{P} > P$ denote the predetermined maximum number of reservations that the airline company will accept during the booking period. Observe $\bar{P}$ can also attain the value ∞ and this means that there is no a priory bound set on this maximum number. The continuous arrival process of fare classes is now defined as follows. Let T be the length of the booking period and $(T_i, L_i)_{\{i \in \mathbb{N}\}}$ a marked point process with T_i denoting the arrival time of the ith request and $L_i \in \{1, 2, \cdots, m\}$ the marker representing the type of booking request. The random counting measure of this marked point process is given by

$$\eta((0,t) \times \{j\}) = \sum_{i \in \mathbb{N}} 1_{(0,t] \times \{j\}}(T_i, L_i), \quad t \in [0,T], j \in \{1, 2, \cdots, m\}. \tag{2.1}$$

It is assumed that this counting measure is a Poisson random measure with mean

$$\begin{aligned} \nu((0,t) \times \{j\}) &= \mathbb{E}(\eta((0,t),\{j\}) \\ &= \int_{(0,t]} \lambda(u) q(u,j) du \end{aligned} \tag{2.2}$$

for $t \in [0, T]$ and $j \in \{1, 2, ..., m\}$ and $\lambda : [0, T] \mapsto \mathbb{R}_+$ a continuous intensity function. Also introducing the domain set $\triangle : [0, T] \times \{1, 2, \cdots, \bar{P}\}$ the function $q(t, j) : \triangle \mapsto [0, 1]$ represents the conditional probability of an arrival of a fare class j requests at time t given that an arrival occurs at time t. By this interpretation it is obvious for each $t \in [0, T]$ that

$$\sum\nolimits_{j=1}^{m} q(t,j) = 1. \tag{2.3}$$

Hence by construction the arrival process of each fare class is a nonhomogeneous Poisson process and these processes are independent. For a detailed overview on the theory of Poisson random measures the reader should consult Cinlar [12]. Also let r_i, $i \in \{1, ..., m\}$ denotes the price of fare class i. Without loss of generality we assume that

$$0 < r_1 < r_2 < ... < r_m \tag{2.4}$$

and so fare class 1 is the cheapest and fare class m the most expensive. This means that at an arrival of a fare class i request the airline will receive r_i if this request is accepted. This means that total revenue received after accepting an arriving request is given by $r(L_i)$ where

$$r(l) := \sum_{j \leq m} r_j 1_{\{l=j\}}. \tag{2.5}$$

Airline management has the option to reject or accept a request. In the sequel, the random vector $\mathcal{A} \equiv (A_i)_{i \in \mathbb{N}}$, keeps track of the accept and reject decisions for each booking request $i \in \mathbb{N}$. The event $\{A_i = 1\}$ shows that the ith booking request is accepted, while $\{A_i = 0\}$ denotes rejection. By the definition of $\bar{P}$ a booking request is rejected when the current number of reserved seats is equal to $\bar{P}$ and so in this case A_i is assigned the value zero.

We also allow cancellations in our model. It is assumed that each customer can later cancel his/her reservation independently of the other reservations and the random time to cancellation of each customer has an exponential distribution with common parameter μ. This means that all customers independent of their reserved fare class show probabilistically the same cancellation behavior. When a cancelation occurs, the airline company refunds the cancelling customer an amount κ and this amount is independent of the fare class. Denoting by X_i the exponentially distributed random time to cancellation of the ith arriving request and by $\mathcal{C} = \{\mathcal{C}_t : t \leq T\}$ the cancellation process it follows that the total number $\mathcal{C}_t$ of cancellations up to time t is given by

$$\mathcal{C}_t = \sum_{i \in \mathbb{N}} 1_{\{T_i + X_i \leq t\}} \cdot 1_{\{A_i = 1\}} \tag{2.6}$$

As previously mentioned the random variables X_i, $i \in \mathbb{N}$ are independent.

The *Accept* and *Reject* decisions are based on the information gathered over time by observing the arrival and cancelation process up to the arrival time of a request. Such policies will be referred as *admissible* in the sequel. The collection of admissible decisions $\{A_1, A_2, \cdots\}$ forms a non-decreasing and piece-wise constant jump process and let $D = \{\mathcal{D}_t\}_{t \in [0,T]}$ note the collectin of admissible policies.

$$\mathcal{D}_t = \sum_{i \leq N_t} A_i 1_{\{T_i \leq t\}}, \quad t \in [0, T]. \tag{2.7}$$

In terms of the cancellation process $\mathcal{C}$ and requests process $\mathcal{D}$, the controlled seat process $\mathcal{S} = \{\mathcal{S}_t\}_{t \in [0,T]}$ can be written as

$$S_t = S_0 + D_t - C_t \tag{2.8}$$

where $S_0 = 0$ is the initial number of seat. Clearly S_T denotes the number of reserved seats at the departure time. However, each customer with a reserved seat may not show-up at the boarding time. To model this behavior we assume that each customer independent of the fare class has a show up probability p. Hence, introducing the collection of independent Bernoulli random variables $(B_1, B_2, \cdots)$ each with success probability p, the total number of boarding request or show-ups is equal to $\sum_{i=1}^{S_T} B_i$. Since it might happen that customers are denied boarding due to the arrival at departure of more customers than the actual capacity of the plane, the airline pays for any denied boarding customer a penalty $\gamma > r_m$. Hence the total overbooking penalty is given by

$$\gamma \left(\sum_{i=1}^{S_T} B_i - P \right)^+ . \tag{2.9}$$

Combining all revenues, refunds and penalties, the net expected revenue of a given admissible policy $\mathcal{A}$ is given by

$$\mathbb{E} \left[\sum_{i=1}^{N_T} A_i r(L_i) - \kappa C_T - \gamma \left(\sum_{i=1}^{S_T} B_i - P \right)^+ \right], \tag{2.10}$$

where $N \equiv \{N_t\}_{t \geq 0}$ counts the cumulative reservation requests. Hence the objective of airline management is to evaluate

$$\sup_{\mathcal{A} \in \mathcal{D}} \mathbb{E} \left[\sum_{i=1}^{N_T} A_i r(L_i) - \kappa C_T - \gamma \left(\sum_{i=1}^{S_T} B_i - P \right)^+ \right] \tag{2.11}$$

and to find an admissible policy (if it exists) attaining this value. In the next sections we will show that such a policy indeed exists and also show how to evaluate such a policy.

2.1 On the Dynamic Programming Operator and its Properties.

To start with our analysis introduce the set

$$\Delta := \begin{cases} [0,T] \times \{0,1,...,\overline{P}\} & \text{if } \overline{P} < \infty \\ [0,T] \times \mathbb{Z}_+ & \text{if } \overline{P} = \infty \end{cases} \tag{2.12}$$

and introduce the optimal value function $V : \Delta \to \mathbb{R}$ given by

$$V(t,s) := \sup_{\mathcal{A} \in \mathcal{D}} G^{(A)}(t,s), \quad \text{for } (t,s) \in \triangle \tag{2.13}$$

where

$$G^{(\mathcal{A})}(t,s) := \mathbb{E}^{(t,s)} \left[\sum_{i=1}^{N_t} [A_i \, r(L_i)] - \kappa C_t - \gamma \left(\sum_{i=1}^{S_t} B_i - P \right)^+ \right] \tag{2.14}$$

In relation (2.14), the expectation operator $\mathbb{E}^{(t,s)}$ corresponds to the probability measure $\mathbb{P}^{(t,s)}$ under which

ii) the point process $(T_i, L_i)_{i \in \mathbb{N}}$ has the shifted compensator $\lambda^{(t)}(u)du \times q^{(t)}(u,j)$, where $\lambda^{(t)}(u) = \lambda(T-t+u)$ and $q^{(t)}(u) = q(T-t+u,j)$, for $u \geq 0$ and $1 \leq j \leq m$.

i) $S_0 = s$ with probability one and the random variable $\mathcal{C}_t$ also includes the total number of cancellations among the first s reservations.

Note by taking $t = T$ and $s = 0$ in (2.13) we obtain the optimal expected revenue. Due to the properties of a nonhomogeneous Poisson process the value $G^{(\mathcal{A})}(t,s)$ can also be seen as the total expected net revenue generated after time $T - t$ up to time T by the considered policy when there are s reservations at time $T - t$. Hence the value $V(t,s)$ denotes the optimal expected revenue over all admissible policies generated after time

$T-t$ when at time $T-t$ there are s reservations. Clearly this value depends on the used upper bound $\bar{P}$ but for notational convenience this dependence is not shown. At the same time the reader should note that in the above alternative interpretation there is a slight misuse of the original definition of T_i since now the random variable T_i represents the ith arrival after time $T-t$ of the original nonhomogeneous Poisson arrival process starting at time 0. To analyze the function V and derive its properties we introduce a sequence of functions V_n, $n \in \mathbb{N}$. The function V_n represent a truncated version of the optimal value function considered in (2.13). It only applies to the first n booking request the decision of acceptance or rejection and it rejects all the remaining arrivals. By definition it is given by

$$V_n(t,s) = \sup_{\mathcal{A} \in \mathcal{D}_n} G^{(\mathcal{A})}(t,s), \quad \text{for } n \in \mathbb{N} \text{ and } (t,s) \in \triangle, \tag{2.15}$$

where $\mathcal{D}_n$ is the set of all admissible controls with $A_i = 0$ for every $i > n$. The next result is obvious and requires no proof.

Corollary 1 *The functions $V_n : \Delta \to \mathbb{R}, n \in \mathbb{Z}_+$ are monotone in n and satisfy*

$$-\gamma \max\{s-P,0\} - \kappa s \leq V_0(t,s) \leq V_1(t,s) \leq ... \leq V(t,s) \leq r_m \Lambda(T) \tag{2.16}$$

for all $(t,s) \in \triangle$ and $\Lambda(T) = \int_0^T \lambda(u)du$.

For any Borel measurable function $f : \Delta \to \mathbb{R}$ we introduce for $\overline{P}$ finite and $n \in \{0,1,...,\overline{P}\}$ or $\overline{P} = \infty$ and $n \in \mathbb{N}$ the so-called supnorms

$$\| f \|_n := \sup_{t \in [0,T], s \in \{0,1,...n\}} | f(t,s) |$$

Clearly

$$\| f \|_\infty := \sup_{t \in [0,T], s \in \mathbb{Z}_+} | f(t,s) |$$

if $\bar{P} = \infty$, a Borel measurable function $f : \Delta \to \mathbb{R}$ is called locally bounded if $\| f \|_n$ is finite for every n. Also for $\overline{P}$ finite let $\mathcal{S}$ denote the linear space of Borel measurable bounded functions on Δ while for $\overline{P}$ infinite it represents the linear space of Borel measurable locally bounded functions on Δ satisfying $\lim_{n\uparrow\infty} c^n \parallel f \parallel_n = 0$ for every $0 < c < 1$. It is now obvious from Corollary 1 that the optimal value function V_n and V belong to $\mathcal{S}$. In the next lemma we show the intuitively clear result that the functions V_n converge in the supnorm to V and give an upperbound on the error.

Lemma 2 The sequence of functions $V_n : \Delta \to \mathbb{R}, n \in \mathbb{Z}_+$ converges in the supnorm to V. In particular it holds for every $n \geq 1$ that

$$\|V - V_n\|_{\overline{P}} \leq r_m \frac{\Delta(T)^{n+1}}{(n-1)!} \tag{2.17}$$

Proof. Let $\mathcal{A} \in \mathcal{D}$ and $\mathcal{A}^{(n)} \in \mathcal{D}_n$ be two admissible policies such that $\mathcal{A}^{(n)}$ coincides with the $\mathcal{A}$ until (and including) the first n jumps. Note that

$$\begin{aligned}
\mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_t} A_i \cdot r(L_i)\right] &= \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{n\wedge N_t} A_i \cdot r(L_i) + \sum_{i=n+1}^{\infty} A_i \cdot r(L_i)\, 1_{\{T_i \leq t\}}\right] \\
&\leq \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{n\wedge N_t} A_i \cdot r(L_i)\right] + r_m\, \mathbb{E}^{(t,s)}\left[\sum_{i=n+1}^{\infty} 1_{\{T_i \leq t\}}\right] \\
&\leq \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{n\wedge N_t} A_i \cdot r(L_i)\right] + r_m\, \frac{[\Lambda(T)]^{n+1}}{(n-1)!},
\end{aligned} \tag{2.18}$$

where the last inequality is due to Remark 20. Also, when we start with the same number s of reserved seats it is easy to see using the interpretation of $\mathcal{A}$ and $\mathcal{A}^{(n)}$ that the expected total cancellation fee and overbooking cost are both separately higher for $\mathcal{A}$ then for $\mathcal{A}^{(n)}$. This yields applying also (2.18)

$$G^{(\mathcal{A})}(t,s) \leq G^{(\mathcal{A}^{(n)})}(t,s) + r_m\, \frac{[\Lambda(T)]^{n+1}}{(n-1)!} \leq V_n(t,s) + r_m\, \frac{[\Lambda(T)]^{n+1}}{(n-1)!}.$$

Since this is true for any $\mathcal{A} \in \mathcal{D}$, we have

$$V(t,s) \leq V_n(t,s) + r_m\, \frac{[\Lambda(T)]^{n+1}}{(n-1)!}$$

Applying now Corollary 1 we also know that

$$V(t,s) \geq V_n(t,s)$$

for every $(t,s) \in \Delta$ and this shows the desired result. $\square$

To determine the properties of the function V we introduce the operator $\mathcal{L} : \mathcal{S} \to \mathcal{S}$ given by

$$\mathcal{L}[f](t,s) = \begin{cases} \sup_{A\in\{0,1\}} \mathcal{L}_A[f](t,s) & \text{if } s \in \{0,1,...,\overline{P}-1\} \\ \mathcal{L}_0[f](t,s) & \text{if } s = \overline{P} \end{cases} \tag{2.19}$$

with

$$\mathcal{L}_A[f](t,s) := \mathbb{E}^{(t,s)}\Bigg\{ -\kappa C_{t\wedge T_1} - 1_{\{T_1>t\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ + 1_{\{T_1\leq t\}}\Big[A\cdot r(L_1) + f(t-T_1, S_{T_1-}+A)\Big]\Bigg\}$$

For notational convenience we suppress the dependence of this operator on $\overline{P}$. This dependence will always be clear from the used context. In the next remark we will give an alternative description of the operator $\mathcal{L}$.

Remark 3 It is easy to see that the supremum in (2.19) is attained if we set

$$A = \begin{cases} 1, & \text{if } T_1 \leq t \text{ and } r(L_1) + f(t-T_1, S_{T_1-}+1) \geq f(t-T_1, S_{T_1-}), \\ 0, & \text{otherwise.} \end{cases}$$

Hence we obtain the equivalent representation $\mathcal{L}[f](t,s)$

$$= \mathbb{E}\Bigg\{ -\kappa C_{t\wedge T_1} - 1_{\{T_1>t\}} \cdot \gamma \cdot \left(\sum_{i=1}^{S_t} B_i - P\right)^+ + 1_{\{T_1 \le t\}} \cdot \mathcal{M}_{r(L_1)}[f](t-T_1, S_{T_1-}) \Bigg\}, \tag{2.20}$$

where

$$\mathcal{M}_r[f](t,s) = \begin{cases} \max\{r + f(t,s+1), f(t,s)\} & \text{if } s \in \{0,1,...,\overline{P}-1\}, \\ f(t,s) & \text{if } s = \overline{P} \end{cases} \tag{2.21}$$

In the next result we will show some important properties of the dynamic operator $\mathcal{L}$. As shown by the next result this operator preserves concavity and continuity of the function f.

Lemma 4 *If the function $s \mapsto f(t,s)$ is decreasing and discrete concave in s for each given t then the function $s \mapsto \mathcal{L}[f](t,s)$ also satisfies this property.*

Proof. We first will analyze the terms $\mathbb{E}C_{t\wedge T_1}$ and $\mathbb{E}1_{\{T_1>t\}}\left(\sum_{i=1}^{S_t} B_i - P\right)^+$ in (2.20). It is easy to see using the assumptions of the cancellation process and conditioning on T_1 that

$$\begin{aligned} \mathbb{E}C_{t\wedge T_1} &= s\mathbb{E}(1 - e^{-\mu(t\wedge T_1)}) \\ &= s[1 - e^{-\mu t}]e^{-\Lambda^t(t)} + \int_{(0,t]} \lambda^{(t)}(u) e^{-\Lambda^{(t)}(u)}(1 - e^{-\mu u})du \end{aligned} \tag{2.22}$$

where $\lambda^{(t)}(\cdot) = \lambda(T - t + \cdot)$ and $\Lambda^{(t)}(\cdot) = \int_0^{\cdot} \lambda^{(t)}(u)du = \Lambda(T-t+\cdot) - \Lambda(T-t)$. Also by the properties of independent binomial distributed random variables with the same success probability we obtain

$$\mathbb{E}1_{\{T_1>t\}}\left(\sum_{i=1}^{S_t} B_i - P\right)^+ = e^{-\Lambda^{(t)}(t)} \sum_{i=0}^{s} \binom{s}{i} (pe^{-\mu t})^i (1 - pe^{-\mu t})^{s-i}(i-P)^+, \tag{2.23}$$

From (2.22) and (2.23) this shows that the number of expected cancelation and denied boarding customers are increasing functions of s for each given t. Using now that the maximum of two decreasing functions is again decreasing and the definition of the operator $\mathcal{L}$ in (2.20), yields the monotonicity property in s. To prove the discrete concavity we first note that the expression in (2.22) is linear in s. Also applying Lemma B.2 in Aydin et al [2] we know that the expression for the expected number of denied boarding customer in (2.23) is discrete convex in s. Hence by the negative coefficients the first two terms in (2.20) are discrete concave in s. In addition, according to Lemma 1 of Lautenbacher and Stidham [17] the operator $\mathcal{M}$ preserves the concavity of the function $f(t,s)$ in s and so the function $s \mapsto \mathcal{L}[f](t,s)$ is discrete concave in s. Since the binomial selection scheme preserves concavity as shown in Lemma B.3 Aydın et al. [2] this implies that the expectation of $1_{\{T_1 \leq t\}}\mathcal{M}[f](t,s)$ is discrete concave in s and using the above observations we have verified the result. $\square$

We observe the following remarks whose proofs are given in the Appendix.

Remark 5 *If the mapping $t \mapsto f(t,s)$ is continuous for each s then the mapping $t \mapsto \mathcal{L}[f](t,s)$ also satisfies this property.*

Remark 6 *If the function $f : \Delta \to \mathbb{R}$ satisfies*

$$\kappa(1 - e^{-\mu t}) \leq f(t,s) - f(t,s+1) \leq \kappa(1 - e^{-\mu t}) + \gamma p e^{-\mu t} \tag{2.24}$$

for $(t, s+1) \in \triangle$, then these upper and lower bounds also hold for $\mathcal{L}[f](t,s) - \mathcal{L}[f](t,s+1)$ over the same domain.

In the next section we will analyze in detail the relation between the function V and the operator $\mathcal{L}$. It will be shown that V is a fixed point of this operator in the set S. There are also other properties like continuity in t (and also see Remark 6)

2.2 A Useful Sequence of Functions.

In the previous section we introduce the dynamic programming operator $\mathcal{L}$ and show that this operator preserves concavity and monotonicity in s. Using this operator $\mathcal{L}$ introduce now the function sequence $(U_n)_{n\in\mathbb{N}}$ given by

$$U_0(\cdot,\cdot) = V_0(\cdot,\cdot) \ and \ U_n(\cdot,\cdot) = \mathcal{L}[U_{n-1}](\cdot,\cdot), \ \ for \ \ n \geq 1 \tag{2.25}$$

Lemma 7 *The sequences in* (2.15) *and* (2.25) *coincide and we have*

$$U_n(t,s) = V_n(t,s) = G^{(\widetilde{\mathcal{A}}^{(n)}(t,s))}(t,s), \qquad \textit{for } (t,s) \in \triangle \textit{ and } n \geq 1, \tag{2.26}$$

where the dynamic control policy $\widetilde{\mathcal{A}}^{(n)}(t,s)$ *is determined in terms of the decision variables*

$$\widetilde{A}_i^{(n)}(t,s) = \begin{cases} 1, & \textit{if } T_i \leq t \ and \ r(L_i) + U_{n-i}(t - T_i, S_{T_i-} + 1) > U_{n-i}(t - T_i, S_{T_i-}), \\ 0, & \textit{otherwise,} \end{cases} \tag{2.27}$$

for $i \leq n$.

The proof of Lemma 7 is given in the appendix. Applying Lemma 2 and the previous result one can recover the optimal value function V by applying the operator $\mathcal{L}$ to the functions U_n successively. Hence, using the error bound in Lemma 2 and some preset $\epsilon \geq 0$, one can find some number n satisfying

$$\|V - U_n\|_{\overline{P}} \leq \epsilon. \tag{2.28}$$

This means we can approximate V up to a certain accuracy. The next lemma shows that V is a unique fixed point of the operator $\mathcal{L}$. This result yields an alternative approach for computing V. However this approach will not be pursued in the computational section.

Lemma 8 *Within the set $\mathcal{S}$ the optimal value function V is a unique fixed point of the operator $\mathcal{L}$.*

Proof. We already know that V belongs to $\mathcal{S}$ and first show that V is a fixed point of the operator $\mathcal{L}$. By Lemma 7 it follows that $V_n(\cdot,\cdot) = \mathcal{L}[V_{n-1}](\cdot,\cdot)$. Applying now $V_n \nearrow V$ and the monotone convergence theorem we obtain

$$\begin{aligned} V(t,s) &= \sup_{n\in\mathbb{N}} V_n(t,s) \\ &= \sup_{n\in\mathbb{N}} \sup_A \mathcal{L}_A[V_{n-1}](t,s) \\ &= \sup_A \sup_{n\in\mathbb{N}} \mathcal{L}_A[V_n](t,s) \\ &= \sup_A \mathcal{L}_A[V](t,s) \\ &= \mathcal{L}[V](t,s) \end{aligned} \tag{2.29}$$

and so V is a fixed point of the operator $\mathcal{L}$. To show the uniqueness of the fixed point within the set $\mathcal{S}$, let W be another function belonging to $\mathcal{S}$ satisfying $W = \mathcal{L}[W]$. Hence by Remark 3 we obtain

$$\begin{aligned} &W(t,s) - V(t,s) = \mathcal{L}[W](t,s) - \mathcal{L}[V](t,s) \\ &= \mathbb{E}^{(t,s)}(1_{\{T_1 \le t\}}\{\mathcal{M}_{r(L_1)}[W](t-T_1, S_{T_1-}) - \mathcal{M}_{r(L_1)}[V](t-T_1, S_{T_1-})\}). \end{aligned} \tag{2.30}$$

It is easy to see for $\overline{P}$ finite and $(t,s) \in \triangle$ that

$$\mathcal{M}_r[W](t,s) - \mathcal{M}_r[V](t,s) \le \|W - V\|_{\overline{P}}, \tag{2.31}$$

while for $\overline{P}$ infinite we have

$$\mathcal{M}_r[W](t,s) - \mathcal{M}_r[V](t,s) \le \|W - V\|_{s+1} \tag{2.32}$$

Applying (2.31) to relation (2.30) yields for $\overline{P}$ finite

$$\begin{aligned} W(t,s)-V(t,s) &\leq \mathbb{E}^{(t,s)}(1_{\{T_1\leq t\}}\|V-W\|_{\overline{P}}) \\ &= (1-e^{-(\Lambda(T)-\Lambda(T-t))})\|V-W\|_{\overline{P}} \\ &\leq (1-e^{-\Lambda(T)})\|V-W\|_{\overline{P}}. \end{aligned} \tag{2.33}$$

By reversing the roles of W and V we obtain

$$V(t,s)-W(t,s)\leq(1-e^{-\Lambda(T)})\|V-W\|_{\overline{P}}$$

and hence

$$\|V-W\|_{\overline{P}}\leq(1-e^{-\Lambda(T)})\|V-W\|_{\overline{P}}. \tag{2.34}$$

This shows $V=W$ and we have shown the result for $\overline{P}$ finite. For $\overline{P}$ infinite we obtain by a similar approach using (2.32) applied to (2.30) that for every $s\leq n$ and $0\leq t\leq T$

$$V(t,s)-W(t,s)\leq(1-e^{-\Lambda(T)})\|V-W\|_{n+1}.$$

Again we reverse the roles of V and W in the above inequality and this yields for $\overline{P}$ infinite that

$$\|V-W\|_n\leq(1-e^{-\Lambda(T)})\|V-W\|_{n+1}.$$

Iterating the above inequality q times we obtain

$$\|V-W\|_n\leq(1-e^{-\Lambda(T)})^q\|V-W\|_{n+q+1}.$$

Since W and V belongs to $\mathcal{S}$ it follows for every fixed n that

$$\lim_{q\uparrow\infty}(1-e^{-\Lambda(T)})^q\|V-W\|_{n+q+1}=0$$

and so $W = V$ on $[0,T] \times \{0,1,...,n\}$. Since n is arbitrary this yields $W = V$ on Δ. $\square$

Until this point we have introduced our objective function V and its relation to the operator $\mathcal{L}$. We have shown that V is a unique fixed point of this operator within the set $\mathcal{S}$ and for $\overline{P}$ finite the operator $\mathcal{L}$ is a contraction mapping in the supnorm $\| \, . \, \|_{\overline{P}}$ with contraction constant $1 - e^{-\Lambda(T)}$. In the sequel of this section we will use the functions V_n to derive global properties of the function V. To do so we first verify these properties for the functions V_n.

Lemma 9 *For every $n \in \mathbb{N}$ and $t \in [0,T]$ the function V_n is discrete concave and decreasing in s and satisfies for every $(t, s+1) \in \triangle$ and $s \in \{0,1,...,\overline{P}-1\}$*

$$\kappa(1 - e^{-\mu t}) \leq V_n(t,s) - V_n(t,s+1) \leq \kappa(1 - e^{-\mu t}) + \gamma p e^{-\mu t}. \tag{2.35}$$

Also for each s the function V_n is continuous in t.

Proof. We first show the desired result for $n = 0$. From (2.15) it is easy to see that

$$\begin{aligned} V_0(t,s) &= \mathbb{E}^{(t,s)}\left[-\kappa C_t - \gamma\left(\textstyle\sum_{i=1}^{s-C_t} B_i - P\right)^+\right] \\ &= -s\kappa(1-e^{-\mu t}) - \gamma \textstyle\sum_{i=0}^{s} \binom{s}{i} (pe^{-\mu t})^i (1-pe^{-\mu t})^{s-i} (i-P)^+ \end{aligned} \tag{2.36}$$

Hence by the first expression in relation (2.36) it is obvious that function V_0 is decreasing in s for each fixed t, while by Lemma B.3 in Aydin et al. [2] it follows that the function V_0 is concave in s for each fixed $t \leq T$. By the second expression in relation (2.36) it is also obvious that the function V_0 is continuous in t for fixed s. Applying now Remark 5 and Lemma 4 the result follows by induction for the functions V_n. To show relation (2.35) for $n = 0$, it follows as in the proof of Remark 6 that the difference $V_0(t,s) - V_0(t,s+1)$

can be written as

$$
\begin{aligned}
& \mathbb{E}^{(t,s)}\left[-\kappa C_t-\gamma\left(\sum_{i=1}^{s-C_t}B_i-P\right)^+\right]-\mathbb{E}^{(t,s)}\left[-\kappa C_t-\gamma\left(\sum_{i=1}^{s+1-C_t}B_i-P\right)^+\right] \\
= \ & \kappa(1-e^{-\mu t})+\gamma\mathbb{E}^{(t,s)}\left[\left(\sum_{i=1}^{s-C_t}B_i+Z-P\right)^+-\left(\sum_{i=1}^{s-C_t}B_i-P\right)\right]
\end{aligned}
$$

for an independent Bernoulli random variable Z with success probability $pe^{-\mu t}$. The last expectation above is non-negative and bounded above by $\mathbb{E}^{(t,s)}1_{\{Z=1\}}=pe^{-\mu t}$. Therefore, we have the bounds

$$
\kappa(1-e^{-\mu t})\leq V_0(t,s)-V_0(t,s+1)\leq\kappa(1-e^{-\mu t})+\gamma pe^{-\mu t}.
$$

This shows the inequality for $n=0$ and by Remark 6 and induction the same bounds hold for $V_n(t,s)-V_n(t,s+1)$ for all $n\in\mathbb{N}$. □

An immediate consequence of Lemma 2 and Lemma 9 is given by the next result.

Lemma 10 *The function V is discrete concave and decreasing in s and satisfies for every $(t,s+1)\in\triangle$*

$$
\kappa(1-e^{-\mu t})\leq V(t,s)-V(t,s+1)\leq\kappa(1-e^{-\mu t})+\gamma pe^{-\mu t}. \tag{2.37}
$$

Also for each s the function V is continuous in t.

Proof. Since the functions V_n converge in the supnorm to V the desired result follows by Lemma 9 and Theorem 7.12 of Rudin [29]. □

In this section we have shown that the optimal value function V can be easily computed. However we did not prove that an optimal policy exists. This will be the topic of the next section.

2.3 On the Existence of an Optimal Policy

Introduce the policy $\mathcal{A}^*(t,s) \equiv (A_i^*(t,s))_{i\in\mathbb{N}}$ as follows;

$$A_i^*(t,s) = \begin{cases} 1, & \text{if } r(L_i) + V(t-T_i, S_{T_i-}+1) > V(t-T_i, S_{T_i-}) \text{ and } T_i \le t, \\ 0, & \text{otherwise,} \end{cases} \tag{2.38}$$

for $i \in \mathbb{N}$. At this point, we still need to show that the policy given in (2.38) is an optimal policy. To verify this we prove the following results.

Proposition 11 *For all $(t,s) \in \triangle$ and $n \ge 1$, we have*

$$V(t,s) = \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_{t\wedge T_n}} A_i^* \cdot r(L_i) - \kappa C_{t\wedge T_n} + V\left(t - t\wedge T_n, S_{t\wedge T_n}\right)\right]. \tag{2.39}$$

Proof. For $n = 1$, the right hand side in (2.39) can be written as

$$\begin{aligned}
&\mathbb{E}^{(t,s)}\Big[-\kappa C_{t\wedge T_1} + 1_{\{t<T_1\}} V\left(0, S_t\right) + 1_{\{T_1\le t\}}\Big[A_1^* \cdot r(L_1) + V\left(t - T_1, S_{T_1}\right)\Big]\Big] \\
&= \mathbb{E}^{(t,s)}\left[-\kappa C_{t\wedge T_1} - 1_{\{t<T_1\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ + 1_{\{T_1\le t\}}\Big[A_1^* \cdot r(L_1) + V\left(t - T_1, S_{T_1}\right)\Big]\right] \\
&= \mathcal{L}[V](t,s) = V(t,s),
\end{aligned}$$

where the last equality is due to Lemma 8.

Now suppose (2.39) holds for some $n \geq 1$. Let us then decompose the right side in (2.39) with $n+1$ as

$$\mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_{t\wedge T_{n+1}}} A_i^* \cdot r(L_i) - \kappa C_{t\wedge T_{n+1}} + V\left(t - t\wedge T_{n+1}, S_{t\wedge T_{n+1}}\right)\right]$$

$$= \mathbb{E}^{(t,s)}\Big[- \kappa C_{t\wedge T_1} + 1_{\{t<T_1\}} V\left(0, S_t\right) + 1_{\{T_1 \leq t\}}\Big[- \kappa (C_{t\wedge T_{n+1}} - C_{T_1})$$

$$+ \sum_{i=1}^{N_{t\wedge T_{n+1}}} A_i^* r(L_i) + V\left(t - t\wedge T_{n+1}, S_{t\wedge T_{n+1}}\right)\Big]\Big]$$

$$= \mathbb{E}^{(t,s)}\Bigg[- \kappa C_{t\wedge T_1} + 1_{\{t<T_1\}} V\left(0, S_t\right) + 1_{\{T_1\leq t\}} A_1^* \cdot r(L_1)$$

$$+ 1_{\{T_1\leq t\}}\mathbb{E}^{(t,s)}\left[- \kappa (C_{t\wedge T_{n+1}} - C_{T_1}) + \sum_{i=2}^{N_{t\wedge T_{n+1}}} A_i^* r(L_i) + V\left(t - t\wedge T_{n+1}, S_{t\wedge T_{n+1}}\right) \middle| \mathcal{F}_{T_1}\right]\Bigg].$$

where $\mathcal{F}_{T_1}$ is the information generated by the arrival and cancelation processes as time T_1 Note that on the event $\{T_1 \leq t\}$ we have $t - t\wedge T_{n+1} = (t-T_1) - (t-T_1)\wedge(T_n \circ \theta_{T_1} - T_1)$. Hence, the conditional expectation above can be replaced with $V(t-T_1, S_{T_1})$ thanks to the induction hypothesis and the strong Markov property. Carrying out this substitution, we obtain

$$\mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_{t\wedge T_{n+1}}} A_i^* \cdot r(L_i) - \kappa C_{t\wedge T_{n+1}} + V\left(t - t\wedge T_{n+1}, S_{t\wedge T_{n+1}}\right)\right]$$

$$= \mathbb{E}^{(t,s)}\Big[- \kappa C_{t\wedge T_1} + 1_{\{t<T_1\}} V\left(0, S_t\right) + 1_{\{T_1\leq t\}}[A_1^* \cdot r(L_1) + V(t - T_1, S_{T_1})]\Big]$$

$$= \mathbb{E}^{(t,s)}\Big[\sum_{i=1}^{N_{t\wedge T_1}} A_i^* \cdot r(L_i) - \kappa C_{t\wedge T_1} + V(t - t\wedge T_1, S_{t\wedge T_1})\Big] = V(t,s)$$

where we used the result for $n = 1$ in the last equality. This proves (2.39) for $n+1$ and we have verified the result. □

Proposition 12 *The policy $\mathcal{A}^*(t,s)$ attains the supremum in* (2.13) *and we have*

$$V(t,s) = G^{(\mathcal{A}^*(t,s))}(t,s) \tag{2.40}$$

for all $(t,s) \in \triangle$.

Proof. Let us take the limsup as $n \to \infty$ of the right hand side in (2.39). This shows

$$\begin{aligned} V(t,s) &= \limsup_{n\uparrow\infty} \mathbb{E}^{(t,s)} \left[\sum_{i=1}^{N_{t\wedge T_n}} A_i^* r(L_i) - \kappa C_{t\wedge T_n} + V(t - t\wedge T_n, S_{t\wedge T_n}) \right] \\ &\leq \limsup_{n\uparrow\infty} \mathbb{E}^{(t,s)} \left[\sum_{i=1}^{N_{t\wedge T_n}} A_i^* r(L_i) - \kappa C_{t\wedge T_n} \right] \\ &\qquad + \limsup_{n\uparrow\infty} \mathbb{E}^{(t,s)} \left[V(t - t\wedge T_n, S_{t\wedge T_n}) \right]. \end{aligned} \tag{2.41}$$

Using $\lim_n T_n = \infty$ it follows $N_{t\wedge T_n} \nearrow N_t$, $C_{t\wedge T_n} \nearrow C_t$, $S_{t\wedge T_n} \to S_t$ and $V(t - t \wedge T_n, S_{t\wedge T_n}) \to V(0, S_t)$ with probability one. Also by Corollary 1 we obtain

$$V(\cdot,\cdot) \leq r_m \Lambda(T) \tag{2.42}$$

and this implies by Fatou lemma that the second term in (2.41) satisfies

$$\begin{aligned} \limsup_{n\uparrow\infty} \mathbb{E}^{(t,s)} \left[V(t - t\wedge T_n, S_{t\wedge T_n}) \right] &\leq \mathbb{E}^{(t,s)} (\limsup_{n\uparrow\infty} V(t - t\wedge T_n, S_{t\wedge T_n})) \\ &= \mathbb{E}^{(t,s)} V(0, S_t). \end{aligned} \tag{2.43}$$

The first term of (2.41) consists of the difference of two increasing sequences of positive random variables each being bounded by an integrable random variable. Hence we may apply the monotone convergence theorem and this yields

$$\limsup_{n\uparrow\infty} \mathbb{E}^{(t,s)} \left[\sum_{i=1}^{N_{t\wedge T_n}} A_i^* r(L_i) - \kappa C_{t\wedge T_n} \right] = \mathbb{E}^{(t,s)} \left[\sum_{i=1}^{N_t} A_i^* r(L_i) - \kappa C_t \right]. \tag{2.44}$$

Using (2.43),(2.44) and (2.41) we finally obtain

$$\begin{aligned} V(t,s) &\leq \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_t} A_i^* r(L_i) - \kappa C_t) + V(0,S_t)\right] \\ &= \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_t} A_i^* r(L_i) - \kappa C_t) - \gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+\right] \\ &= G^{(A^*(t,s)}(t,s) \\ &\leq V(t,s) \end{aligned}$$

and the desired result is proved. □

Finally we relate the optimal value function V to the accept-reject decisions given by the optimal policy. As already observed a policy decides whether to accept or reject a booking request according to the arrival time of a request, its fare class and the number of reservation at this arrival. An ith fare class booking request arriving at time $T-t$ and s seats are reserved at time $T-t$ is accepted if and only if

$$r_j + V(t,s+1) \geq V(t,s).$$

Since the mapping $s \mapsto V(t,s)$ is concave, the difference $V(t,s) - V(t,s+1)$ is a decreasing function in s for fixed $t \leq T$. Hence we can introduce a booking limit of fare class j when there is still t time units to departure as

$$s^*_{t,j} := \max\{s \in \{0,1,\cdots,\bar{P}\} : r_j \geq V(t,s) - V(t,s+1)\} \text{ for } j \in \{1,\cdots,m\} \tag{2.45}$$

Airline management now accept a booking request of a fare class type j arriving at time $T-t$ if and only if the number of reserved seat at time $T-t$ is less than this booking limit $s^*_{t,j}$. Clearly these booking limits are monotone in j, i.e for any $t \leq T$

$$s^*_{t,1} \leq \cdots \leq s^*_{t,m} \tag{2.46}$$

Finally we mention some possible extensions of the above analysis which can be done by similar techniques. First of all is also possible to analyse batch arrivals of the same fare class under the assumption that all of the requests in this batch are either rejected or accepted. In this case we model the arrival process of requests by a nonhomogeneous compound Poisson process. Also it is possible to consider the case that the cancellation rate μ is a function of the current time. Finally if the cancellation rate and the show up probability are fare class dependent one need to extend the state space to $m + 1$-dimensions. In this case the i component represent the current number of reservations of fare class i. However, computing the optimal policy is computationally very expensive.

Chapter 3

Computation of the Optimal Policy

In this chapter, we present the solution approaches for computing value function V. First note that the optimal net revenue that the airline can generate is a function of $\bar{P}$. Therefore initially we demonstrate the choice of $\bar{P}$. Then numerical evaluation of function V is revealed.

3.1 Setting the Value of $\bar{P}$

Needless to say, the optimal expected net revenue that the airline can generate is a non-decreasing function of $\bar{P}$. Earlier work on continuous and discrete time problems generally assumes that $\bar{P}$ is a given parameter, for example a fixed multiple of the capacity P. One exception is Feng (2006), which treats $\bar{P}$ as a decision variable and gives a characterization for the *effective* overbooking limit beyond which no improvement is obtained in the optimal expected revenue; see Section 4 in Feng 2006. However, this characterization involves the value function itself, and therefore it can be computationally expensive to calculate. Moreover, in cases where it is optimal to accept every request, the effective overbooking limit becomes infinity.

If the booking limit is a decision variable, it becomes important to control the loss in the expected revenue by choosing the value of $\bar{P}$ properly. Let $V^{(\bar{P})}(T,0)$ and $V^{(\infty)}(T,0)$

denote the optimal expected net revenue respectively with an overbooking limit $\bar{P}$ and without any limit. Lemma 13 below gives an error bound on the convergence of $V^{(\bar{P})}(T,0)$ to $V^{(\infty)}(T,0)$, and therefore allows us to select the value of $\bar{P}$ that would keep the loss in the revenue within acceptable tolerance limits.

Lemma 13 *As $\bar{P}$ goes to ∞, $V^{(\bar{P})}(T,0)$ converges to $V^{(\infty)}(T,0)$ and we have*

$$0 \leq V^{(\infty)}(T,0) - V^{(\bar{P})}(T,0) \leq r_m \frac{[\Lambda(T)]^{\bar{P}+1}}{(\bar{P}-1)!}. \tag{3.1}$$

Proof. We only need to prove the second inequality (3.1) as the first one is immediate. Let $\mathcal{A}^{(\infty)} = (A_i^{(\infty)})_{i\in\mathbb{N}}$ be the optimal policy for $V^{(\infty)}(T,0)$. Under $\mathcal{A}^{(\infty)}$ that among the first $N_T \wedge \bar{P}$-many requests some of them may be rejected. Among those which are accepted, let $C_T^{\bar{P}}$ and $S_T^{\bar{P}}$ denote respectively the number of cancellations and number of remaining reservations as of the departure time. Note that $C_T^{\bar{P}} \leq C_T$, $S_T^{\bar{P}} \leq S_T$ and $C_T^{\bar{P}} + S_T^{\bar{P}} = \sum_{i=1}^{N_T\wedge\bar{P}} A_i^{(\infty)}$. Then using the optimality of $\mathcal{A}^{(\infty)} = (A_i^{(\infty)})_{i\in\mathbb{N}}$ we write

$$\begin{aligned}
V^{(\infty)}(T,0) &= \mathbb{E}^{(T,0)}\left[\sum_{i=1}^{N_T}\left[A_i^{(\infty)} r(L_i)\right] - \kappa C_T - \gamma\left(\sum_{i=1}^{S_T} B_i - P\right)^+\right] \\
&\leq \mathbb{E}^{(T,0)}\left[\sum_{i=1}^{N_T\wedge\bar{P}}\left[A_i^{(\infty)} r(L_i)\right] - \kappa C_T^{\bar{P}} - \gamma\left(\sum_{i=1}^{S_T^{\bar{P}}} B_i - P\right)^+\right] \\
&\qquad + \mathbb{E}^{(T,0)}\left[\sum_{i=\bar{P}+1}^{\infty} 1_{\{T_i\leq T\}} A_i^{(\infty)} r(L_i)\right] \\
&\leq V^{(\bar{P})}(T,0) + r_m\,\mathbb{E}^{(T,0)}\left[\sum_{i=\bar{P}+1}^{\infty} 1_{\{T_i\leq T\}}\right] \leq V^{(\bar{P})}(T,0) + r_m\frac{[\Lambda(T)]^{\bar{P}+1}}{(\bar{P}-1)!}
\end{aligned}$$

where the last inequality follows from Remark 20 in the Appendix A. □

3.2 Numerical Computation of the Optimal Value Function V

In this section, we present two approaches for computing the optimal value function V considered in Chapter 2. The first procedure is to apply the operator $\mathcal{L}$ successively and the second is to utilize the Hamilton-Jacobi-Bellman equation. In the sequel we will discuss both methods of which the last is implemented on a computer.

Recall our objective function

$$V(t,s) = \sup_{\mathcal{A}\in\mathcal{D}} G^{(\mathcal{A})}(t,s), \qquad \text{for } (t,s) \in \triangle,$$

where

$$G^{(\mathcal{A})}(t,s) := \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_t}[A_i\, r(L_i)] - \kappa C_t - \gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+\right]$$

In the first method, we follow the basic steps for computing V;

- First step is constructing the function,

$$V_0(t,s) = -s\kappa(1-e^{-\mu t}) - \gamma\sum_{i=0}^{s}\binom{s}{i}(pe^{-\mu t})^i(1-pe^{-\mu t})^{s-i}(i-P)^+.$$

- Applying dynamic operator $\mathcal{L}$ iteratively starting with $V_0(t,s)$, we can compute $V_n(t,s),\ n = 1,2,\cdots$ successively.

- Thanks to Lemma 7, we can terminate the iterations for given ϵ. We simply fix a value of n large enough so that $\|V(t,s) - V_n(t,s)\| \leq \epsilon$, for some $\epsilon > 0$, so that the approximation error is at most ϵ

As seen in the operator $\mathcal{L}$, the calculation contains many integrations. Due to these integrals, overall computational complexity for given mesh length $h > 0$ becomes $o(mT^2\bar{P}^2)$ where T is total booking period and $\bar{P}$ is number of seats that offer for reservations.

Alternatively, instead of computing V_n's via the operator $\mathcal{L}$, we can use the infinitesimal transition probabilities of the processes C and N. We know that the optimal policy $\mathcal{A}^*$ in (2.38) implies that

$$\begin{aligned}
V(t,s) &= G^{(\mathcal{A}^*)}(t,s) = \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_t} A_i^* \, r(L_i) - \kappa C_t - \gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+\right] \\
&= \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_h} A_i^* \, r(L_i) - \kappa C_h + \mathbb{E}^{(t,s)}\left[\sum_{i=N_h+1}^{N_t} A_i^* \, r(L_i) - \kappa(C_t - C_h) - \gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ \middle| \mathcal{F}_h\right]\right] \\
&= \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_h} A_i^* \, r(L_i) - \kappa C_h + G^{(\mathcal{A}^*)}(t-h, S_h)\right] = \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_h} A_i^* \, r(L_i) - \kappa C_h + V(t-h, S_h)\right]
\end{aligned}$$

for small $h \leq t$, thanks to the Markov property. Hence, this property allows us to use the usual *infinitesimal first step analysis* as Chapter 4 in Karlin and Taylor [15] with the transition probabilities

$$\begin{aligned}
&\mathbb{P}\left\{C_{t+h} - C_t = 1 \middle| \mathcal{F}_t\right\} = \mu S_t h + o(h) \\
&\mathbb{P}\left\{N_{t+h} - N_t = 1 \ \middle| \mathcal{F}_t\right\} = \lambda(t) h + o(h) \\
&\mathbb{P}\left\{N_{t+h} - N_t = 0 \ \middle| \mathcal{F}_t\right\} = 1 - \mu S_t h - \lambda(t) h + o(h).
\end{aligned}$$

We can then compute $V(t,s)$ after observing that for small $h > 0$

$$\begin{aligned}
V(t,s) = (-\kappa + V(t-h, s-1))\mathbb{P}\left\{C_{t+h} - C_t = 1 \middle| \mathcal{F}_t\right\} \\
+ \left[\sum_{j=1}^{m} q^t(0,j) M_{r_j}[V](t,s)\right] \mathbb{P}\left\{N_{t+h} - N_t = 1 \ \middle| \mathcal{F}_t\right\} \\
+ V(t-h, s)\mathbb{P}\left\{N_{t+h} - N_t = 0 \ \middle| \mathcal{F}_t\right\}
\end{aligned}$$

Then by differentiating in t, we obtain

$$\frac{\partial V(t,s)}{\partial t} = \mu s(-\kappa + V(t,s-1)) + \\ \sum_{j=1}^{m} \lambda^{(t)}(0)\, q^{(t)}(0,j)\, M_{r_j}[V](t,s) - \left[\mu s + \lambda^{(t)}(0)\right] V(t,s), \tag{3.2}$$

which is the well known Hamilton Jacobi Bellman equation for the problem in (2.13). Similar analysis can also repeat for sequence of functions V_n. For small $h \geq 0$, we can approximate $\frac{\delta V(t,s)}{\delta t}$ with $\frac{V_n(t+h,s)-V_n(t,s)}{h}$ and this gives

$$\frac{V_n(t+h,s) - V_n(t,s)}{h} \approx \mu s(-\kappa + V_{n-1}(t,s-1)) - [\mu s + \lambda^{(t)}(0)]V_{n-1}(t,s) \\ + \sum_{j=1}^{m} \lambda^{(t)}(0) q^{(t)}(0,j) \max(r_j + V_{n-1}(t,s+1), V_{n-1}(t,s))$$

After arranging the terms, we get

$$V_n(t+h,s) = h\mu s(-\kappa + V_{n-1}(t,s-1)) + V_{n-1}(t,s)(1 - h[\mu s + \lambda^{(t)}(0)]) \\ + h\sum_{j=1}^{m} \lambda^{(t)}(0) q^{(t)}(0,j) \max(r_j + V_{n-1}(t,s+1), V_{n-1}(t,s)) \tag{3.3}$$

Main steps of the computations summarized as follow:

- For each $s \leq \bar{P}$, $V_0(0,s)$ is boundary condition and can directly be computed as follow;

  $$V_0(0,s) = \mathbb{E}\left(\sum_{i=1}^{s} B_i - P\right)^+ \quad \text{for } s \leq \bar{P}.$$

  and due to only cancelation allowed for V_0, we use following approximations;

  $$V_0(t+h,s) = h\mu s(-\kappa + V_0(t,s-1)) + (1 - h\mu s)V_0(t,s)$$

- Employing the equation (3.3) derived from Hamilton Jacobi Bellman equation first on V_0 and continuing iteratively, we compute $V_n(t,s)$.

- Total iteration number is a function of predetermined error term $\epsilon > 0$. According to Lemma 7, given ϵ, there is a large number n such that $\|V(t,s) - V_n(t,s)\| \leq \epsilon$.

Overall complexity of this method for fixed mesh h is $o(mT\bar{P}^2)$.

An immediate consequence of (3.2) is that $V(t,s)$ is non-decreasing in t when $s = 0$, which is intuitive; when no seat is reserved initially, the airline can perform better when there is more time to departure.

Chapter 4

Computational Experiments

In this section, we give a detailed analysis of the simulation setup and report the behavior of the different policies under this simulation setup. In particular we compare the behavior of the DP policy and three well known EMSR heuristics. Notice that these heuristics use the computed booking limits in the same standard nested way as defined on page 28 in Van Ryzin and Talluri [32] but may differ in their choice of the virtual capacity of the plane .

4.1 Simulation Setup

To start explaining our computational experiments we first give a brief explanation of our simulation setup. The arrival process of requests is given by a nonhomogeneous Poisson process with continuous intensity function $\lambda : \mathbb{R}_+ \to \mathbb{R}$. At an arrival time t the arrival is a type j fare class requests with probability $q(t, j)$. The arrival processes of the different fare classes are now given by independent nonhomogeneous Poisson processes with arrival intensity function $\lambda_i : \mathbb{R}_+ \to \mathbb{R}$ given by

$$\lambda_i(t) = q(t, j)\lambda(t).$$

Then total arrival intensity function can also be written as

$$\lambda(t) = \sum\nolimits_{j=1}^{m} \lambda_j(t), \;\; q(t,j) = \frac{\lambda_j(t)}{\lambda(t)}.$$

Also in the simulation we assume that each reserved seat can be cancelled independently and the time to cancelation is a realization of an exponentially distributed random variable. The parameter of this exponential distribution for all reservations is given by $\mu > 0$. To simulate the arrival and cancelation processes we use discrete event simulation. Let $t_n, n \in \mathbb{N}$ be the realized arrival time of the n'th event with $t_0 = 0$ and the total number of reservation at time t_0 equal to zero. To generate the realization t_1 of the arrival time of the first event (necessarily a fare class request) we simulate using the thinning procedure for a nonhomogeneous Poisson process the first arrival time of the arrival process of requests. In this procedure (see page 693 of [25]) we set $\lambda = \max_{0 \leq t \leq T} \lambda(t)$. After having determined the arrival time t_1 of the first arrival we select its type using a multinomial experiment. Remember this arrival is of type j with probability $q(t_1, j)$. Given now the realized arrival t_n of the $n'th$ event being either a cancelation or a type j fare class arrival we generate the next event at time t_{n+1} as follows. If at time t_n there are $s > 0$ reservations the time until the first cancelation among the s seats is given by $t_n + \gamma_{t_n}^c$ with $\gamma_{t_n}^c$ a realization of an exponentially distributed random variable with parameter $s\mu$. Clearly if $s = 0$ no cancelations occur. Also, independently of this cancelation process, we select the first arrival time of a fare class request after time t_n given by $t_n + \gamma_{t_n}^a$. Notice the realization $t_n + \gamma_{t_n}^a$ is generated according to the already mentioned thinning procedure and given this realization we also determine its type in a similar way as before. We now set $t_{n+1} = \min\{t_n + \gamma_{t_n}^a, t_n + \gamma_{t_n}^c\}$ and determine whether this minimum is attained by either a cancelation or a request arrival. If it is a cancelation we set the number of reserved seats to $s - 1$ and if it is an arrival we apply to this arrival our policy given either by the EMSR heuristic or the DP algorithm. Hence we have generated the next arrival time t_{n+1} and updated immediately after this time our number of reserved seats. Now continue in the same way by selected the next arrival time of the $(n+2)'nd$ event until we hit the departure time T. At time T we then consider the total number s_{total} of reserved sets and generate a realization of a binomial distributed random variable with success probability β^s and s_{total} trials to determine the realization of the total number of customers showing up at the departure time. Clearly for this simulation procedure we need to know the arrival intensity functions and this will be the topic of the next subsection.

4.1.1 Selection of the Intensity Functions

In our computational setup we have selected different arrival intensity function emphasizing the difference between the policies generated by the DP and the EMSR heuristics. To evaluate the policy of the EMSR heuristic it is only important to know the expected number of type j fare class arrivals over the whole booking period. For the evaluation of the DP policy it is also important to know how these arrivals are spread over this booking period. To stress this difference we try to select intensity functions which give the same expected number of arrivals over the whole period but are differently shaped. In particular we are interested for $m = 2$ in the earliest and latest possible time that given an arrival this is more likely to be an arrival of an expensive fare class request. To construct these intensity functions for the different fare classes let T denote the length of the booking period and assume that the fares of the different fare classes $1, \cdots, m$ are given by r_i. Without loss of generality these fares satisfy $r_1 < r_2 < \cdots < r_m$ and so fare class m is the most expensive while fare class 1 is the cheapest one. To select the intensity functions we first consider only two fare classes and generalize the used approach to $m > 2$ cases. We start with the following normalized functions $a_i : \mathbb{R}_+ \mapsto \mathbb{R}$.

Condition 14 *The functions $a_i : \mathbb{R}_+ \to \mathbb{R}, i = 1, 2$ satisfy the following conditions.*

- *The function a_1 is a nonnegative continuously differentiable decreasing function on $[0, T]$ satisfying $\int_0^T a_1(s)ds = 1$.*
- *The function a_2 is a nonnegative continuously differentiable increasing function on $[0, T]$ satisfying $\int_0^T a_2(s)ds = 1$.*

If $\lambda_i : \mathbb{R}_+ \to \mathbb{R}$ denote the intensities functions for type j fare class arrivals then we set

$$\lambda_1(s) := \sigma_1 a_1(s), \lambda_2(s) := \alpha\sigma_1 a_2(s) \tag{4.1}$$

with $\sigma_1 > 0, \alpha > 0$. For these intensity functions it is clear that

$$\mathbb{E}(\text{number of arriving fare class 1 customers}) = \int_0^T \lambda_1(s)ds = \sigma_1$$

and

$$\mathbb{E}(\text{number of arriving fare class 2 customers}) = \int_0^T \lambda_2(s)ds = \alpha\sigma_1.$$

Since on average cheaper fare class 1 requests arrive one should select $0 < \alpha < 1$. To measure the capacity of the plane in comparison with the total number of expected requests the load ρ of the system is by definition

$$\rho := \frac{\mathbb{E}(\text{expected number of arriving customers})}{C} = \frac{\sigma_1(\alpha+1)}{C}$$

By Condition 14 and relation (4.1) the function λ_1 is decreasing and λ_2 is increasing. As shown in the next lemma this monotonicity property represent the tendency of fare class 1 customers to arrive more frequently than fare class 2 customers during the beginning of the booking period while the reverse is true towards at the end of the booking period. To verify this we observe

$$\begin{aligned} p_i(t) &= \mathbb{P}(\text{arrival is fare class } i \text{ request} \mid \text{arrival at time } t) \\ &= \tfrac{\lambda_i(t)}{\lambda_1(t)+\lambda_2(t)} \end{aligned} \tag{4.2}$$

This shows using relation (4.1) that

$$p_1(t) = \frac{a_1(t)}{a_1(t)+\alpha a_2(t)}, p_2(t) = \frac{\alpha a_2(t)}{a_1(t)+\alpha a_2(t)} \tag{4.3}$$

The following result is easy to verify.

Lemma 15 *If the arrival intensity function $\lambda_i, i = 1, 2$ are given by relation (4.1) and the functions $a_i, i = 1, 2$ satisfy condition 14, then the function $t \mapsto p_1(t)$ is decreasing and $t \mapsto p_2(t)$ is increasing. Also $p_2(t) \geq p_1(t)$ if and only if $\alpha a_2(t) \geq a_1(t)$.*

By the above lemma the time

$$t_* = \min\{0 \le t \le T : p_2(t) \ge p_1(t)\} = \min\{0 \le t \le T : \alpha a_2(t) - a_1(t) \ge 0\} \quad (4.4)$$

represents the earliest possible time that an arrival after this time is more likely to be a class 2 type arrival. To guarantee that t_* is well defined the selected intensity functions a_i must satisfy Condition 14 with the additional conditions $a_1(0) \ge \alpha a_2(0)$ and $a_1(T) \le \alpha a_2(T)$. Among the set of functions a_i satisfying all these restrictions we now would like to determine the extremal elements which minimize and maximize the value t_*. Using these extremal elements we can observe that the EMSR heuristics only take in consideration the expected number of customers whereas the policy of the DP algorithm might change due to a changing time t_*. Note it is easy to verify for the selected intensity functions that the feasible set of this optimization problem is described as follows

1. The function $a_1$is a nonnegative continuously differentiable function on $[0, T]$ satisfying
$$\int_0^T a_1(s)ds = 1, \ \max_{0 \le s \le T} a_1'(s) \le 0. \quad (4.5)$$
2. The function a_2 is a nonnegative continuously differentiable function on $[0, T]$ satisfying
$$\int_0^T a_2(s)ds = 1, \ \min_{0 \le s \le T} a_2'(s) \ge 0 \quad (4.6)$$
3. It holds that
$$a_1(0) - \alpha a_2(0) \ge 0, \ \alpha a_2(T) - a_1(T) \ge 0 \quad (4.7)$$

In the following we solve this optimization problem for special cases where $a_i(t)$'s are either linear or quadratic functions. We consider linear intensity functions first.

4.1.1.1 Linear Intensity Functions

Let the functions $a_1(s)$ and $a_2(s)$ given by

$$a_1(s) = a_{11} - a_{12}s \;\; a_2(s) = a_{21} + a_{22}s \tag{4.8}$$

and by relation (4.1) the arrival intensities are;

$$\lambda_1(s) = \sigma(a_{11} - a_{12}s), \quad \lambda_2(s) = \sigma\alpha(a_{21} + a_{22}s).$$

For the linear function case, optimization of t_* can be transformed into simple fractional programming and the solution can be found analytically. In the Appendix C.1 we provide extensive analysis of the solution. The analytical solution reveals that the minimum $t_* = \frac{T}{1+\alpha}$ can be reached by the following two linear functions

$$\lambda_1(s) = \sigma(2T^{-1} - 2sT^{-2}) \text{ and } \lambda_1(s) = 0 + \sigma\alpha(2sT^{-2}).$$

Also the following functions intersect at the maximum point $t_* = T$;

$$\lambda_1(s) = \sigma((2-\alpha)T^{-1} - 2(1-\alpha)T^{-2}) \text{ and } \lambda_2(s) = \sigma T^{-1}$$

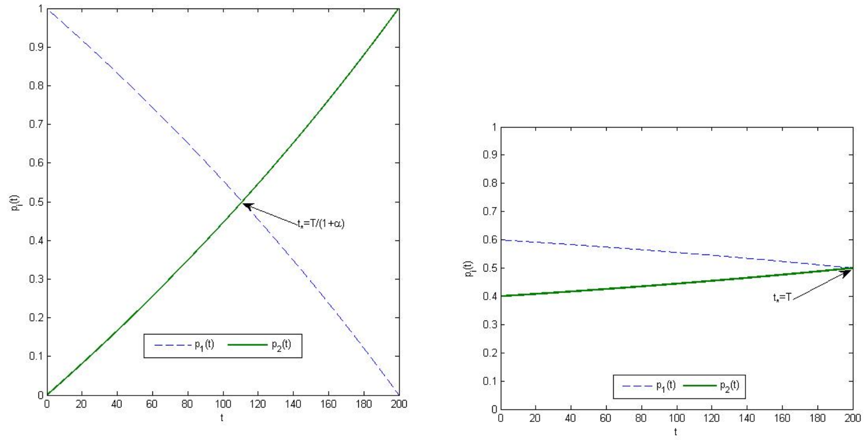
In Figure 4.1, we depict the extreme cases.

4.1.1.2 Quadratic Intensity Functions

Let the functions $a_i : [0, T] \to \mathbb{R}_+, i = 1, 2$ satisfy the parametric representation

$$a_i(s) = a_{i1} + a_{i2}s + a_{i3}s^2 \tag{4.9}$$

We clarify the feasibility conditions of the functions in relations (4.5), (4.6) and (4.7). For construction of feasible region and solution procedure we refer to Appendix C.2. Feasible region of this setting denoted by polytope $\mathcal{P}_q$. Introduce for $\mathbf{a} = (a_1(0), ..., a_2''(0))$



(a) minimum t_* (b) maximum t_*

Figure 4.1: Example of fare class arrival probabilities in extreme linear cases

belonging to $\mathcal{P}_q$ the function $h : P \times \mathbb{R} \to \mathbb{R}$ given by

$$h(\mathbf{a},t) := a_1(t) - \alpha a_2(t)$$

Observe for every t the function $\mathbf{a} \mapsto h(\mathbf{a},t)$ is a linear function. Also we know by Lemma 15 that $p_1(t) \geq p_2(t)$ if and only if $h(\mathbf{a},t) \geq 0$. Now introduce the function $t(\mathbf{a}) : \mathbb{R}^6 \mapsto \mathbb{R}_+$

$$t(\mathbf{a}) = h(t_*) = a_{11} + a_{12}t_* + a_{13}t_*^2 - \alpha(a_{21} + a_{22}t_* + a_{23}t_*^2) = 0.$$

We can easily verify the following lemma;

Lemma 16 *The function $t : P \to [0,T]$ is continuous and quasiconcave.*

The objective function is

$$\min\{t(\mathbf{a});\ \ \mathbf{a} \in \mathcal{P}_q\} \tag{4.10}$$

Unfortunately, this problem does not have an analytical solution. However we know that the minimum t_* is attained at a vertex of $\mathcal{P}_q$ because the function in 4.10 is quasiconcave on the polytope $\mathcal{P}_q$. Hence, enumeration of all vertices will lead us to the optimum point.

4.1.1.3 Comparing Linear and Quadratic Arrival Intensity Functions

Clearly type of intensity function affects on the intersection point t_*. Also as we mention above section, the overall booking demand has an influence on the intersection point. Then, t_* changes according to value of α which is the function of expected number of customers arrivals. Figure 4.2 shows that for quadratic arrival intensities the intersection points t^* are lower compared to those for linear arrival intensity functions.

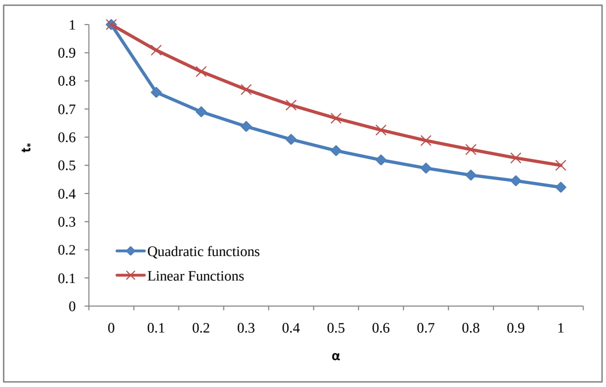


Figure 4.2: Values of t^* according to different α

4.1.1.4 Generalization to More Than 2 Fare Classes

To apply the above procedure to more than 2 fare classes let there be m fare classes and introduce the numbers $1 := \alpha_1 > \alpha_2 > ... > \alpha_m > 0$ and $\sigma_1 > 0$. Also consider two functions $a_i, i = 1, 2$ satisfying Condition 14 and select the intensity functions $\lambda_i : \mathbb{R}_+ \to \mathbb{R}_+, i = 1, ..., m$ satisfying

$$\lambda_i(t) = \alpha_i \sigma_1 a_1(t) \tag{4.11}$$

for $i \leq q$ and

$$\lambda_i(t) = \alpha_i \sigma_1 a_2(t) \tag{4.12}$$

for $i \geq q+1$. This means that the cheaper fare classes $1, \cdot, q$ have decreasing arrival intensity function and the more expensive fare classes $p+1, ..., m$ have increasing arrival intensity functions.

If we select the arrival intensities as in(4.11) and (4.12), both quadratic and linear cases boil down to the two fare classes. Intersection point of $p_1(t)$ and $p_m(t)$ directly implies that the other fare class functions have already intersected with each other. Hence, we only pay attention to select an appropriate $\alpha_i, \ 1 \leq i \leq m$.

4.1.2 Simulation Parameters

In simulation study, though the quadratic arrival intensity function defined in 4.9 provides flexibility, we use linear intensity functions for convenience. Let E refers to the earliest intersection point and L denotes the latest. Then according to section 4.1.1, the arrival intensities of m fare classes are given by;

$$\begin{aligned}
&\lambda_i^E(t) = \sigma_1 \alpha_i (\frac{2}{T} - \frac{2t}{T^2}) \quad i \leq q \\
&\lambda_i^E(t) = \sigma_1 \alpha_i (\frac{2t}{T^2}) \quad i > q+1 \\
&\lambda_i^L(t) = \sigma_1 \alpha_i (\frac{2-\alpha_m}{T} - \frac{2(1-\alpha_m)t}{T^2}) \quad i \leq q \\
&\lambda_i^L(t) = \sigma_1 \alpha_i (\frac{1}{T}) \quad i > q+1
\end{aligned}$$

where σ_1 is the expected demand of fare class 1. Hence total arrival intensities composition of each arrival intensity and;

$$\lambda_T^E(t) = \sigma_1 \sum_{i=1}^{q} \left[\frac{2\alpha_i}{T} - \frac{2\alpha_i t}{T^2} \right] + \sigma_1 \sum_{q+1}^{m} \left[\frac{2\alpha_i t}{T^2} \right]$$
$$\lambda_T^L(t) = \sigma_1 \sum_{i=1}^{q} \left[\frac{\alpha_i(2-\alpha_m)}{T} - \frac{2\alpha_i(1-\alpha_m)t}{T^2} \right] + \sigma_1 \sum_{i=q+1}^{m} \left[\frac{\alpha_i}{T} \right]$$

We assume that each class has its own fare price. If the airline accepts booking request for fare class i among m fare class, the company earns a fix revenue r_i. Without loss of generality, fare class one has lowest fare prices and fares are ordered in the following sequence;

$$r_1 < r_2 < \cdots < r_m \tag{4.13}$$

Also load factor ρ is another important parameter defined as

$$\rho = \frac{\Lambda(T)}{P} = \frac{\sigma_1 \sum_{i=1}^{m} \alpha_i}{P}, \tag{4.14}$$

where P is a plane capacity.

In Figure 4.3, reader can find an graphical representation of cancelation process. Arrows in the horizontal axes indicate the arrival time of booking requests. Each customer whether s/he is accepted or not, cancels his/her reservation independent and according to exponentially distributed with common parameter μ. Stars show the time to cancelation of the each request. Gray part represents the reservation period of the flight. For each arriving of booking request, the gray area shows the period cancellation possibility of customer and this region is denoted by N_C. Introduce

$$N_A := \text{number of points in } N_C$$
$$N_B := \text{number of points outside } N_C$$

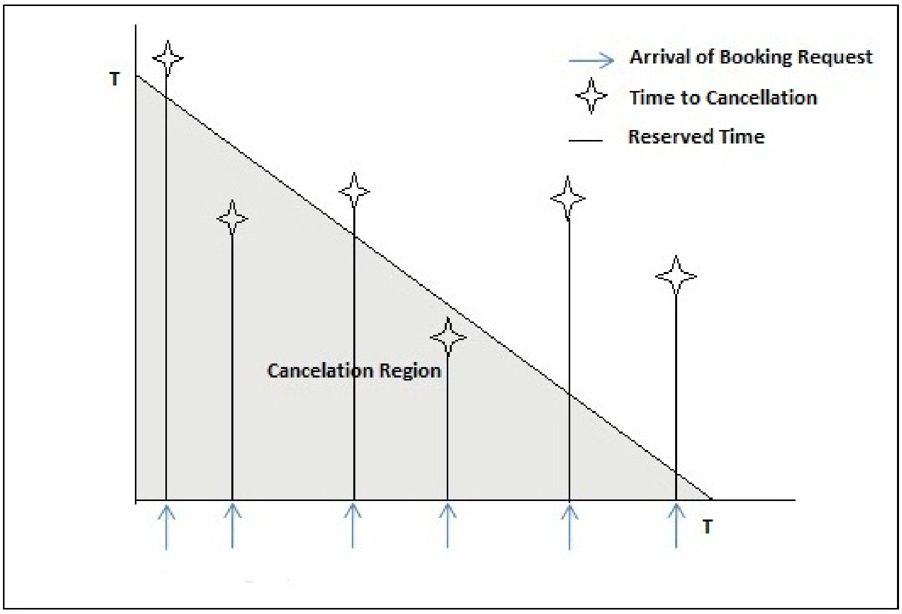


Figure 4.3: Random Poisson Measure

Then by computing following the fraction, we can obtain the probability of cancelling

$$\delta = \mathbb{E}\left(\frac{\mathbf{N}_A}{\mathbf{N}_B + \mathbf{N}_A} \mid \mathbf{N}_A + \mathbf{N}_B \geq 1\right) = \frac{\mathbb{E}\left(\frac{\mathbf{N}_A}{\mathbf{N}_B+\mathbf{N}_A}\mathbf{1}_{\{\mathbf{N}_B+\mathbf{N}_A\geq 1\}}\right)}{\mathbb{P}(\mathbf{N}_A + \mathbf{N}_B \geq 1)}$$

It follows that $\mathbf{N}_B + \mathbf{N}_A$ has the same distribution as the total arrival process $\mathbf{N}_\Lambda(T)$ and by the theory of Poisson random measures (cf.[12]) the random variables $\mathbf{N}_A$ and $\mathbf{N}_B$ are independent and Poisson distributed. Also the probability p that a point in $A \cup B$ will is in A can be computed, i.e,

$$p = \frac{\int_0^T 1 - \exp(-\mu(T-s))d\Lambda(s)}{\Lambda(T)}.$$

Hence conditioned on $\mathbf{N}_B + \mathbf{N}_A \geq 1$ we obtain

$$\mathbf{N}_A \stackrel{d}{=} \mathbf{B}(p, \mathbf{N}_B + \mathbf{N}_A).$$

This shows by conditioning on $\mathbf{N}_B + \mathbf{N}_A$ that

$$\begin{aligned}\mathbb{E}\left(\frac{\mathbf{N}_A}{\mathbf{N}_B+\mathbf{N}_A}\mathbf{1}_{\{\mathbf{N}_B+\mathbf{N}_A\geq 1\}}\right) &= \sum_{k=1}^{\infty}\frac{1}{k}\mathbb{E}(\mathbf{B}(p,k))\mathbb{P}(\mathbf{N}_B+\mathbf{N}_A=k) \\ &= p\mathbb{P}(\mathbf{N}_A+\mathbf{N}_B\geq 1)\end{aligned}$$

and so we obtain

$$\mathbb{E}\left(\frac{\mathbf{N}_A}{\mathbf{N}_B+\mathbf{N}_A} \mid \mathbf{N}_A+\mathbf{N}_B\geq 1\right) = p = \frac{\int_0^T 1-\exp(-\mu(T-s))d\Lambda(s)}{\Lambda(T)}$$

Also it is possible that a customer who keeps his/her reservation until departure time may not show-up the plane with probability $1-\beta^s$. Conditional probability of show up is given by

$$\beta^s := P(\text{reserved customers show-up} \mid \text{reserved customer does not cancel}). \quad (4.15)$$

As mentioned in the problem formulation, $\bar{P}$ denotes the total number seats that airline offer for reservations. In static EMSR policies, the model fixes the overbooking level at the beginning of the period.

- EMSR NO policy does not allow any overbooking so $\bar{P}$ directly is set to P.
- EMSR MP sets the virtual capacity according to basic rule as described in Belobaba[4]. $\bar{P}$ is assigned by the value $\frac{P}{\beta^s}$.
- EMSR Risk policy is a revenue based approach. We refer to Aydin et al. [2] for computation of virtual capacity.

Although DP does not have an actual virtual capacity, a finite value of a $\bar{P}$ must be for computational purpose. Let $V^{(\bar{P})}(T,0)$ and $V^{(\infty)}(T,0)$ denote the optimal expected net revenue respectively with an overbooking limit $\bar{P}$ and without any limit. Then by following formula which already stated in Chapter 3 we can set the $\bar{P}$ for given an error term $\epsilon > 0$. That is,

$$0 \leq V^{(\infty)}(T,0) - V^{(\bar{P})}(T,0) \leq r_m \frac{[\Lambda(T)]^{\bar{P}+1}}{(\bar{P}-1)!} \leq \epsilon. \tag{4.16}$$

In our setup, the following parameters are exogenous: refund amount κ, show-up probability β^s and overbooking penalty γ. We select the refund amount κ half amount of fare class 1 given in 4.13. The overbooking penalty is three times higher than the expected fare price. That is

$$\gamma = 3 \sum_{i=1}^{m} r_i p_i \tag{4.17}$$

where D_i is the aggregate demand of fare class i, and $p_i = \frac{\mathbb{E}(D_i)}{\mathbb{E}(D)}$ $i \in \{1, ..., m\}$

For static models the booking limit type also affects the simulation results. In the literature, there have been two main types of booking limit: nested and non-nested. Non-nested booking limit approach divides the seats into partitions. Each partition serves only for one fare class. Nested booking limits are loosened the in favor of expensive classes. We refer to Talluri and Van Ryzin [32], Lee and Hersh [18] for more information about the booking limits. In our simulation, we use standard nested booking limits. In Figure 4.4, the graphical representation of booking limits can be found.

4.2 Numerical Results

This part is dedicated to testing of the performance of the proposed algorithm. The most widely used heuristic EMSR is also used in the benchmark study.

In numerical experiments, we create varying scenarios to measure the reaction of our dynamic model against the static EMSR models. Table 4.1 presents the parameters used in the setup, with 144 scenarios. In the table, $\lambda(t)$ refers to intensity function. Selection of the arrival intensity functions according to *Early* and *Late* intersection cases is already mentioned in Section 4.1.1. E refers that the earlier intersection point and L means late. We assume that fare class 1 is economy class. We fixed the economy fare

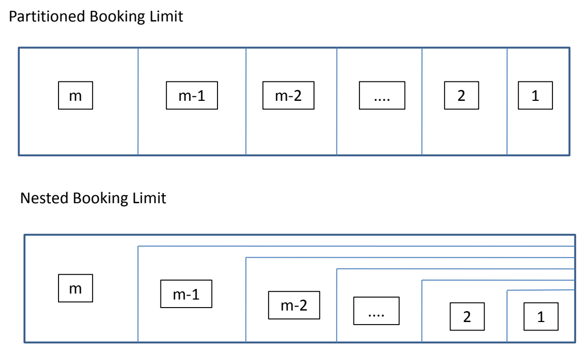


Figure 4.4: Booking Limits

	Low	Medium	High
β^s	0.75	0.85	0.95
μ	0.0005	0.0015	0.0035
m	2	-	4
ρ	1.4	-	1.8
P	150	-	300
$\lambda(t)$	E	-	L

Table 4.1: Simulation Paramaters

class fee as 50 the expensive fare r_m is 200. The fare vectors for $m = 2, 4$, respectively $r = [50, 200]$ and $r = [50, 100, 150, 200]$. In Table 4.1, ρ refers to load factor given in 4.14. We take $\kappa = 25$ and $\gamma = 300$ for all scenarios.

To determine $\bar{P}$, we use the formula given in 4.16 with $\epsilon = 0.1$. The reservation period $T = 200$ is an analogy of 6 months, which is realistic booking periods. Also we take $h = 0.01$. According to given instances in Table 4.1, we perform the simulation. Each simulation runs with same seeds. Sample means and sample deviations are obtained from 1000 replications . In the following, the experiment results are presented for two different plane capacities.

4.2.1 Experiment Results-Small Sized Plane

In this section, the results of the simulation runs for a plane which capacity $P = 150$ are given. Table 4.2 demonstrates the sample means of net revenues. In all scenarios our model results in better revenues compared to EMSR based heuristics. Also, average net revenues grow when load factor ρ increases. This behavior is a result of the select on procedure in our model that is based on more information than static models. Hence DP accepts the reservation requests better when more requests arriving. The table also indicates that, average net revenue tends to increase when cancellation rate increases. According to values given in Table 4.2, number of fare class seems to have slight impact on the average revenues. However, because changing number of fare class alters the average fare price, looking relative difference table 4.3 would be better to interpret the real influence of number of fare class. The table also indicates that show-up probabilities has negative impact on the average net revenue. With low show-up probability more requests are accepted in all policies hence these scenarios concludes with higher average revenues. The effect of different intensity functions seems to be insignificant. We can state that the static EMSR models almost stay same for each case whereas DP seeks best response. Although we could not show it rigourously, for late intersection case our model response better, there is a tendency that the model reacts more than others in late intersection case. Table 4.2 also reveals that the sample deviations of the scenarios. Sample deviations measures about the stability of revenues. In most cases, our model has low sample deviation which reflects that the revenue generated isless than others.Figure 4.5 and 4.6 illustrate the distributions of revenues.

The relative difference revenue gap between EMSRs and our model given in Table 4.3. We define the relative difference $\triangle_R$ as follows

$$\triangle_R = \frac{\text{Sample Mean of Net Revenue}_{DM} - \text{Sample Mean of Net Revenue}_{EMSR}}{\text{Sample Mean of Net Revenue}_{DM}} \quad (4.18)$$

This table highlights the differences between the DM and EMSR based heuristics. The proposed DM policy performs better for all scenarios. The revenue gap between DM and other policies gets large when load factor and cancellation rate are high. Show-up probability has also effect on relative differences; however, this impact is not monotone. When number of fare classes two, increasing the show up probability has positive impact on the relative difference in favor of dynamic model. This behavior also holds when number of fare class is four when the load factor is low. However, when load factor is high, decreasing the show-up probability increases the gap defined in 4.18. The effect of different arrival intensities similarly given in Table 4.2 on relative difference table. The late intersection scenarios (L) have tend to increase the gap in favor of DP.

Also Table 4.3 benchmarks the performances of EMSR heuristics. We observe that EMSR Risk is better static policy according to average net revenue. The difference between EMSR Risk, EMSR MP and EMSR NO comes from the permission of overbooking. EMSR Risk performs slightly better than MP because EMSR MP is a conservative policy in allowing overbooking.

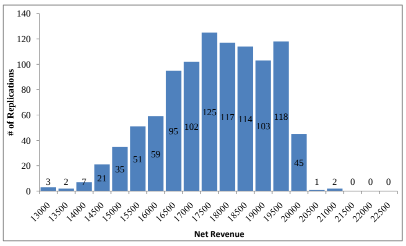
Figures 4.5 and 4.6 present the sample distributions of net revenues in 1000 replications. Observe that DM generate more revenue compared to the static models. It can also be seen that histogram of DM is relatively symmetric compared to others.

Figures 4.7 to 4.12 exhibit sample mean of revenues in varying settings. The figures focus on the differences of the 4 policies with low $\rho_L = 1.4$, high $\rho_H = 1.8$ load factors and high, medium, low cancelation rates respectively $\mu_L = 0.0005$, $\mu_M = 0.0015$ and $\mu_H = 0.0035$. These figures support that the seat allocation with DP policy yields better revenues.

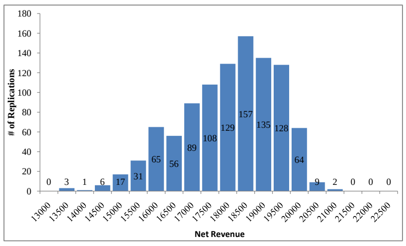
Tables 4.4 and 4.5 present the sample mean of accepted and rejected customers for each fare classes either $m = 2 \ or \ 4$. Indeed, these two table reflect that the why DP policy produce more expected revenue than others. Nearly for all scenarios, the sample mean of accepted number of the expensive fare classes are almost same. The main difference is in the number of economy fare class customers accepted. It can also be seen that DM reacts better when cancelation rate increases. Since economy fare class customers arrive more frequently in the beginning of the period [0,T], DM accepts more considering likelihoods of cancellations whereas static models apply same booking limits irrespective of this the arrival pattern. Tables 4.4 and 4.5 also reveal that, EMSR NO policy is independent from cancellation rate. Also from average accepted booking request information, we can derive

Instances							Sample Means of Revenues				Sample Deviation			
P	ρ	m	μ	β^s	t_*	δ	EMSR NO	EMSR Risk	EMSR MP	DP	EMSR NO	EMSR Risk	EMSR MP	DP
150	1.4	2	0.0005	0.95	E	5.32%	17671.67	18045.1	17998.8	18251.52	1468.33	1332.34	1424.1	1264.33
					L	5.32%	17732.67	18104.35	18052.5	18466.02	1490.92	1374.12	1457.17	1261.26
				0.85	E	5.32%	17671.67	18914.15	18885.97	19096.67	1468.33	1334.16	1409.16	1293.17
					L	5.32%	17732.67	18959.85	18944.7	19250.72	1490.92	1364.8	1437.82	1311.65
				0.75	E	5.32%	17671.67	19982.8	19981	20086.47	1468.33	1436.56	1450.52	1407.277
					L	5.32%	17732.67	20020.65	20023.22	20181.12	1490.92	1437.08	1450.72	1482.83
			0.0015	0.95	E	15.04%	17819.85	18272.8	18190.95	18692	1607.15	1575.25	1596.26	1273.81
					L	15.04%	17859.35	18306.95	18222.67	18920.92	1624.39	1606.03	1629.07	1270.19
				0.85	E	15.04%	17819.85	19252.17	19136.55	19536.32	1607.15	1547.88	1585.28	1390.8
					L	15.04%	17859.35	19316.42	19205.52	19664.9	1624.39	1576.18	1612.64	1399.17
				0.75	E	15.04%	17819.85	20074.72	20039.32	20128.47	1607.15	1621.03	1629.69	1616.27
					L	15.04%	17859.35	20104.17	20069.75	20175.47	1624.39	1642.28	1646.94	1629.77
			0.0035	0.95	E	30.86%	18002.52	18481.65	18377.37	19208.8	1645.16	1647.99	1652.57	1565.89
					L	30.86%	18074.82	18538.17	18437.67	19277.45	1609.42	1602	1601.68	1588.67
				0.85	E	30.86%	18002.52	19211.4	19157.72	19342.17	1645.16	1714.7	1724.99	1700.25
					L	30.86%	18074.82	19252.1	19214.1	19362.52	1609.42	1644.87	1640.2	1679.56
				0.75	E	30.86%	18002.52	19372.1	19371.45	19382.4	1645.16	1763.33	1762.867	1715.62
					L	30.86%	18074.82	19377.3	19375.97	19379.55	1609.42	1696.97	1695.78	1693.04
		4	0.0005	0.95	E	5.32%	18498.6	18835.5	18797.5	19139.92	1405.97	1247.92	1309.75	1095.24
					L	5.52%	18461.1	18799.65	18762.4	19259.25	1448.31	1292.04	1349.94	1117.1
				0.85	E	5.32%	18498.6	19738.77	19726.37	19992.75	1405.97	1274.86	1291.33	1137.97
					L	5.52%	18461.1	19693.42	19671.95	20080.97	1448.31	1331.99	1347.9	1182.25
				0.75	E	5.32%	18498.6	20835.1	20825.42	21014.82	1405.97	1347.73	1355.84	1301.39
					L	5.52%	18461.1	20868.7	20851.67	21060.8	1448.31	1365.05	1368.31	1353.482
			0.0015	0.95	E	15.04%	18321.72	18806.67	18696.82	19605.4	1580.833	1550.595	1564.82	1120.66
					L	15.48%	18292.15	18759.72	18649.27	19745.82	1603.26	1558.43	1576.11	1152.82
				0.85	E	15.04%	18321.72	19783.15	19733.17	20484.6	1580.833	1517.69	1516.51	1249.21
					L	15.48%	18292.15	19733.17	19684.07	20549.75	1603.26	1583.48	1588.47	1293.88
				0.75	E	15.04%	18321.72	20896.9	20866.45	21104.5	1580.833	1515.46	1518.03	1520.91
					L	15.48%	18292.15	20845.47	20821.87	21081.47	1603.26	1564.73	1567.83	1566.81
			0.0035	0.95	E	30.86%	18009.12	18556.9	18415.23	20177.42	1623.608	1585.13	1585.387	1445.95
					L	31.69%	17937.25	18459.37	18358.2	20182.55	1584.14	1542.4	1549.18	1489.093
				0.85	E	30.86%	18009.12	19541.35	19489.93	20311.05	1623.608	1554.76	1545.48	1589.63
					L	31.69%	17937.25	19524.62	19461.72	20273.62	1584.14	1511.7	1513.99	1609.23
				0.75	E	30.86%	18009.12	20320.32	20313.2	20322.25	1623.608	1644.283	1645	1609.37
					L	31.69%	17937.25	20274.75	20268.07	20278.82	1584.14	1606.71	1604.16	1620.6
	1.8	2	0.0005	0.95	E	5.32%	20477.28	20861.78	20824.8	21159.25	1661.377	1533.887	1606.203	1425.74
					L	5.32%	20516.35	20873.6	20819.72	21390.72	1699.92	1581.4	1678.94	1462.51
				0.85	E	5.32%	20477.28	21709.73	21688.53	22001.77	1661.377	1492.021	1570.825	1460.74
					L	5.32%	20516.35	21770.95	21716.12	22201.57	1699.92	1555.45	1622	1500.43
				0.75	E	5.32%	20477.28	22898.13	22862.33	23105.27	1661.377	1552.466	1593.074	1537.5
					L	5.32%	20516.35	22953.05	22931.07	23273.97	1699.92	1536.52	1604.67	1560.88
			0.0015	0.95	E	15.04%	20721.6	21195.48	21091.1	21615.3	1835.801	1785.326	1821.837	1457.32
					L	15.04%	20728.42	21193.6	21117.32	21917.6	1841.3	1822.24	1836.01	1439.73
				0.85	E	15.04%	20721.6	22169.75	22077.43	22522.12	1835.801	1771.868	1848.836	1501.03
					L	15.04%	20728.42	22264.05	22149.95	22792.72	1841.3	1757.89	1811.4	1492.18
				0.75	E	15.04%	20721.6	23433.95	23311.1	23711.65	1835.801	1750.994	1809.312	1571.6
					L	15.04%	20728.42	23527.2	23406.92	23916.15	1841.3	1733.92	1805.52	1566.54
			0.0035	0.95	E	30.86%	20842.43	21381.8	21262.83	22715.9	1842.392	1847.745	1858.343	1469.56
					L	30.86%	20857.22	21381.77	21259.6	23078.3	1864.81	1865.73	1856.73	1459.313
				0.85	E	30.86%	20842.43	22450.7	22339.18	23658.65	1842.392	1873.024	1880.52	1535.76
					L	30.86%	20857.22	22550.92	22450.05	23918.75	1864.81	1829.74	1836.37	1535.27
				0.75	E	30.86%	20842.43	23838.48	23664.9	24609.77	1842.392	1877.74	1884.601	1722.06
					L	30.86%	20857.22	23989.07	23770.27	24719.2	1864.81	1822.76	1837.57	1740.58
		4	0.0005	0.95	E	5.32%	21107.55	21636.85	21604.47	21938.07	1232.39	1083.26	1139.34	984.71
					L	5.52%	21235.87	21779.2	21735.72	22176.47	1245.06	1107.8	1154.45	965.42
				0.85	E	5.32%	21107.55	22747.8	22724.2	23125.42	1232.39	1482.29	1505.59	1250.21
					L	5.52%	21235.87	22827.47	22802.2	23316.2	1245.06	1500.88	1516.96	1235.37
				0.75	E	5.32%	21107.55	23936.15	23915.12	24273	1232.39	1494.7	1500.15	1377.56
					L	5.52%	21235.87	23983.75	23965.65	24423.65	1245.06	1495.61	1506.95	1340.83
			0.0015	0.95	E	15.04%	21752.37	22271.85	22204.32	22681.25	1469.11	1488.85	1494.09	1206.77
					L	15.48%	21759.87	22290.17	22214.95	22919.7	1503.34	1559.1	1553.08	1203.571
				0.85	E	15.04%	21752.37	22784.53	22728.65	23663.4	1469.11	1782.52	1767.72	1295.71
					L	15.48%	21759.87	22681.47	22642.6	23857.27	1503.34	1789.88	1790.97	1279.23
				0.75	E	15.04%	21752.37	24070.25	24032.17	24864.77	1469.11	1780.17	1791.48	1396.14
					L	15.48%	21759.87	24042.72	23992.77	25043.92	1503.34	1748.81	1766.25	1355.84
			0.0035	0.95	E	30.86%	21875.95	22032.15	22018.2	23850.27	1689.9	1747.37	1737.04	1290.95
					L	31.69%	21767.75	21874.07	21866.02	24161.62	1656.51	1705.77	1699.46	1243.81
				0.85	E	30.86%	21875.95	22180.95	22106.52	24858.57	1689.9	1747.7	1751.21	1368.28
					L	31.69%	21767.75	22025.82	21963.97	25134.57	1656.51	1724.59	1717.84	1380.43
				0.75	E	30.86%	21875.95	23691.35	23641.17	25102.47	1689.9	1837.26	1849.46	1401.55
					L	31.69%	21767.75	23590.15	23535.62	25898.87	1656.51	1743.15	1749.64	1608.47

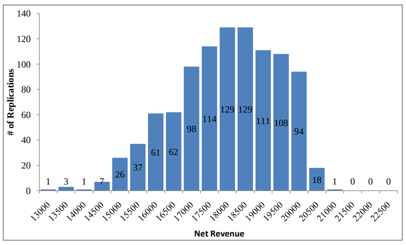
Table 4.2: Sample Means of Net Revenues and Sample Deviations ($P = 150$)



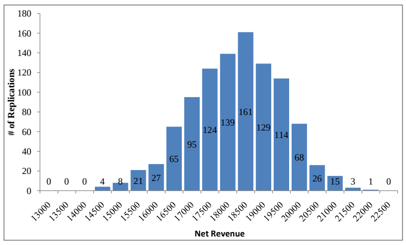
(a) EMSR NO



(b) EMSR Risk

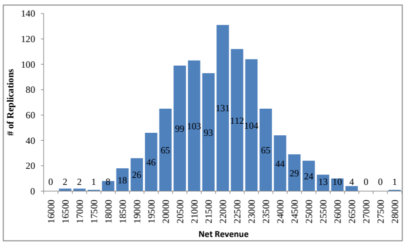


(c) EMSR MP

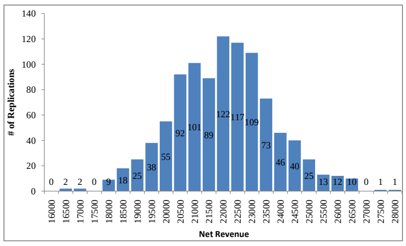


(d) DP

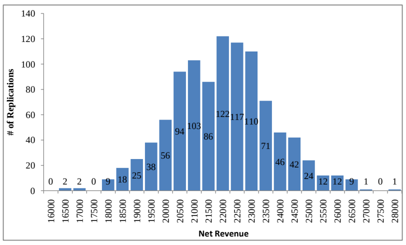
Figure 4.5: Histogram of Revenues.
$(P = 150, \;\; m = 2, \;\; \beta_H^s = 0.95, \;\; \mu_H = 0.0035, \;\; \rho_L = 1.4)$



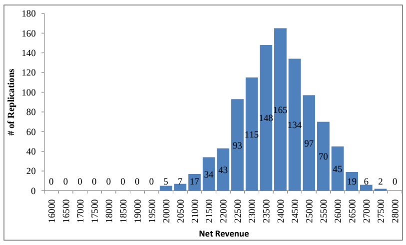
(a) EMSR NO



(b) EMSR Risk

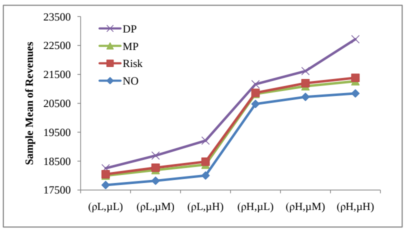


(c) EMSR MP

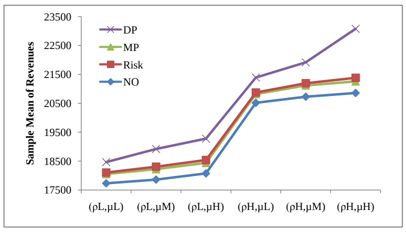


(d) DP

Figure 4.6: Histogram of Revenues
$(P = 150,\ m = 4,\ \beta_H^s = 0.95,\ \mu_H = 0.0035,\ \rho_H = 1.8)$.

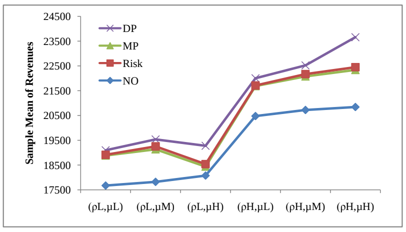


(a) $t_* = E$

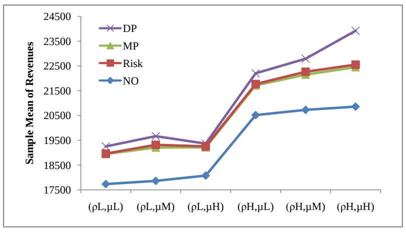


(b) $t_* = L$

Figure 4.7: Sample Mean of Revenues $(P = 150,\ m = 2,\ \beta_H^s = 0.95)$.

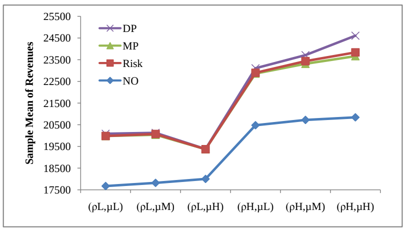


(a) $t_* = E$

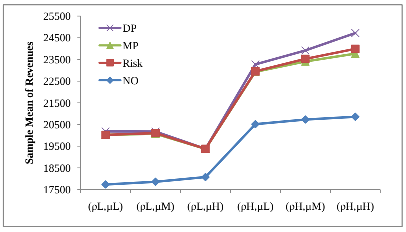


(b) $t_* = L$

Figure 4.8: Sample Mean of Revenues ($P = 150,\ \ m = 2,\ \ \beta^s_M = 0.85$).

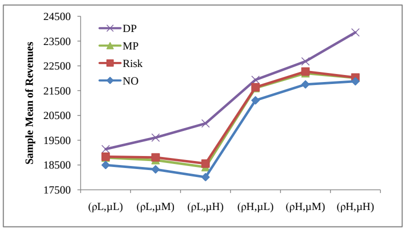


(a) $t_* = E$

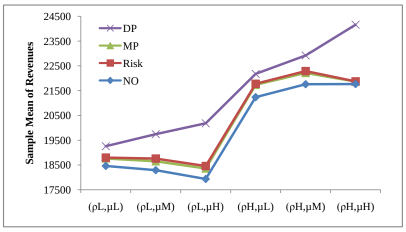


(b) $t_* = L$

Figure 4.9: Sample Mean of Revenues ($P = 150,\ \ m = 2,\ \ \beta^s_L = 0.75$).

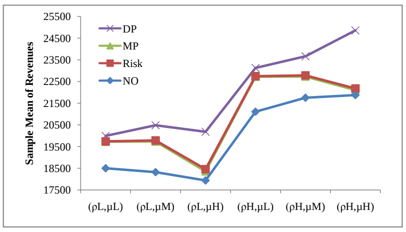


(a) $t_* = E$

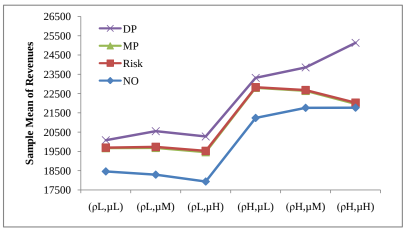


(b) $t_* = L$

Figure 4.10: Sample Mean of Revenues ($P = 150,\ \ m = 4,\ \ \beta^s_H = 0.95$).

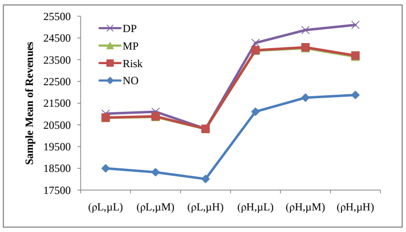


(a) $t_* = E$

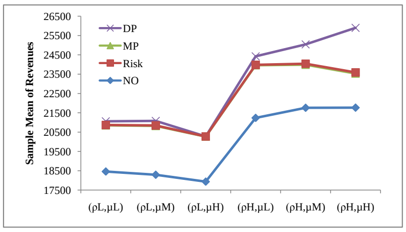


(b) $t_* = L$

Figure 4.11: Sample Mean of Revenues ($P = 150,\ \ m = 4,\ \ \beta_M^s = 0.85$).



(a) $t_* = E$



(b) $t_* = L$

Figure 4.12: Sample Mean of Revenues ($P = 150,\ \ m = 4,\ \ \beta_L^s = 0.75$).

Instances							$\triangle_{revenue}$ See relation 4.18			Instances						$\triangle_{revenue}$ See relation 4.18		
P	ρ	m	μ	β^s	t_*	δ	EMSR NO	EMSR Risk	EMSR MP	ρ	m	μ	β^s	t_*	δ	EMSR NO	EMSR Risk	EMSR MP
150	1.4	2	0.0005	0.95	E	0.0532	3.18%	1.13%	1.38%	1.8	2	0.0005	0.95	E	0.0532	3.22%	1.41%	1.58%
					L	0.0532	3.97%	1.96%	2.24%					L	0.0532	4.09%	2.42%	2.67%
				0.85	E	0.0532	7.46%	0.96%	1.10%				0.85	E	0.0532	6.93%	1.33%	1.42%
					L	0.0532	7.89%	1.51%	1.59%					L	0.0532	7.59%	1.94%	2.19%
				0.75	E	0.0532	12.02%	0.52%	0.53%				0.75	E	0.0532	11.37%	0.90%	1.05%
					L	0.0532	12.13%	0.80%	0.78%					L	0.0532	11.85%	1.38%	1.47%
			0.0015	0.95	E	0.1504	4.67%	2.24%	2.68%			0.0015	0.95	E	0.1504	4.13%	1.94%	2.43%
					L	0.1504	5.61%	3.24%	3.69%					L	0.1504	5.43%	3.30%	3.65%
				0.85	E	0.1504	8.79%	1.45%	2.05%				0.85	E	0.1504	7.99%	1.56%	1.97%
					L	0.1504	9.18%	1.77%	2.34%					L	0.1504	9.06%	2.32%	2.82%
				0.75	E	0.1504	11.47%	0.27%	0.44%				0.75	E	0.1504	12.61%	1.17%	1.69%
					L	0.1504	11.48%	0.35%	0.52%					L	0.1504	13.33%	1.63%	2.13%
			0.0035	0.95	E	0.3086	6.28%	3.79%	4.33%			0.0035	0.95	E	0.3086	8.25%	5.87%	6.40%
					L	0.3086	6.24%	3.83%	4.36%					L	0.3086	9.62%	7.35%	7.88%
				0.85	E	0.3086	6.93%	0.68%	0.95%				0.85	E	0.3086	11.90%	5.11%	5.58%
					L	0.3086	6.65%	0.57%	0.77%					L	0.3086	12.80%	5.72%	6.14%
				0.75	E	0.3086	7.12%	0.05%	0.06%				0.75	E	0.3086	15.31%	3.13%	3.84%
					L	0.3086	6.73%	0.01%	0.02%					L	0.3086	15.62%	2.95%	3.84%
		4	0.0005	0.95	E	0.0532	3.35%	1.59%	1.79%		4	0.0005	0.95	E	0.0532	3.79%	1.37%	1.52%
					L	0.0552	4.14%	2.39%	2.58%					L	0.0552	4.24%	1.79%	1.99%
				0.85	E	0.0532	7.47%	1.27%	1.33%				0.85	E	0.0532	8.73%	1.63%	1.73%
					L	0.0552	8.07%	1.93%	2.04%					L	0.0552	8.92%	2.10%	2.20%
				0.75	E	0.0532	11.97%	0.86%	0.90%				0.75	E	0.0532	13.04%	1.39%	1.47%
					L	0.0552	12.34%	0.91%	0.99%					L	0.0552	13.05%	1.80%	1.88%
			0.0015	0.95	E	0.1504	6.55%	4.07%	4.63%			0.0015	0.95	E	0.1504	4.10%	1.81%	2.10%
					L	0.1548	7.36%	4.99%	5.55%					L	0.1548	5.06%	2.75%	3.07%
				0.85	E	0.1504	10.56%	3.42%	3.67%				0.85	E	0.1504	8.08%	3.71%	3.95%
					L	0.1548	10.99%	3.97%	4.21%					L	0.1548	8.79%	4.93%	5.09%
				0.75	E	0.1504	13.19%	0.98%	1.13%				0.75	E	0.1504	12.52%	3.20%	3.35%
					L	0.1548	13.23%	1.12%	1.23%					L	0.1548	13.11%	4.00%	4.20%
			0.0035	0.95	E	0.3086	10.75%	8.03%	8.73%			0.0035	0.95	E	0.3086	8.28%	7.62%	7.68%
					L	0.3169	11.12%	8.54%	9.04%					L	0.3169	9.91%	9.47%	9.50%
				0.85	E	0.3086	11.33%	3.79%	4.04%				0.85	E	0.3086	12.00%	10.77%	11.07%
					L	0.3169	11.52%	3.69%	4.00%					L	0.3169	13.40%	12.37%	12.61%
				0.75	E	0.3086	11.38%	0.01%	0.04%				0.75	E	0.3086	12.85%	5.62%	5.82%
					L	0.3169	11.55%	0.02%	0.05%					L	0.3169	15.95%	8.91%	9.12%

Table 4.3: Relative Difference of EMSR with DP ($P = 150$)

the total revenues without any refund and penalties.

Number of rejected customers given in Table 4.4 and 4.5 are also important for airline service quality. Rejecting more customer reduces the airline reliability and the image on the market. Also, offering more promotion ticket increases the airline marketing strategy.

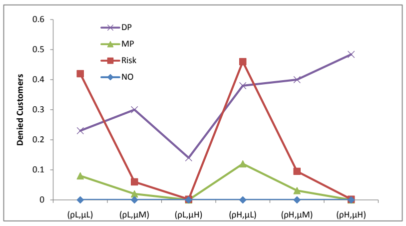
Instances						Sample Mean of Accepted								Sample Mean of Rejected							
						No		Risk		MP		DP		No		Risk		MP		DP	
m	ρ	μ	β^s	t_*	δ	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
2	1.4	0.0005	0.95	E	0.0532	78.64	69.79	89.22	69.71	86.07	69.74	91.62	69.83	61.19	0.47	50.6	0.51	53.78	0.49	47.87	0.38
				L	0.0532	78.77	70.14	89.31	70.08	86.17	70.1	94.16	70.41	60.75	0.47	50.18	0.49	53.32	0.47	45.07	0.03
			0.85	E	0.0532	78.64	69.79	109.08	69.71	105.95	69.73	109.35	70.12	61.19	0.47	30.71	0.58	33.85	0.563	30.15	0.07
				L	0.0532	78.77	70.14	109.07	70.08	105.97	70.1	111.57	70.41	60.75	0.47	30.37	0.5	33.44	0.5	27.66	0
			0.75	E	0.0532	78.64	69.79	132.27	69.88	129.31	69.82	130	70.12	61.19	0.47	7.49	0.37	10.45	0.43	9.51	0.01
				L	0.0532	78.77	70.14	132.13	70.23	129.22	70.18	130.97	70.42	60.75	0.47	7.35	0.31	10.2	0.36	8.29	0
		0.0015	0.95	E	0.1504	89.32	69.87	99.59	69.88	97.3	69.93	110.88	69.73	50.38	0.07	40.13	0.07	42.41	0.083	29.01	0.38
				L	0.1504	90.02	70.14	100.24	70.07	97.97	70.11	113.83	70.28	49.51	0.03	39.22	0.04	41.47	0.04	25.833	0.05
			0.85	E	0.1504	89.32	69.87	121.52	69.8	118.43	69.81	128.39	70.03	50.38	0.07	18.09	0.07	21.21	0.07	11.51	0.07
				L	0.1504	90.02	70.14	122.11	70.1	119	70.12	129.76	70.33	49.51	0.03	17.29	0.03	20.39	0.03	9.9	0
			0.75	E	0.1504	89.32	69.87	138.11	70.02	136.91	70.031	139.03	70.08	50.38	0.07	1.43	0.02	2.63	0.03	0.86	0
				L	0.1504	90.02	70.14	138.04	70.16	136.96	70.17	138.87	70.31	49.51	0.03	1.32	0.01	2.4	0.01	0.78	0
		0.00035	0.95	E	0.3086	107.99	69.96	119.04	70.03	116.65	70.01	138.1	69.77	31.94	0	20.82	0	23.2	0	1.58	0.11
				L	0.3086	110.64	70.08	121.57	69.9	119.2	70.02	138.1	70.02	29.16	0	18.23	0	20.59	0	1.47	0.01
			0.85	E	0.3086	107.99	69.96	136.2	69.99	135.09	69.97	139.65	69.9	31.94	0	3.69	0	4.79	0	0.03	0
				L	0.3086	110.64	70.08	137.33	69.98	136.45	70	139.5	70.02	29.16	0	2.57	0	3.46	0	0.06	0
			0.75	E	0.3086	107.99	69.96	139.88	69.98	139.85	69.99	139.68	69.9	31.94	0	0.03	0	0.07	0	0	0
				L	0.3086	110.64	70.08	139.87	70.01	139.85	70.01	139.56	70.02	29.16	0	0.01	0	0.037	0	0	0
	1.8	0.0005	0.95	E	0.0532	57.573	88.979	68.365	89.005	65.101	89.097	71.301	89.58	122.121	0.691	111.443	0.724	114.627	0.719	108.3	0.27
				L	0.0532	57.77	89.29	68.54	89.08	65.33	89.15	74.163	90.05	122	0.55	111.29	0.5	114.49	0.51	105.29	0.02
			0.85	E	0.0532	57.573	88.979	88.698	88.936	85.497	88.981	89.99	89.74	122.121	0.691	91.145	0.653	94.364	0.695	89.72	0.06
				L	0.0532	57.77	89.29	88.81	89.17	85.64	89.08	92.277	90.11	122	0.55	91.02	0.5	94.18	0.5	87.22	0
			0.75	E	0.0532	57.573	88.979	115.137	89.076	110.928	88.995	114.25	89.71	122.121	0.691	64.71	0.663	68.928	0.637	65.53	0.003
				L	0.0532	57.77	89.29	115.19	89.34	110.98	89.37	115.81	90.1	122	0.55	64.48	0.53	68.7	0.53	63.62	0
		0.0015	0.95	E	0.1504	66.62	89.864	77.465	89.919	75.05	89.851	90.64	89.45	113.289	0.147	102.235	0.172	104.67	0.147	89.08	0.31
				L	0.1504	67.13	90.18	78.11	90.09	75.73	90.21	95.51	90.06	112.31	0.06	101.47	0.07	103.8	0.06	83.87	0.06
			0.85	E	0.1504	66.62	89.864	101.334	89.669	97.78	89.776	111.93	89.71	113.289	0.147	78.592	0.116	82.048	0.112	67.8	0.07
				L	0.1504	67.13	90.18	102.13	90.14	98.6	90.19	115.66	90.25	112.31	0.06	77.3	0.07	80.91	0.07	63.75	0.003
			0.75	E	0.1504	66.62	89.864	130.205	89.722	125.646	89.708	138.68	89.89	113.289	0.147	49.485	0.074	54.166	0.1	41.05	0.004
				L	0.1504	67.13	90.18	131.01	90.04	126.49	90.17	141.14	90.31	112.31	0.06	48.25	0.06	52.81	0.07	38.22	0
		0.00035	0.95	E	0.3086	83.017	89.91	96.081	89.966	93.184	89.964	135.68	89.5	96.732	0.006	83.649	0.001	86.489	0.003	44.03	0.53
				L	0.3086	85.34	90.235	98.73	90.09	95.79	90.08	141.52	90.26	94.09	0	80.76	0	83.703	0	37.8	0.11
			0.85	E	0.3086	83.017	89.91	122.464	89.925	119.765	89.909	159.02	89.7	96.732	0.006	57.21	0	59.877	0	20.74	0.11
				L	0.3086	85.34	90.235	125.8	90.23	123.05	90.31	161.74	90.28	94.09	0	53.64	0	56.44	0	17.54	0.007
			0.75	E	0.3086	83.017	89.91	155.602	89.93	150.867	90.064	177.07	89.81	96.732	0.006	24.094	0	28.901	0	2.698	0.001
				L	0.3086	85.34	90.235	158.91	90.19	154.14	90.14	176.63	90.32	94.09	0	20.53	0	25.34	0	2.63	0

Table 4.4: Sample Means of Accepted and Rejected Booking Requests $(P = 150,\ m = 2)$

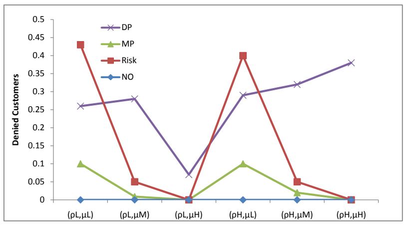
						Expected Accepted																Expected Rejected															
Instances						EMSR NO				EMSR Risk				EMSR MP				DP				EMSR NO				EMSR Risk				EMSR MP				DP			
m	ρ	μ	β^s	t_*	δ	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
4	1.4	0.0005	0.95	E	0.0532	14.12	61.7	42.48	27.177	23.79	61.76	42.51	27.17	21.67	61.77	42.49	27.16	29.64	61.67	42.3	27.24	63.49	4.02	1.457	0.288	53.86	0.36	0.302	0.27	55.96	0.37	0.3	0.275	47.77	0.4	0.526	0.03
				L	0.0552	14.21	61.59	42.66	27	23.94	61.63	42.68	27.019	21.8	61.68	42.65	27.014	31.74	61.24	42.84	27.22	63.47	0.39	0.22	0.19	53.82	0.412	0.207	0.19	55.92	0.4	0.21	0.18	45.95	0.54	0.12	0.008
			0.85	E	0.0532	14.12	61.7	42.48	27.177	42.961	61.87	42.5	27.167	41.9	61.85	42.54	27.163	47.88	61.71	42.61	27.3	63.49	4.02	1.457	0.288	34.63	0.36	0.28	0.297	35.72	0.36	0.274	0.299	29.6	0.31	0.26	0.001
				L	0.0552	14.21	61.59	42.66	27	43.17	61.82	42.67	26.94	42.1	61.81	42.66	26.93	49.5	61.51	42.9	27.24	63.47	0.39	0.22	0.19	34.47	0.39	0.2	0.19	35.55	0.39	0.2	0.2	28.18	0.29	0.02	0
			0.75	E	0.0532	14.12	61.7	42.48	27.177	67.39	61.89	42.48	27.16	66.48	61.88	42.48	27.16	68.93	61.82	42.74	27.32	63.49	4.02	1.457	0.288	10.12	0.332	0.31	0.25	11.037	0.341	0.341	0.25	8.62	0.13	0.04	0
				L	0.0552	14.21	61.59	42.66	27	67.55	61.89	42.74	26.96	66.67	61.88	42.73	26.966	69.39	61.79	42.97	27.23	63.47	0.39	0.22	0.19	10.03	0.325	0.192	0.189	10.92	0.333	0.19	0.19	8.27	0.08	0	0
		0.0015	0.95	E	0.1504	16.29	62.05	42.58	27.24	27.35	62.04	42.57	27.29	24.88	62.12	42.49	27.25	48.8	61.89	42.05	27.33	61.3	0.03	0.041	0.02	50.27	0.03	0.031	0.03	52.61	0.031	0.04	0.028	28.72	0.43	0.67	0.038
				L	0.1548	16.5	62.11	42.6	27.23	27.66	62.1	42.59	27.14	25.2	62.05	42.62	27.12	51.68	61.43	42.83	27.16	61.034	0.051	0.019	0.017	49.82	0.04	0.02	0.02	52.3	0.04	0.023	0.02	25.78	0.68	0.18	0.008
			0.85	E	0.1504	16.29	62.05	42.58	27.24	48.6	62.16	42.58	27.28	47.42	62.152	42.55	27.3	66.82	62.11	42.56	27.33	61.3	0.03	0.041	0.02	28.869	0.04	0.03	0.02	30.04	0.04	0.03	0.023	10.72	0.25	0.23	0
				L	0.1548	16.5	62.11	42.6	27.23	49.19	61.99	42.61	27.12	48.02	62	42.63	27.12	68.11	61.8	43.03	27.19	61.034	0.051	0.019	0.017	28.27	0.07	0.018	0.01	29.47	0.06	0.01	0.022	9.34	0.28	0.01	0
			0.75	E	0.1504	16.29	62.05	42.58	27.24	72.74	62.13	42.6	27.32	72.06	62.13	42.6	27.32	76.91	62.3	42.79	27.29	61.3	0.03	0.041	0.02	4.636	0.03	0.032	0.02	5.33	0.03	0.035	0.025	0.64	0.01	0.006	0
				L	0.1548	16.5	62.11	42.6	27.23	73.17	62.01	42.53	27.18	72.54	62.01	42.54	27.18	76.76	62.07	43.05	27.19	61.034	0.051	0.019	0.017	4.2	0.03	0.01	0.01	4.83	0.04	0.01	0.01	0.691	0.02	0	0
		0.00035	0.95	E	0.3086	20.4	62.36	42.76	27.31	33.912	62.36	42.79	27.33	30.96	62.39	42.68	27.27	76.25	62.04	42.45	27.19	56.96	0.002	0.001	0	43.67	0.003	0	0	46.54	0.004	0	0	1.29	0.09	0.2	0.01
				L	0.3169	20.97	61.94	43.1	27.26	34.92	61.88	42.89	27.2	31.92	61.92	42.94	27.27	76.57	61.77	42.859	27.16	56.7	0	0	0	42.8	0	0	0	45.87	0	0	0	0.978	0.1	0.05	0.005
			0.85	E	0.3086	20.4	62.36	42.76	27.31	58.43	62.24	42.7	27.34	57.19	62.27	42.7	27.34	77.51	62.14	42.66	27.22	56.96	0.002	0.001	0	19.11	0	0	0	20.33	0	0	0	0.02	0.003	0	0
				L	0.3169	20.97	61.94	43.1	27.26	60.08	62.07	42.83	27.22	58.78	62.08	42.76	27.22	77.51	61.87	42.92	27.16	56.7	0	0	0	17.57	0	0	0	18.86	0	0	0	0.03	0.003	0.002	0
			0.75	E	0.3086	20.4	62.36	42.76	27.31	77	62.23	42.67	27.3	76.82	62.23	42.68	27.3	77.53	62.14	42.66	27.21	56.96	0.002	0.001	0	0.666	0	0	0	0.85	0	0	0	0	0	0	0
				L	0.3169	20.97	61.94	43.1	27.26	77.16	61.99	42.83	27.24	77.04	61.98	42.82	27.24	77.55	61.87	42.92	27.16	56.7	0	0	0	0.414	0	0	0	0.53	0	0	0	0	0	0	0
	1.8	0.0005	0.95	E	0.0532	0	64.85	53.64	33.81	0	72.91	53.84	33.88	0	71.32	53.75	33.85	1.61	73.16	53.29	34.72	99.69	15.11	1.34	0.85	99.69	7.06	1.13	0.75	99.7	8.65	1.219	0.79	98.17	6.63	1.67	0.06
				L	0.0552	0	65	54.09	34.19	0	72.88	54.31	34.28	0	71.36	54.242	34.25	2.38	72.32	54.78	35.04	99.73	14.7	1.07	0.68	99.71	6.82	0.848	0.6	99.71	8.35	0.92	0.63	97.28	7.34	0.52	0.03
			0.85	E	0.0532	0	64.85	53.64	33.81	2.17	79.45	54.67	34.41	1.08	79.47	54.65	34.36	10.24	78.98	54.59	34.79	99.69	15.11	1.34	0.85	97.51	0.52	0.34	0.24	98.56	0.53	0.33	0.25	89.53	0.86	0.37	0
				L	0.0552	0	65	54.09	34.19	2.18	79.29	54.99	34.71	1.08	79.26	54.97	34.71	11.77	78.67	55.22	35.07	99.73	14.7	1.07	0.68	97.51	0.52	0.18	0.2	98.6	0.52	0.17	0.2	87.86	1	0.08	0
			0.75	E	0.0532	0	64.85	53.64	33.81	28.19	79.49	54.69	34.35	27.12	79.49	54.68	34.34	33.07	79.61	54.93	34.77	99.69	15.11	1.34	0.85	71.54	0.52	0.314	0.24	72.59	0.51	0.31	0.24	66.67	0.22	0.08	0
				L	0.0552	0	65	54.09	34.19	28.34	79.26	54.97	34.61	27.27	79.27	54.96	34.61	34.27	79.5	55.35	35.03	99.73	14.7	1.07	0.68	71.37	0.48	0.2	0.21	72.43	0.47	0.2	0.21	65.32	0.21	0.007	0
		0.0015	0.95	E	0.1504	0	71.54	54.78	34.65	0	77.4	54.86	34.74	0	76.44	54.82	34.7	8.74	78.82	54.14	34.82	99.63	8.59	0.42	0.25	99.64	2.73	0.23	0.19	99.64	3.69	0.26	0.2	90.98	1.08	0.84	0.04
				L	0.1548	0	71.71	55.03	34.79	0	77.36	55.17	34.81	0	76.47	55.13	34.8	12.22	78.41	54.96	35	99.76	8.05	0.261	0.13	99.73	2.4	0.112	0.1	99.75	3.26	0.15	0.108	87.43	1.26	0.214	0.01
			0.85	E	0.1504	0	71.54	54.78	34.65	2.57	80.03	55.14	34.91	1.28	80.11	55.06	34.92	29.13	79.56	54.6	34.98	99.63	8.59	0.42	0.25	97.11	0.09	0.02	0.02	98.28	0.1	0.02	0.01	70.57	0.52	0.4	0.003
				L	0.1548	0	71.71	55.03	34.79	2.606	79.61	55.2	34.88	1.3	79.69	55.22	34.92	33.17	79.03	55.11	35.04	99.76	8.05	0.261	0.13	97.16	0.04	0.01	0.004	98.38	0.04	0.01	0.003	66.44	0.63	0.05	0
			0.75	E	0.1504	0	71.54	54.78	34.65	32.47	80.07	54.9	34.94	31.27	80.03	54.92	34.98	57.6	79.69	54.68	34.95	99.63	8.59	0.42	0.25	67.298	0.093	0.01	0.02	68.49	0.08	0.01	0.02	42.12	0.3	0.14	0
				L	0.1548	0	71.71	55.03	34.79	32.98	79.74	55.07	35.02	31.78	79.69	55.08	35.04	60.55	79.444	55.2	35.1	99.76	8.05	0.261	0.13	66.69	0.04	0	0.01	67.9	0.04	0.007	0.01	38.95	0.35	0.01	0
		0.00035	0.95	E	0.3086	0	78.13	55.03	34.88	0	79.74	55.06	34.88	0	79.59	55.05	34.89	53.99	79.53	53.78	35.06	99.68	1.81	0.01	0	99.65	0.2	0.002	0	99.66	0.36	0.004	0	45.42	0.79	1.11	0.09
				L	0.3169	0	78.71	54.93	34.76	0	79.78	54.96	34.75	0	79.71	54.95	34.75	61.67	78.64	54.89	35.04	99.551	1.16	0.002	0	99.52	0.09	0	0	99.527	0.17	0	0	37.76	1.09	0.31	0.03
			0.85	E	0.3086	0	78.13	55.03	34.88	3.26	79.93	55.01	34.93	1.66	79.91	54.94	34.93	78.5	79.93	54.39	34.93	99.68	1.81	0.01	0	96.5	0	0	0	98.07	0	0	0	20.93	0.431	0.48	0.01
				L	0.3169	0	78.71	54.93	34.76	3.34	79.89	55	34.74	1.68	79.91	55.05	34.74	83.31	79.22	55.16	35.13	99.551	1.16	0.002	0	96.11	0	0	0	97.76	0	0	0	16.18	0.51	0.09	0
			0.75	E	0.3086	0	78.13	55.03	34.88	40.54	80.04	55	34.99	39.09	80.1	55.02	34.99	78.5	79.93	54.39	34.93	99.68	1.81	0.01	0	59.052	0	0	0	60.48	0	0	0	20.93	0.43	0.48	0
				L	0.3169	0	78.71	54.93	34.76	41.68	79.89	55.08	34.79	40.15	79.82	55.12	34.85	97.59	79.67	55.24	35.09	99.551	1.16	0.002	0	57.9	0	0	0	59.45	0	0	0	1.89	0.05	0	0

Table 4.5: Sample Means of Accepted and Rejected Booking Requests $(P = 150,\ m = 4)$

Table 4.6 demonstrates the sample averages, sample standard deviation of denied boarding customers, number of show-up customers at the departure time and under that four policies. The show-up numbers are other results that support why the DM produces more revenue than others. DM fills the plane at the departure time and therefore this reduces the opportunity cost coming from empty seats. Denied boarding numbers also important for airline managements. It can be seen from the table applying DM and EMSR Risk policies results in denied boarding than other policies. However the difference between DP and EMSR Risk policies are significantly insignificant. Tables 4.7 and 4.8 give the empirical distribution of denied boarding customers.

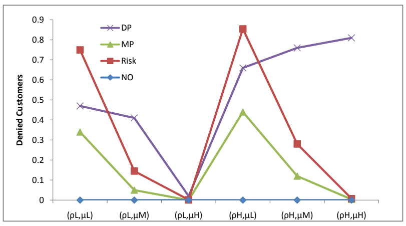


(a) $t_* = E$

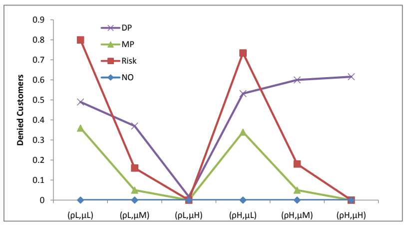


(b) $t_* = L$

Figure 4.13: Sample Mean of Denied Customers ($P = 150,\ m = 2,\ \beta^s_H = 0.95$).



(a) $t_* = E$



(b) $t_* = L$

Figure 4.14: Sample Mean of Denied Customers ($P = 150,\ m = 2,\ \beta^s_M = 0.85$).

Instances							Sample Mean of Denied				Denied Sample Deviation				Sample Mean of Show-up			
P	ρ	m	μ	β^s	t_*	δ	NO	Risk	MP	DP	NO	Risk	MP	DP	NO	Risk	MP	DP
150	1.4	2	0.0005	0.95	E	0.0532	0	0.42	0.08	0.23	0	1.18	0.45	0.761	132.9	142.31	139.47	144.83
					L	0.0532	0	0.43	0.1	0.26	0	1.21	0.535	0.753	132.6	142.04	139.21	147.12
				0.85	E	0.0532	0	0.75	0.34	0.47	0	1.95	1.3	1.475	118.92	143.25	140.71	144
					L	0.0532	0	0.8	0.36	0.49	0	2.07	1.46	1.35	118.64	143.26	140.72	145.71
				0.75	E	0.0532	0	1.1	0.57	0.54	0	2.67	1.85	1.789	105.03	143.28	141.06	141.91
					L	0.0532	0	1.17	0.64	0.57	0	2.77	1.92	1.69	104.77	143.36	141.17	142.72
			0.0015	0.95	E	0.1504	0	0.06	0.02	0.3	0	0.48	0.25	0.952	127.56	135.78	134.01	144.73
					L	0.1504	0	0.05	0.009	0.28	0	0.378	0.12	0.837	126.61	135.03	133.16	146.68
				0.85	E	0.1504	0	0.145	0.05	0.41	0	0.77	0.409	1.325	114.11	137.37	135.05	142.63
					L	0.1504	0	0.16	0.05	0.37	0	0.819	0.41	1.19	113.27	137.38	134.98	143.6
				0.75	E	0.1504	0	0.17	0.1	0.17	0	1.05	0.778	1.105	100.57	132.48	131.7	133.068
					L	0.1504	0	0.15	0.096	0.144	0	0.966	0.69	0.967	99.85	132.52	131.74	133.1
			0.00035	0.95	E	0.3086	0	0.002	0	0.14	0	0.063	0	0.69	116.2	123.42	121.87	136.31
					L	0.3086	0	0	0	0.07	0	0	0	0.42	110.64	123.41	121.68	136.43
				0.85	E	0.3086	0	0.002	0	0.02	0	0.06	0	0.399	103.94	120.867	120.09	123.28
					L	0.3086	0	0	0	0.017	0	0	0	0.347	103.41	121.33	120.65	123.26
				0.75	E	0.3086	0	0	0	0	0	0	0	0	91.58	108.66	108.65	108.69
					L	0.3086	0	0	0	0	0	0	0	0	91.15	108.68	108.66	108.69
		4	0.0005	0.95	E	0.0532	0	0.44	0.21	0.36	0	1.37	0.89	1.01	131.05	139.54	137.67	145.12
					L	0.0552	0	0.46	0.24	0.33	0	1.48	1.001	0.982	130.3	138.85	137	146.34
				0.85	E	0.0532	0	0.54	0.42	0.6	0	1.77	1.527	1.649	117.26	140.09	139.23	144.36
					L	0.0552	0	0.58	0.46	0.58	0	1.92	1.71	1.55	116.65	139.39	138.56	145.2
				0.75	E	0.0532	0	0.84	0.733	0.72	0	2.4	2.19	2.12	103.56	141.02	140.39	142.46
					L	0.0552	0	0.72	0.623	0.65	0	2.338	2.15	1.94	102.95	140.35	139.69	142.3
			0.0015	0.95	E	0.1504	0	0.06	0.02	0.38	0	0.56	0.324	1.06	120.44	128.83	126.99	145.11
					L	0.1548	0	0.04	0.028	0.34	0	0.49	0.339	0.989	118.91	127.33	125.38	146.06
				0.85	E	0.1504	0	0.06	0.04	0.53	0	0.463	0.371	1.569	107.73	129.965	129.167	143.11
					L	0.1548	0	0.035	0.03	0.46	0	0.419	0.404	1.43	106.26	128.85	128.01	143.22
				0.75	E	0.1504	0	0.11	0.1	0.203	0	0.88	0.81	1.21	94.97	130.24	129.81	133.19
					L	0.1548	0	0.09	0.08	0.17	0	0.82	0.74	0.976	93.75	129.53	129.12	132.32
			0.00035	0.95	E	0.3086	0	0	0	0.13	0	0	0	0.6	102.36	110.11	108.26	136.38
					L	0.3169	0	0	0	0.101	0	0	0	0.52	98.99	107.16	105.34	134.96
				0.85	E	0.3086	0	0.001	0.001	0.03	0	0.031	0.032	0.41	91.73	111.62	110.88	123.27
					L	0.3169	0	0.001	0	0.01	0	0.031	0	0.14	88.49	110.41	109.58	121.47
				0.75	E	0.3086	0	0	0	0	0	0	0	0	80.89	108.28	108.17	108.21
					L	0.3169	0	0	0	0	0	0	0	0	77.95	107.02	106.95	106.99
	1.8	2	0.0005	0.95	E	0.0532	0	0.46	0.12	0.38	0	1.3	0.559	1.055	131.81	141.33	138.54	145.05
					L	0.0532	0	0.4	0.1	0.29	0	1.23	0.53	0.832	131.03	140.39	137.5	146.98
				0.85	E	0.0532	0	0.855	0.44	0.66	0	2.15	1.395	1.8	117.83	142.36	139.91	144.51
					L	0.0532	0	0.735	0.34	0.532	0	2	1.251	1.431	117.18	141.65	139.1	145.73
				0.75	E	0.0532	0	1.25	0.636	0.87	0	2.936	1.97	2.31	103.77	144.22	141.16	144.27
					L	0.0532	0	1.18	0.6	0.75	0	2.7	1.89	1.93	103.16	143.78	140.81	145.04
			0.0015	0.95	E	0.1504	0	0.095	0.031	0.4	0	0.55	0.272	1.1	126.55	134.97	133.01	145.06
					L	0.1504	0	0.05	0.02	0.32	0	0.464	0.24	0.889	124.21	132.73	130.92	146.91
				0.85	E	0.1504	0	0.28	0.12	0.76	0	1.27	0.76	1.9	113.28	137.2	134.74	144.72
					L	0.1504	0	0.18	0.05	0.6	0	0.89	0.43	1.481	111.17	135.89	133.29	145.91
				0.75	E	0.1504	0	0.51	0.21	1.01	0	1.81	1.092	2.53	99.89	139.33	136.43	144.69
					L	0.1504	0	0.37	0.16	0.86	0	1.43	0.806	2.1	98.08	138.51	135.7	145.435
			0.00035	0.95	E	0.3086	0	0.002	0	0.484	0	0.06	0	0.484	115.35	123.06	121.33	144.68
					L	0.3086	0	0	0	0.38	0	0	0	1.09	111.48	119.65	117.87	146.69
				0.85	E	0.3086	0	0.008	0.004	0.81	0	0.14	0.077	2.14	103.291	124.37	122.97	143.76
					L	0.3086	0	0	0	0.616	0	0	0	1.7	99.76	122.92	121.33	145.09
				0.75	E	0.3086	0	0.01	0.009	0.42	0	0.41	0.202	1.664	91.26	126.71	124.15	138.14
					L	0.3086	0	0	0	0.31	0	0	0	1.341	88.08	127.29	124.43	138.08
		4	0.0005	0.95	E	0.0532	0	1.038	0.56	0.66	0	2.09	1.48	1.33	137.62	145.26	143.69	147.24
					L	0.0552	0	0.97	0.55	0.59	0	2.088	1.53	1.22	137.43	145.02	143.49	147.95
				0.85	E	0.0532	0	0.6	0.48	0.73	0	1.934	1.67	1.86	123.07	138.09	137.23	144.69
					L	0.0552	0	0.57	0.46	0.67	0	1.94	1.71	1.66	123	137.79	136.94	145.82
				0.75	E	0.0532	0	0.77	0.66	0.93	0	2.33	2.14	2.36	108.35	139.5	138.79	144
					L	0.0552	0	0.8	0.695	0.911	0	2.43	2.25	2.17	108.55	139.66	138.91	145.09
			0.0015	0.95	E	0.1504	0	0.27	0.13	0.44	0	1.083	0.7	1.099	132.01	137.22	136.32	145.37
					L	0.1548	0	0.15	0.09	0.38	0	0.84	0.58	0.984	129.78	134.88	134.07	146.57
				0.85	E	0.1504	0	0.05	0.04	0.76	0	0.462	0.432	1.876	118.16	126.96	126.17	144.66
					L	0.1548	0	0.04	0.04	0.654	0	0.61	0.581	1.6	116.29	124.47	123.69	145.61
				0.75	E	0.1504	0	0.108	0.08	1.062	0	0.81	0.756	2.522	104.22	129.32	128.68	144.61
					L	0.1548	0	0.097	0.086	1.01	0	0.96	0.872	2.35	102.77	127.99	127.21	145.34
			0.00035	0.95	E	0.3086	0	0.005	0.003	0.45	0	0.15	0.094	1.15	115.29	116.53	116.41	145.07
					L	0.3169	0	0	0	0.38	0	0	0	1.06	110.76	111.63	111.56	146.13
				0.85	E	0.3086	0	0	0	0.81	0	0	0	2.04	103.19	105.99	105.18	144.12
					L	0.3169	0	0	0	0.61	0	0	0	1.68	99.13	101.64	100.88	144.65
				0.75	E	0.3086	0	0	0	0	0	0	0	0	91.18	110.4	109.76	127.37
					L	0.3169	0	0	0	0.39	0	0	0	1.63	87.51	107.69	106.89	136.88

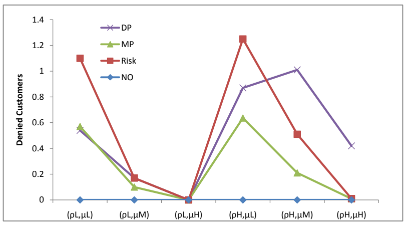
Table 4.6: Sample Mean of Denied Boarding and Show Up Customers ($P = 150$)

	Number of Replications								
Number of Denied Boarding	0	1	2	3	4	5	6	7	8
Policies — EMSR Risk	842	45	38	34	16	14	6	4	1
Policies — EMSR MP	953	24	13	4	5	1	0	0	0
Policies — DP	887	45	33	21	9	5	0	0	0

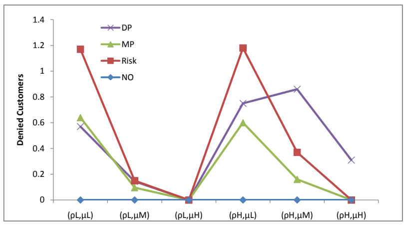
Table 4.7: Histogram of Denied Boarding
$(P = 150,\ m = 4,\ \beta_H^s = 0.95,\ \mu_L = 0.0005,\ \rho_H = 1.4,\ t_* = E)$

	Number of Replications								
Number of Denied Boarding	0	1	2	3	4	5	6	7	8
Policy — EMSR Risk	999	0	0	0	0	1	0	0	0
Policy — EMSR MP	999	0	0	1	0	0	0	0	0
Policy — DP	813	66	41	45	19	9	5	1	1

Table 4.8: Histogram of Denied Boarding
$(P = 150,\ m = 4,\ \beta_H^s = 0.95,\ \mu_H = 0.0035,\ \rho_H = 1.8,\ t_* = E)$.

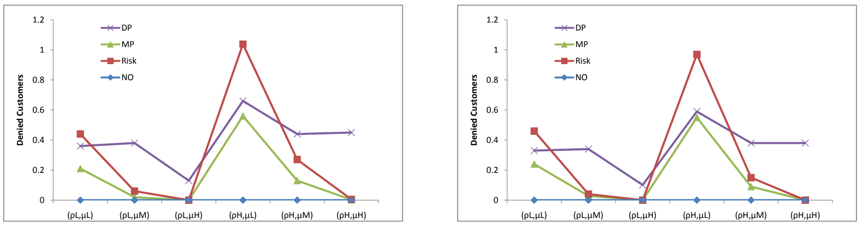


(a) $t_* = E$



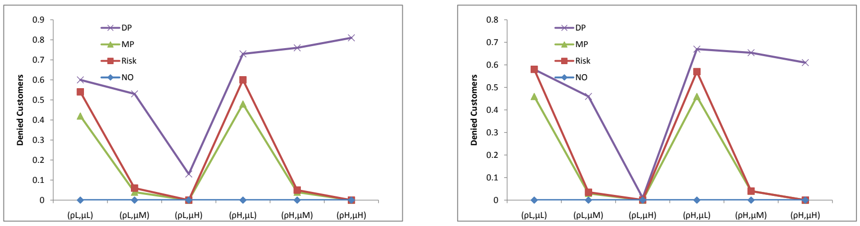
(b) $t_* = L$

Figure 4.15: Sample Mean of Denied Customers $(P = 150,\ m = 2,\ \beta_L^s = 0.75)$.



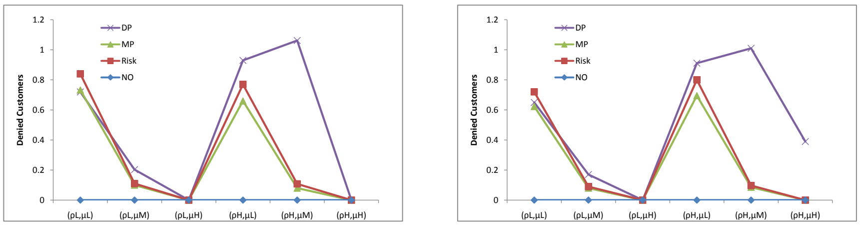
(a) $t_* = E$ (b) $t_* = L$

Figure 4.16: Sample Mean of Denied Customers ($P = 150,\ \ m = 4,\ \ \beta_H^s = 0.95$).



(a) $t_* = E$ (b) $t_* = L$

Figure 4.17: Sample Mean of Denied Customers ($P = 150,\ \ m = 4,\ \ \beta_M^s = 0.85$).



(a) $t_* = E$ (b) $t_* = L$

Figure 4.18: Sample Mean of Denied Customers ($P = 150,\ \ m = 4,\ \ \beta_L^s = 0.75$).

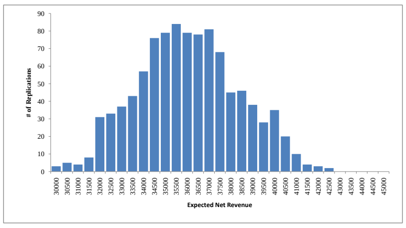
4.2.2 Experiment Results-Medium-Sized Plane

Here, we provide the results for a plane with capacity $P = 300$. Table 4.9 presents sample mean of their net revenues and sample deviations. Table 4.10 similarly shows the relative differences as defined in 4.18. Changing plane capacity does not seem to alter the difference between DM and EMSR based heuristics, there is only a slight positive effect in favor of DP.

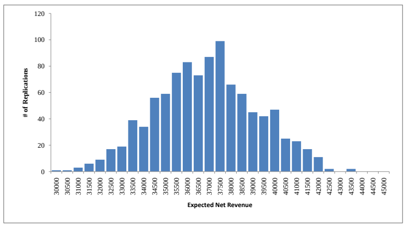
Histograms given in Figures 4.19 and 4.20 and charts given in Figure 4.21 to 4.26 demonstrate the revenue results for medium size plane ($P = 300$). In Tables 4.11 and 4.12, the average number of accepted and rejected booking requests are presented. Sample mean of denied boarding and show-up customers are given in Table 4.13. Also denied boarding numbers are shown in histograms in Table 4.14 and 4.15 and charts in Figure 4.27 to 4.29.

Instances							Sample Mean of Revenue				Sample Deviation			
P	ρ	m	μ	β^s	t_*	δ	EMSR NO	EMSR Risk	EMSR MP	DP	EMSR NO	EMSR Risk	EMSR MP	DP
150	1.4	2	0.0005	0.95	E	5.32%	35586.9	36302.75	36270.92	36734.92	2145.48	2007.86	2073.591	1809.14
					L	5.32%	35627.02	36396.85	36347.85	37029.25	2226.43	2118.64	2171.7	1830.62
				0.85	E	5.32%	35586.9	38141.77	38100	38454	2145.48	1970.8	2032.53	1916.6
					L	5.32%	35627.02	38175.92	38130.27	38639.97	2226.43	2067.88	2125.627	1906.2
				0.75	E	5.32%	35586.9	40402.72	40382.05	40591.8	2145.48	2116.15	2139.46	2127.69
					L	5.32%	35627.02	40437.15	40399.25	40704.02	2226.43	2165.6	2193.32	2115.98
			0.0015	0.95	E	15.04%	36048.57	36899.1	36802.5	37731.6	2362.79	2322.39	2340.49	1853.33
					L	15.04%	36092.22	36964.7	36871.32	38051.75	2385.39	2345.89	2352.543	1856.78
				0.85	E	15.04%	36048.57	38842.1	38701.75	39415.92	2362.79	2277.26	2316.13	2015.27
					L	15.04%	36092.22	38979.35	38850.1	39591.92	2385.39	2331.39	2360.32	2049.868
				0.75	E	15.04%	36048.57	40379.82	40360.25	40387.47	2362.79	2440.36	2440.02	2430.78
					L	15.04%	36092.22	40488.02	40464.8	40499.2	2385.39	2497.54	2495.62	2444.692
			0.00035	0.95	E	30.86%	36363.05	37286.3	37156.75	38639.17	2346.78	2332.93	2339.32	2331.1
					L	30.86%	36567.25	37469.95	37377.85	38786.65	2241.67	2291.69	2291.03	2319.65
				0.85	E	30.86%	36363.05	38615.87	38567	38718.77	2346.78	2412.127	2405.39	2430.96
					L	30.86%	36567.25	38702.77	38670	38837.13	2241.67	2393.86	2382.266	2384.22
				0.75	E	30.86%	36363.05	38719.12	38719.12	38720.92	2346.78	2440.58	2440.58	2431.33
					L	30.86%	36567.25	38752.57	38752.57	38837.42	2241.67	2413.02	2413.02	2384.8
		4	0.0005	0.95	E	5.32%	37292.37	38017.2	37981.4	38405.8	2122.63	2021.65	2050.287	1772.26
					L	5.52%	37272.75	37992.75	37966.17	38808.87	2218.78	2071.03	2102.01	1606.86
				0.85	E	5.32%	37292.37	39833.87	39809.92	40169.4	2122.63	2016.82	2034.16	1776.98
					L	5.52%	37272.75	39862.82	39837.57	40489.87	2218.78	2044.77	2067.19	1735.15
				0.75	E	5.32%	37292.37	42201.22	42183.75	42406.8	2122.63	2058.38	2065.49	1863.895
					L	5.52%	37272.75	42196.1	42180.12	42576.95	2218.78	2085.45	2098.96	1997.25
			0.0015	0.95	E	15.04%	37033.5	37905.32	37839.17	39365.85	2323.18	2319.81	2316.15	1755.68
					L	15.48%	36931.35	37820.02	37763.55	39795.32	2241.72	2256.2	2260.96	1613.35
				0.85	E	15.04%	37033.5	39864	39817.75	41175.3	2323.18	2306.485	2324.2	1839.18
					L	15.48%	36931.35	39813.95	39766.02	41431.87	2241.72	2242.28	2253.47	1852.99
				0.75	E	15.04%	37033.5	42153.15	42123.52	42252.82	2323.18	2343.53	2342.72	2295.75
					L	15.48%	36931.35	42096.72	42068.92	42270.07	2241.72	2307.31	2303.96	2280.31
			0.00035	0.95	E	30.86%	36168.57	37134.67	37093.45	40542.25	2221.87	2195.28	2208.98	2175.07
					L	31.69%	36024.22	37034.32	36973.65	40577.4	2260.181	2202.13	2212.08	2166.6
				0.85	E	30.86%	36168.57	39188.7	39137.6	40604.62	2221.87	2216.61	2226.57	2263.87
					L	31.69%	36024.22	39218.02	39169	40618.93	2260.181	2230.68	2240.82	2236.04
				0.75	E	30.86%	36168.57	40636.7	40636.7	40674.85	2221.87	2313.23	2313.23	2264.51
					L	31.69%	36024.22	40596.35	40596.35	40619.02	2260.181	2288.59	2288.59	2236.313
	1.8	2	0.0005	0.95	E	5.32%	41563.35	42334.37	42287.95	42788.2	2377.52	2270.75	2323.59	2084.54
					L	5.32%	41576.72	42375.05	42314.4	43099.85	2500.71	2372.13	2413.57	2040.99
				0.85	E	5.32%	41563.35	44137.67	44088.82	44523.82	2377.52	2235.55	2305.59	2174.34
					L	5.32%	41576.72	44244.02	44183.1	44777.85	2500.71	2315.19	2383.84	2113.3
				0.75	E	5.32%	41563.35	46485.15	46442.62	46759.4	2377.52	2210.79	2266.68	2234.78
					L	5.32%	41576.72	46569.92	46510.17	46981.97	2500.71	2264	2333.7	2159.86
			0.0015	0.95	E	15.04%	41709.8	42637.97	42542.32	43598.3	2624.31	2585.05	2604.14	2072.18
					L	15.04%	41686.7	42638.8	42523.02	44077.92	2602.61	2653.3	2648.5	2054.32
				0.85	E	15.04%	41709.8	44662.35	44517.35	45471.82	2624.31	2614.03	2635.189	2152.27
					L	15.04%	41686.7	44725.55	44561.4	45871.1	2602.61	2648.5	2685.08	2097.347
				0.75	E	15.04%	41709.8	47249.53	47065.12	47882.52	2624.31	2578.13	2597.11	2217.86
					L	15.04%	41686.7	47319.07	47115.5	48177.22	2602.61	2633.46	2660.33	2190
			0.00035	0.95	E	30.86%	41972.82	42986.92	42828.77	45762.95	2692.9	2656.9	2663.27	2102.8
					L	30.86%	41841.65	42891.97	42765.62	46308.17	2654.67	2646.08	2656.39	2119.5
				0.85	E	30.86%	41972.82	45219.27	45016.9	47777.6	2692.9	2663.6	2640.97	2202.9
					L	30.86%	41841.65	45183.55	45007.95	48131.45	2654.67	2622.35	2614.25	2238
				0.75	E	30.86%	41972.82	47838.05	47622.77	49567.07	2692.9	2606.78	2610.69	2566.84
					L	30.86%	41841.65	47994.4	47776.62	49665.22	2654.67	2618.33	2619.13	2592.24
		4	0.0005	0.95	E	5.32%	42788.4	43970.55	43951.32	44459.32	1769.5	1547.9	1576.58	1318.96
					L	5.52%	42774.2	44010.37	43981.15	44706.42	1883.19	1660.5	1688.8	1359.26
				0.85	E	5.32%	42788.4	46285.67	46245.1	46892.12	1769.5	2204.66	2234.3	1857.9
					L	5.52%	42774.2	46153.95	46132.82	46973.2	1883.19	2334.8	2356.31	1847.38
				0.75	E	5.32%	42788.4	48648	48629.47	49195.05	1769.5	2193.93	2199.6	1953.46
					L	5.52%	42774.2	48611.52	48588.15	49285.55	1883.19	2322.76	2340.18	1939.22
			0.0015	0.95	E	15.04%	43822.42	44956.42	44916.92	45835.35	2192.4	2249.14	2252.38	1765.65
					L	15.48%	43877.37	44979.02	44936.97	46144.75	2291.24	2398.1	2392.89	1767.68
				0.85	E	15.04%	43822.42	45778.85	45735.87	47809.27	2192.4	2539.66	2536.24	1885.68
					L	15.48%	43877.37	45622.85	45567.57	48073.85	2291.24	2567.29	2578.64	1870.05
				0.75	E	15.04%	43822.42	48447.52	48394.77	50281.65	2192.4	2561.67	2560.13	1964.35
					L	15.48%	43877.37	48355.77	48303.15	50516.12	2291.24	2617.85	2627.55	2002.86
			0.00035	0.95	E	30.86%	44079.8	44211.32	44210.52	48013.93	2402.59	2472	2470.83	1837.51
					L	31.69%	43817.85	43886.95	43886.65	48626.52	2482.22	2543.69	2543.25	1869.6
				0.85	E	30.86%	44079.8	44566.97	44528.15	50207.08	2402.59	2526.35	2517.291	2007.66
					L	31.69%	43817.85	44311.65	44266.05	50570.2	2482.22	2562.55	2584.01	2040.99
				0.75	E	30.86%	44079.8	47494.8	47494.8	52084.7	2402.59	2542.2	2542.208	2391.7
					L	31.69%	43817.85	47370.55	47370.55	52054.67	2482.22	2532.14	2532.14	2454.75

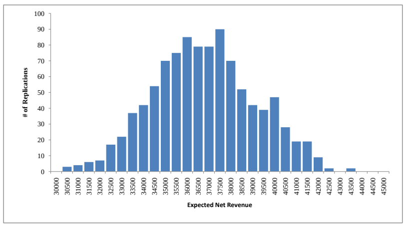
Table 4.9: Sample Mean of Revenue and Sample Deviations ($P = 300$)



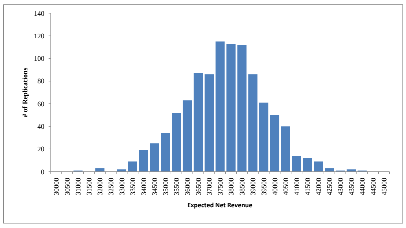
(a) EMSR NO



(b) EMSR Risk

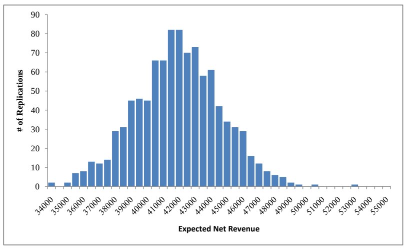


(c) EMSR MP

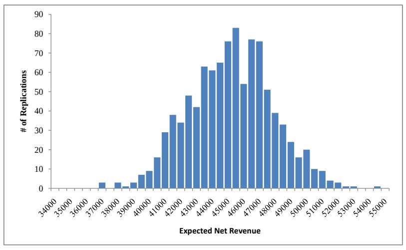


(d) DP

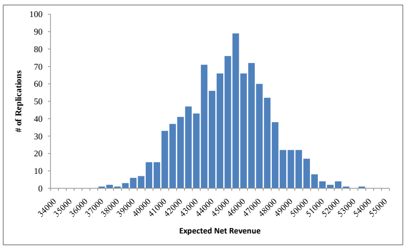
Figure 4.19: Histogram Of Revenues
$(P = 300,\ \ m = 2,\ \ \beta_H^s = 0.95,\ \ \mu_M = 0.015,\ \ \rho_L = 1.4,\ \ t_* = E)$.



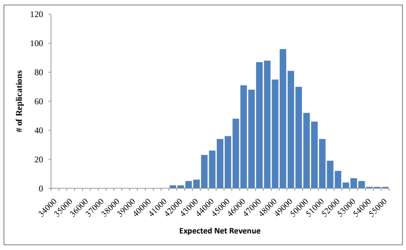
(a) EMSR NO



(b) EMSR Risk

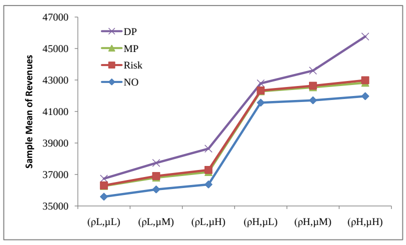


(c) EMSR MP

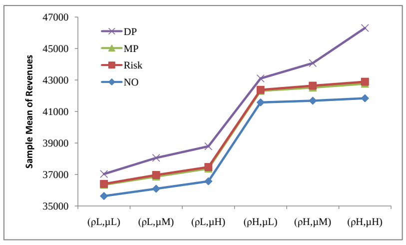


(d) DP

Figure 4.20: Histogram Of Revenues
$(P = 300,\ m = 2,\ \beta^s_M = 0.85,\ \mu_H = 0.035,\ \rho_H = 1.8,\ t_* = E)$.

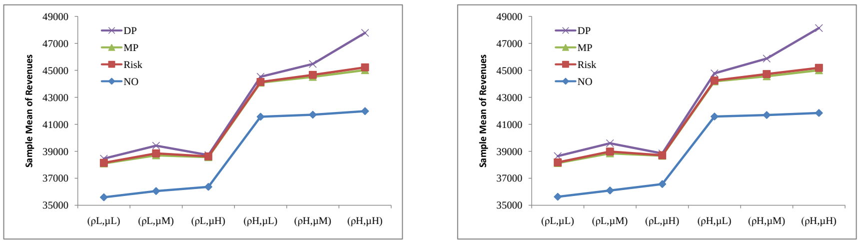


(a) $t_* = E$



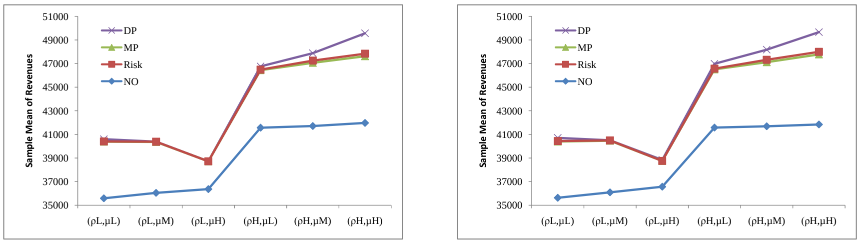
(b) $t_* = L$

Figure 4.21: Sample Mean of Revenues $(P = 300,\ m = 2,\ \beta^s_H = 0.95)$.



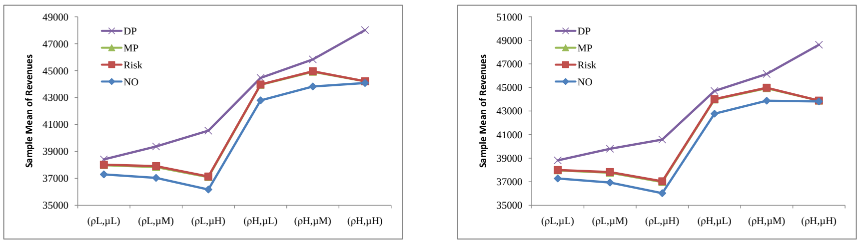
(a) $t_* = E$ (b) $t_* = L$

Figure 4.22: Sample Mean of Revenues ($P = 300,\ \ m = 2,\ \ \beta_M^s = 0.85$).



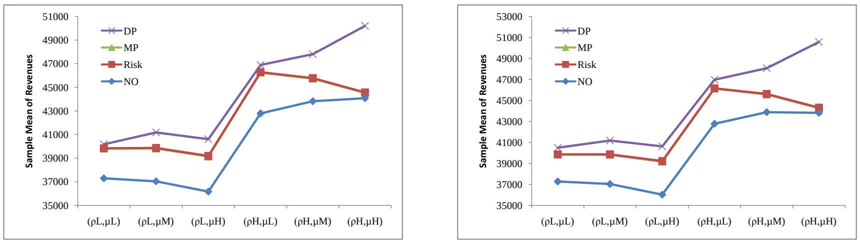
(a) $t_* = E$ (b) $t_* = L$

Figure 4.23: Sample Mean of Revenues ($P = 300,\ \ m = 2,\ \ \beta_L^s = 0.75$).



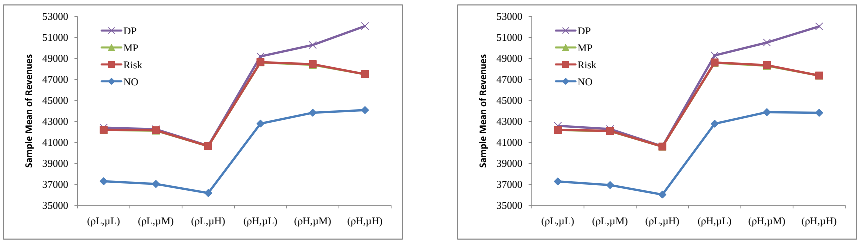
(a) $t_* = E$ (b) $t_* = L$

Figure 4.24: Sample Mean of Revenues ($P = 300,\ \ m = 4,\ \ \beta_H^s = 0.95$).



(a) $t_* = E$ (b) $t_* = L$

Figure 4.25: Sample Mean of Revenues ($P = 300,\ \ m = 4,\ \ \beta_M^s = 0.85$).



(a) $t_* = E$ (b) $t_* = L$

Figure 4.26: Sample Mean of Revenues ($P = 300,\ \ m = 4,\ \ \beta_L^s = 0.75$).

Instances						$\triangle_R$ See relation 4.18			Instances						$\triangle_R$ See relation 4.18		
ρ	m	μ	β^s	t_*	δ	EMSR NO	EMSR Risk	EMSR MP	ρ	m	μ	β^s	t_*	δ	EMSR NO	EMSR Risk	EMSR MP
1.4	2	0.0005	0.95	E	5.32%	3.13%	1.18%	1.26%	1.8	2	0.0005	0.95	E	5.32%	2.86%	1.06%	1.17%
				L	5.32%	3.79%	1.71%	1.84%					L	5.32%	3.53%	1.68%	1.82%
			0.85	E	5.32%	7.46%	0.81%	0.92%				0.85	E	5.32%	6.65%	0.87%	0.98%
				L	5.32%	7.80%	1.20%	1.32%					L	5.32%	7.15%	1.19%	1.33%
			0.75	E	5.32%	12.33%	0.47%	0.52%				0.75	E	5.32%	11.11%	0.59%	0.68%
				L	5.32%	12.47%	0.66%	0.75%					L	5.32%	11.50%	0.88%	1.00%
		0.0015	0.95	E	15.04%	4.46%	2.21%	2.46%			0.0015	0.95	E	15.04%	4.33%	2.20%	2.42%
				L	15.04%	5.15%	2.86%	3.10%					L	15.04%	5.42%	3.26%	3.53%
			0.85	E	15.04%	8.54%	1.46%	1.81%				0.85	E	15.04%	8.27%	1.78%	2.10%
				L	15.04%	8.84%	1.55%	1.87%					L	15.04%	9.12%	2.50%	2.86%
			0.75	E	15.04%	10.74%	0.02%	0.07%				0.75	E	15.04%	12.89%	1.32%	1.71%
				L	15.04%	10.88%	0.03%	0.08%					L	15.04%	13.47%	1.78%	2.20%
		0.0035	0.95	E	30.86%	5.89%	3.50%	3.84%			0.0035	0.95	E	30.86%	8.28%	6.07%	6.41%
				L	30.86%	5.72%	3.39%	3.63%					L	30.86%	9.65%	7.38%	7.65%
			0.85	E	30.86%	6.08%	0.27%	0.39%				0.85	E	30.86%	12.15%	5.35%	5.78%
				L	30.86%	5.84%	0.35%	0.43%					L	30.86%	13.07%	6.12%	6.49%
			0.75	E	30.86%	6.09%	0.00%	0.00%				0.75	E	30.86%	15.32%	3.49%	3.92%
				L	30.86%	5.85%	0.22%	0.22%					L	30.86%	15.75%	3.36%	3.80%
	4	0.0005	0.95	E	5.32%	2.90%	1.01%	1.11%		4	0.0005	0.95	E	5.32%	3.76%	1.10%	1.14%
				L	5.52%	3.96%	2.10%	2.17%					L	5.52%	4.32%	1.56%	1.62%
			0.85	E	5.32%	7.16%	0.84%	0.89%				0.85	E	5.32%	8.75%	1.29%	1.38%
				L	5.52%	7.95%	1.55%	1.61%					L	5.52%	8.94%	1.74%	1.79%
			0.75	E	5.32%	12.06%	0.48%	0.53%				0.75	E	5.32%	13.02%	1.11%	1.15%
				L	5.52%	12.46%	0.89%	0.93%					L	5.52%	13.21%	1.37%	1.42%
		0.0015	0.95	E	15.04%	5.92%	3.71%	3.88%			0.0015	0.95	E	15.04%	4.39%	1.92%	2.00%
				L	15.48%	7.20%	4.96%	5.11%					L	15.48%	4.91%	2.53%	2.62%
			0.85	E	15.04%	10.06%	3.18%	3.30%				0.85	E	15.04%	8.34%	4.25%	4.34%
				L	15.48%	10.86%	3.91%	4.02%					L	15.48%	8.73%	5.10%	5.21%
			0.75	E	15.04%	12.35%	0.24%	0.31%				0.75	E	15.04%	12.85%	3.65%	3.75%
				L	15.48%	12.63%	0.41%	0.48%					L	15.48%	13.14%	4.28%	4.38%
		0.0035	0.95	E	30.86%	10.79%	8.41%	8.51%			0.00035	0.95	E	30.86%	8.19%	7.92%	7.92%
				L	31.69%	11.22%	8.73%	8.88%					L	31.69%	9.89%	9.75%	9.75%
			0.85	E	30.86%	10.92%	3.49%	3.61%				0.85	E	30.86%	12.20%	11.23%	11.31%
				L	31.69%	11.31%	3.45%	3.57%					L	31.69%	13.35%	12.38%	12.47%
			0.75	E	30.86%	11.08%	0.09%	0.09%				0.75	E	30.86%	15.37%	8.81%	8.81%
				L	31.69%	11.31%	0.06%	0.06%					L	31.69%	15.82%	9.00%	9.00%

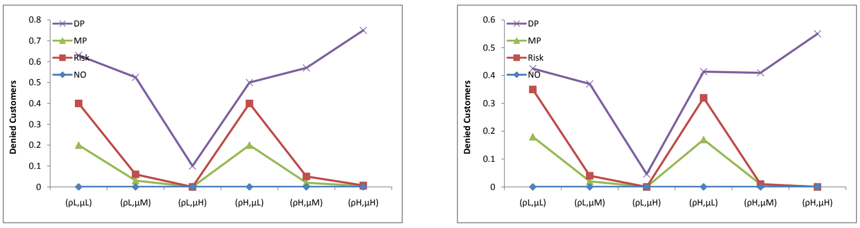
Table 4.10: Relative Differences of EMSRs with DP ($P = 300$)

Instances						Accepted Customers								Rejected Customers							
						EMSR NO		EMSR Risk		EMSR MP		DP		EMSR NO		EMSR Risk		EMSR MP		DP	
m	ρ	μ	β^s	t_*	δ	1	2	1	2	1	2	1	2	1	2	1	2	1	2	1	2
2	1.4	0.0005	0.95	E	0.0532	162.8	139.38	180.76	139.21	178.63	139.27	187.61	140.01	116.4	0.68	98.45	0.66	100.58	0.67	91.8	0.22
				L	0.0532	162.85	139.76	180.74	139.78	178.6	139.79	190.66	140.56	116.1	0.56	98.16	0.56	100.26	0.56	88.3	0.01
			0.85	E	0.0532	162.8	139.38	220.6	139.48	217.49	139.44	224.64	140.26	116.4	0.68	58.58	0.6	61.74	0.61	54.66	0.02
				L	0.0532	162.85	139.76	220.3	139.76	217.19	139.79	226.62	140.55	116.1	0.56	58.62	0.57	61.71	0.57	52.36	0
			0.75	E	0.0532	162.8	139.38	268.38	139.67	264.74	139.58	267.37	140.2	116.4	0.68	10.8	0.39	14.45	0.45	11.92	0
				L	0.0532	162.85	139.76	267.77	140.01	264.1	139.91	267.41	140.59	116.1	0.56	11.15	0.36	14.82	0.42	11.51	0
		0.0015	0.95	E	0.1504	185.5	140.22	205.05	140.09	202.72	140.08	226.63	139.97	94.09	0.07	74.66	0.06	76.97	0.05	52.52	0.31
				L	0.1504	186.264	140.72	205.78	140.59	203.53	140.62	230.79	140.61	92.95	0.02	73.38	0.01	75.62	0.01	47.96	0.04
			0.85	E	0.1504	185.5	140.22	247.42	140.12	244.22	140.05	263.55	140.23	94.09	0.07	32.33	0.01	35.55	0.02	15.65	0.04
				L	0.1504	186.264	140.72	247.85	140.86	244.7	140.8	264.44	140.6	92.95	0.02	31.37	0.02	34.54	0.02	14.37	0
			0.75	E	0.1504	185.5	140.22	278.83	140.18	278.25	140.19	278.99	140.18	94.09	0.07	0.97	0	1.56	0	0.23	0
				L	0.1504	186.264	140.72	278.36	140.83	277.79	140.83	278.59	140.57	92.95	0.02	0.793	0	1.373	0	0.23	0
		0.00035	0.95	E	0.3086	225.1	139.83	246.08	140	243.68	139.85	278.8	139.79	54.32	0	33.39	0	35.8	0	0.794	0.04
				L	0.3086	230.18	140.32	251.18	140.19	248.8	140.25	278.06	140.64	49.05	0	28.1	0	30.42	0	0.744	0
			0.85	E	0.3086	225.1	139.83	276.99	139.91	275.98	139.89	279.59	139.85	54.32	0	2.43	0	3.46	0	0.003	0
				L	0.3086	230.18	140.32	278.08	140.13	277.45	140.1	278.8	140.64	49.05	0	1.15	0	1.803	0	0.006	0
			0.75	E	0.3086	225.1	139.83	279.44	139.88	279.44	139.88	279.59	139.85	54.32	0	0	0	0	0	0	0
				L	0.3086	230.18	140.32	279.21	140.11	279.21	140.11	278.8	140.64	49.05	0	0	0	0	0	0	0
	1.8	0.0005	0.95	E	0.0532	119.41	179.95	137.74	180	135.61	179.99	145.63	180.4	239.85	0.85	221.6	0.852	223.76	0.84	213.86	0.33
				L	0.0532	119.742	180.26	138.01	180.35	135.85	180.34	150.18	181.02	239.4	0.55	221.17	0.53	223.326	0.545	208.81	0.01
			0.85	E	0.0532	119.41	179.95	178.42	179.87	175.23	179.84	183.864	180.46	239.85	0.85	180.875	0.82	184.069	0.81	175.65	0.07
				L	0.0532	119.742	180.26	178.57	180.39	175.36	180.4	187.25	180.93	239.4	0.55	180.56	0.49	183.73	0.482	171.74	0.004
			0.75	E	0.0532	119.41	179.95	231.51	179.86	226.23	179.87	232.79	180.39	239.85	0.85	127.88	0.73	133.14	0.72	126.7	0
				L	0.0532	119.742	180.26	231.38	180.41	226.16	180.35	235.04	180.87	239.4	0.55	127.69	0.462	132.98	0.45	123.98	0
		0.0015	0.95	E	0.1504	138.02	179.86	158.65	179.93	156.18	179.96	185.54	179.47	221.78	0.07	201.18	0.07	203.6	0.07	174.07	0.39
				L	0.1504	138.94	180.37	159.73	180.44	157.31	180.4	194.43	180.3	220.46	0.007	199.54	0.001	202	0.003	164.61	0.06
			0.85	E	0.1504	138.02	179.86	204.13	179.8	200.61	179.8	228.97	179.6	221.78	0.07	155.73	0.04	159.3	0.04	130.68	0.119
				L	0.1504	138.94	180.37	205.19	180.51	201.66	180.46	235.87	180.33	220.46	0.007	154.1	0.006	157.66	0.006	123.14	0.009
			0.75	E	0.1504	138.02	179.86	261.47	179.93	256.91	179.88	283.327	179.83	221.78	0.07	98.25	0.04	102.85	0.03	76.19	0.004
				L	0.1504	138.94	180.37	262.17	180.47	257.598	180.4	287.35	180.36	220.46	0.007	96.85	0.009	101.44	0.005	71.6	0
		0.00035	0.95	E	0.3086	173.62	179.61	198.59	179.65	195.64	179.43	277.16	179.1	185.74	0.003	160.63	0	163.65	0.003	81.96	0.65
				L	0.3086	177.19	179.7	202.61	179.66	199.7	179.65	287.19	180.04	182.03	0	156.6	0	159.5	0	71.43	0.14
			0.85	E	0.3086	173.62	179.61	251.85	179.89	247.81	179.71	324.22	179.52	185.74	0.003	107.46	0	111.55	0	34.94	0.12
				L	0.3086	177.19	179.7	257.32	179.6	253.16	179.6	328.69	180.14	182.03	0	101.96	0	106.11	0	29.96	0.003
			0.75	E	0.3086	173.62	179.61	315.07	179.78	310.2	179.73	357.63	179.74	185.74	0.003	44.2	0	49.12	0	1.557	0
				L	0.3086	177.19	179.7	321.44	179.68	316.66	179.67	357.09	180.26	182.03	0	38.075	0	42.81	0	1.573	0

Table 4.11: Sample Mean of Accepted and Rejected Customers ($P = 300,\ m = 2$)

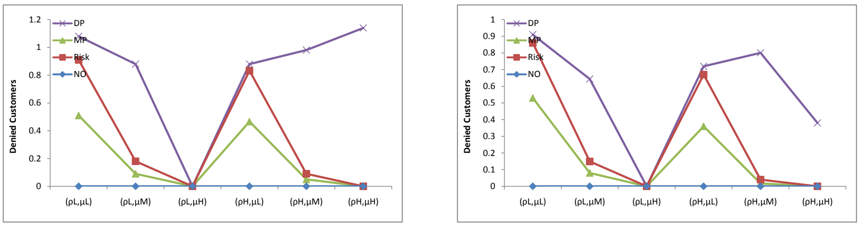
Instances						Accepted Customers																Rejected Customers															
						EMSR NO				EMSR Risk				EMSR MP				DP				EMSR NO				EMSR Risk				EMSR MP				DP			
m	ρ	μ	β^s	t_*	δ	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4	1	2	3	4
4	1.4	0.0005	0.95	E	0.0532	31.56	124.52	85.17	54.32	48.82	124.34	85.27	54.434	47.77	124.34	85.22	54.41	53.19	124.34	85.53	54.55	122.84	0.387	0.232	0.29	105.64	0.399	0.217	0.258	106.72	0.407	0.21	0.26	101.41	0.1	0.03	0
				L	0.0552	31.65	123.75	85.6	54.44	48.984	123.7	85.71	54.38	47.9	123.7	85.73	54.36	65.08	123.51	85.71	54.59	123.15	0.35	0.135	0.16	105.8	0.37	0.13	0.18	106.89	0.37	0.13	0.18	89.71	0.5	0.1	0.002
			0.85	E	0.0532	31.56	124.52	85.17	54.32	88.3	124.4	85.17	54.46	87.246	124.41	85.17	54.45	90.36	124.49	85.62	54.53	122.84	0.387	0.232	0.29	66.155	0.36	0.224	0.22	67.2	0.36	0.22	0.22	64.19	0.02	0.002	0
				L	0.0552	31.65	123.75	85.6	54.44	88.52	123.74	85.79	54.36	87.47	123.73	85.77	54.38	102.07	123.8	85.79	54.6	123.15	0.35	0.135	0.16	66.27	0.37	0.15	0.18	67.34	0.37	0.15	0.18	52.63	0.26	0.02	0
			0.75	E	0.0532	31.56	124.52	85.17	54.32	137.87	124.47	85.12	54.48	136.92	124.47	85.12	54.478	136.69	124.48	85.71	54.56	122.84	0.387	0.232	0.29	16.49	0.36	0.208	0.216	17.44	0.36	0.204	0.21	17.86	0.004	0	0
				L	0.0552	31.65	123.75	85.6	54.44	137.9	123.76	85.72	54.38	136.97	123.76	85.73	54.37	143.66	124.04	85.82	54.63	123.15	0.35	0.135	0.16	16.87	0.346	0.153	0.17	17.8	0.344	0.15	0.17	11.06	0.06	0	0
		0.0015	0.95	E	0.1504	36.62	124.84	85.43	54.93	56.26	125.03	85.41	54.83	55.04	125	85.41	54.79	92.57	124.69	85.16	54.58	117.82	0.03	0.002	0	98.32	0.02	0.004	0.006	99.56	0.02	0.004	0.006	61.9	0.13	0.072	0
				L	0.1548	36.9	124.57	85.85	54.68	56.79	124.54	85.92	54.65	55.55	124.54	85.91	54.64	106.6	123.6	85.35	54.57	117.61	0.007	0.001	0.001	97.61	0.007	0.002	0.005	98.91	0.007	0.003	0.005	48	0.68	0.17	0.007
			0.85	E	0.1504	36.62	124.84	85.43	54.93	100.28	125.04	85.36	54.75	99.12	125.01	85.45	54.72	132.58	124.6	85.28	54.61	117.82	0.03	0.002	0	54.35	0.011	0	0.005	55.528	0.01	0	0.005	21.88	0.05	0.02	0
				L	0.1548	36.9	124.57	85.85	54.68	101.3	124.37	85.92	54.54	100.127	124.4	85.91	54.56	141.5	124.01	85.56	54.6	117.61	0.007	0.001	0.001	53.25	0.006	0.001	0.004	54.41	0.006	0.001	0.004	13.17	0.23	0.02	0
			0.75	E	0.1504	36.62	124.84	85.43	54.93	149.41	124.9	85.47	54.735	148.81	124.91	85.45	54.72	154.02	124.62	85.3	54.64	117.82	0.03	0.002	0	5.484	0.009	0	0	6.08	0.009	0	0	0.47	0.001	0	0
				L	0.1548	36.9	124.57	85.85	54.68	149.63	124.41	86.03	54.44	149.08	124.4	86.02	54.44	154.54	124.22	85.5	54.59	117.61	0.007	0.001	0.001	4.994	0.003	0	0	5.547	0.003	0	0	0.17	0	0	0
		0.00035	0.95	E	0.3086	46.26	124.73	85.48	54.29	70.42	124.41	85.74	54.24	68.93	124.46	85.8	54.26	153.65	124.7	85.35	54.68	108.7	0	0	0	84.646	0	0	0	86.11	0	0	0	1.02	0.01	0.02	0
				L	0.3169	47.344	124.16	85.83	54.29	72.31	124.03	85.97	54.24	70.85	123.99	85.99	54.24	154.48	123.96	85.78	54.609	107.87	0	0	0	82.88	0	0	0	84.35	0	0	0	0.413	0.052	0.014	0.001
			0.85	E	0.3086	46.26	124.73	85.48	54.29	121.4	124.3	85.8	54.05	120.16	124.31	85.81	54.05	154.67	124.71	85.4	54.69	108.7	0	0	0	33.74	0	0	0	34.98	0	0	0	0.005	0	0	0
				L	0.3169	47.344	124.16	85.83	54.29	124.77	123.96	85.79	54.37	123.4	123.97	85.79	54.41	154.89	124.01	85.79	54.61	107.87	0	0	0	30.42	0	0	0	31.75	0	0	0	0.002	0	0	0
			0.75	E	0.3086	46.26	124.73	85.48	54.29	154.94	124.35	85.84	54.03	154.94	124.35	85.84	54.037	154.67	124.71	85.409	54.69	108.7	0	0	0	0.137	0	0	0	0.13	0	0	0	0	0	0	0
				L	0.3169	47.344	124.16	85.83	54.29	155	124.11	85.76	54.46	155	124.11	85.76	54.46	154.89	124.01	85.79	54.61	107.87	0	0	0	0.049	0	0	0	0.04	0	0	0	0	0	0	0
	1.8	0.0005	0.95	E	0.0532	0	130.18	108.92	69.05	0	145.86	109.05	69.14	0	144.97	109.03	69.14	1.68	148.37	107.69	70.28	198.95	30.31	1.578	1.21	198.96	14.63	1.42	1.152	198.95	15.53	1.43	1.16	197.34	12.3	2.65	0.04
				L	0.0552	0	130.36	108.94	69.14	0	146	109.12	69.25	0	145.1	109.09	69.22	2.98	147.68	109.84	70.15	199.3	29.402	1.272	0.97	199.3	13.76	1.12	0.87	199.29	14.64	1.141	0.89	196.35	11.67	0.55	0.009
			0.85	E	0.0532	0	130.18	108.92	69.05	6.54	160.08	110.13	70.02	5.46	160.01	110.15	69.98	20.8	160.02	109.93	70.34	198.95	30.31	1.578	1.21	192.33	0.45	0.28	0.28	193.42	0.45	0.26	0.294	178.21	0.65	0.375	0.001
				L	0.0552	0	130.18	108.92	69.05	6.56	159.5	109.98	69.97	5.47	159.4	110.02	70	24.86	158.81	110.31	70.17	199.3	29.402	1.272	0.97	192.61	0.35	0.17	0.16	193.77	0.34	0.16	0.164	174.45	0.55	0.02	0
			0.75	E	0.0532	0	130.18	108.92	69.05	58.55	160.14	110	69.93	57.48	160.17	110.01	69.93	69.86	160.34	110.15	70.29	198.95	30.31	1.578	1.21	140.32	0.44	0.23	0.25	141.39	0.43	0.23	0.27	129.08	0.24	0.082	0
				L	0.0552	0	130.18	108.92	69.05	58.793	159.53	110.11	69.9	57.72	159.55	110.13	69.92	73.51	159.24	110.34	70.14	199.3	29.402	1.272	0.97	140.38	0.34	0.172	0.18	141.466	0.345	0.169	0.187	125.82	0.15	0.001	0
		0.0015	0.95	E	0.1504	0	144.61	109.86	69.87	0	156.38	109.99	69.98	0	155.89	110	69.97	17.79	159.53	109.05	70.07	199.1	15.86	0.2	0.16	199.1	4.058	0.097	0.1	199.1	4.55	0.101	0.114	181.1	1.074	1.06	0.02
				L	0.1548	0	145.63	109.93	70.33	0	156.63	110.02	70.41	0	156.21	110.01	70.39	26.21	158.27	109.9	70.22	199.49	14.16	0.1	0.07	199.5	3.15	0.05	0.02	199.5	3.56	0.05	0.03	173.32	1.236	0.234	0.01
			0.85	E	0.1504	0	144.61	109.86	69.87	7.67	160.42	110.04	70.12	6.36	160.44	110.08	70.15	63.04	159.79	109.48	70.05	199.1	15.86	0.2	0.16	191.31	0.01	0.01	0.005	192.65	0.014	0.012	0.004	136.03	0.67	0.44	0.003
				L	0.1548	0	145.63	109.93	70.33	7.75	159.69	110.12	70.27	6.44	159.79	110.05	70.28	71.1	158.61	110.21	70.12	199.49	14.16	0.1	0.07	191.77	0.014	0	0	193.03	0.01	0	0	128.37	0.74	0.05	0
			0.75	E	0.1504	0	144.61	109.86	69.87	67.61	160.57	109.76	70.23	66.41	160.59	109.75	70.22	120.3	160.07	109.56	70.08	199.1	15.86	0.2	0.16	131.62	0	0.007	0.005	132.79	0	0.005	0.002	78.79	0.33	0.14	0
				L	0.1548	0	145.63	109.93	70.33	68.45	159.75	110.43	70.11	67.2	159.77	110.45	70.11	126.78	158.92	110.2	70.08	199.49	14.16	0.1	0.07	130.77	0.004	0.001	0.001	132.07	0.005	0.001	0.001	72.71	0.354	0.015	0
		0.00035	0.95	E	0.3086	0	159.12	109.82	70.16	0	160.52	109.81	70.16	0	160.51	109.81	70.16	113.43	159.16	108.11	69.86	198.9	1.43	0	0	198.88	0.026	0	0	198.88	0.03	0	0	85.57	1.05	1.5	0.07
				L	0.3169	0	158.9	110.25	69.89	0	159.67	110.24	69.897	0	159.67	110.24	69.897	127.27	157.873	109.97	70.1	199.28	0.7	0	0	199.26	0.004	0	0	199.26	0.007	0	0	72.01	1.48	0.32	70.1
			0.85	E	0.3086	0	159.12	109.82	70.16	9.92	160.42	109.7	70.09	8.29	160.58	109.73	70.13	163.24	159.72	109.07	69.98	198.9	1.43	0	0	189.01	0	0	0	190.69	0	0	0	35.67	0.52	0.57	0
				L	0.3169	0	158.9	110.25	69.89	10.15	159.56	110.33	69.99	8.48	159.69	110.3	70.06	172.22	158.76	110.21	70.04	199.28	0.7	0	0	189.1	0	0	0	190.8	0	0	0	27.15	0.581	0.05	0
			0.75	E	0.3086	0	159.12	109.82	70.16	83.72	160.28	109.83	69.99	83.72	160.28	109.83	69.99	197.5	160.26	109.68	70.01	198.9	1.43	0	0	115.38	0	0	0	115.38	0	0	0	1.39	0.03	0.02	0
				L	0.3169	0	158.9	110.25	69.89	85.67	159.85	110.35	69.91	85.67	159.85	110.35	69.91	198.39	159.33	110.22	70.05	199.28	0.7	0	0	113.64	0	0	0	113.64	0	0	0	0.94	0.015	0	0

Table 4.12: Sample Mean of Accepted and Rejected Customers ($P = 300,\ m = 4$)



(a) $t_* = E$ (b) $t_* = L$

Figure 4.27: Sample Mean of Denied Customers ($P = 30,\ m = 2,\ \beta_H^s = 0.95$).



(a) $t_* = E$ (b) $t_* = L$

Figure 4.28: Expected Denied Customers ($P = 300,\ m = 2,\ \beta_M^s = 0.85,\ t_* = E$).

4.2.3 Results Summary

The simulation results are classified into two main groups: small-sized and medium-size plane see section 4.2.1 and 4.2.2. According to 72 different scenarios are presented in table 4.1, the experiments conduct for two varying plane capacity. As seen from results, generally behaviors of four policies almost stay same considering plane capacities.

We can report some common patterns after given all results. For all policies, load factor has positive impact on average net revenue. However, other parameters such that

Instances							Expected Denied				Denied Standard Deviation				Expected Show-up			
P	ρ	m	μ	β^s	t_*	δ	NO	Risk	MP	DP	NO	Risk	MP	DP	NO	Risk	MP	DP
150	1.4	2	0.0005	0.95	E	0.0532	0	0.4	0.2	0.63	0	1.33	0.86	1.62	270.76	286.67	284.84	293.93
					L	0.0532	0	0.35	0.18	0.425	0	1.27	0.82	1.19	269.82	285.96	284.07	296.13
				0.85	E	0.0532	0	0.91	0.51	1.08	0	2.54	1.79	2.714	242.21	288.84	286.3	293.1
					L	0.0532	0	0.86	0.53	0.91	0	2.55	1.91	2.27	241.4	288.23	285.7	294.44
				0.75	E	0.0532	0	1.32	0.73	0.903	0	3.44	2.47	2.78	213.67	289.66	286.96	289.34
					L	0.0532	0	1.32	0.78	0.77	0	3.6	2.66	2.38	212.92	289.36	286.59	289.56
			0.0015	0.95	E	0.1504	0	0.06	0.03	0.525	0	0.49	0.29	1.43	261.05	276.6	274.71	293.67
					L	0.1504	0	0.04	0.02	0.37	0	0.411	0.289	1.11	258.73	274.7	272.86	295.95
				0.85	E	0.1504	0	0.18	0.09	0.88	0	1.04	0.719	2.55	233.79	278.88	276.41	291.01
					L	0.1504	0	0.149	0.08	0.644	0	1.05	0.74	1.98	231.85	278.46	275.95	291.53
				0.75	E	0.1504	0	0.05	0.039	0.084	0	0.71	0.586	0.89	206.2	267.14	266.72	267.34
					L	0.1504	0	0.05	0.03	0.073	0	0.56	0.37	0.806	204.51	267.23	266.82	267.33
			0.00035	0.95	E	0.3086	0	0	0	0.1	0	0	0	0.74	237.98	252.01	250.36	275
					L	0.3086	0	0	0	0.046	0	0	0	0.409	237.64	252.88	251.21	275.033
				0.85	E	0.3086	0	0	0	0	0	0	0	0	213.19	245.06	244.38	246.96
					L	0.3086	0	0	0	0	0	0	0	0	212.84	245.72	245.23	246.96
				0.75	E	0.3086	0	0	0	0	0	0	0	0	188.12	217.65	217.65	217.87
					L	0.3086	0	0	0	0	0	0	0	0	187.87	217.65	217.65	217.87
		4	0.0005	0.95	E	0.0532	0	0.4	0.32	0.03	0	1.64	1.45	0.33	266.47	281.52	280.52	285.78
					L	0.0552	0	0.37	0.28	0.404	0	1.48	1.25	1.18	264.82	280.06	279.11	295.29
				0.85	E	0.0532	0	0.68	0.59	0.198	0	2.41	2.2	1.12	238.47	283.15	282.31	285.3
					L	0.0552	0	0.54	0.46	0.88	0	2.12	1.95	2.31	237.06	282.15	281.28	293.88
				0.75	E	0.0532	0	0.853	0.75	0.29	0	2.94	2.74	1.56	210.3	285.28	284.59	284.33
					L	0.0552	0	0.81	0.71	0.81	0	2.71	2.52	2.71	209.05	284.46	283.79	289.35
			0.0015	0.95	E	0.1504	0	0.003	0.002	0.06	0	0.094	0.06	0.5	245.6	260.23	259.25	285.82
					L	0.1548	0	0.005	0.002	0.42	0	0.158	0.063	1.2	241.82	256.85	255.9	295.04
				0.85	E	0.1504	0	0.026	0.02	0.195	0	0.36	0.3	1.11	219.85	263.42	262.64	284.42
					L	0.1548	0	0.014	0.01	0.671	0	0.32	0.282	2.05	216.64	261.21	260.41	290.81
				0.75	E	0.1504	0	0.01	0.01	0.016	0	0.3	0.25	0.189	193.86	264.04	263.622	265.23
					L	0.1548	0	0.02	0.02	0.04	0	0.379	0.37	0.42	191.03	262.57	262.23	265.85
			0.00035	0.95	E	0.3086	0	0	0	0.01	0	0	0	0.22	207.86	221.62	220.9	270.2
					L	0.3169	0	0	0	0.04	0	0	0	0.414	201.19	215.88	214.99	271.76
				0.85	E	0.3086	0	0	0	0	0	0	0	0	186.38	226.1	225.4	242.58
					L	0.3169	0	0	0	0	0	0	0	0	180.22	222.93	222.067	243.64
				0.75	E	0.3086	0	0	0	0	0	0	0	0	164.35	217.56	217.56	217.82
					L	0.3169	0	0	0	0	0	0	0	0	159.1	214.74	214.74	215.09
	1.8	2	0.0005	0.95	E	0.0532	0	0.4	0.2	0.5	0	1.39	0.92	1.479	269.33	285.38	283.48	293.63
					L	0.0532	0	0.32	0.17	0.414	0	1.27	0.872	1.14	267.4	283.53	281.61	296.05
				0.85	E	0.0532	0	0.832	0.466	0.88	0	2.45	1.73	2.54	240.79	287.28	284.71	292.71
					L	0.0532	0	0.67	0.36	0.72	0	2.16	1.5	2.02	239.14	285.96	283.35	294.01
				0.75	E	0.0532	0	1.58	0.87	1.26	0	3.93	2.88	3.41	212.52	291	287.23	292.58
					L	0.0532	0	1.49	0.8	1.082	0	3.76	2.66	2.788	211.1	290.02	286.26	293.36
			0.0015	0.95	E	0.1504	0	0.05	0.02	0.57	0	0.44	0.22	1.59	257.84	273.63	271.81	293.53
					L	0.1504	0	0.01	0.009	0.41	0	0.26	0.17	1.24	252.72	268.93	267	296.02
				0.85	E	0.1504	0	0.09	0.05	0.981	0	0.803	0.545	2.576	230.4	275.98	273.59	292.52
					L	0.1504	0	0.04	0.017	0.801	0	0.48	0.294	2.0478	225.82	272.64	270.03	294.18
				0.75	E	0.1504	0	0.28	0.16	1.417	0	1.81	1.28	3.584	203.53	279.41	276.58	292.9
					L	0.1504	0	0.18	0.097	1.165	0	1.18	0.86	2.835	199.43	277.2	274.15	293.72
			0.00035	0.95	E	0.3086	0	0.007	0.004	0.75	0	0.17	0.12	1.947	235.63	250.37	248.53	293.44
					L	0.3086	0	0	0	0.55	0	0	0	1.469	226.73	242.82	240.891	296
				0.85	E	0.3086	0	0	0	1.14	0	0	0	3.03	210.85	253.16	250.75	292.46
					L	0.3086	0	0	0	0.819	0	0	0	2.209	202.9	248.93	246.48	293.78
				0.75	E	0.3086	0	0	0	0.38	0	0	0	1.9071	186	255.14	252.53	278.69
					L	0.3086	0	0	0	0.3	0	0	0	1.654	179	255.66	252.83	278.59
		4	0.0005	0.95	E	0.0532	0	1.36	1.12	0.933	0	3.01	2.72	1.866	278.34	292.88	292.04	296.72
					L	0.0552	0	1.21	0.991	0.91	0	2.79	2.47	1.831	276.73	291.37	290.52	297.39
				0.85	E	0.0532	0	0.52	0.44	0.96	0	1.99	1.78	2.53	248.89	280.49	279.56	292.57
					L	0.0552	0	0.5	0.436	0.87	0	2.22	2.03	2.32	247.42	278.35	277.56	293.74
				0.75	E	0.0532	0	0.84	0.74	1.3	0	2.95	2.73	3.34	219.81	283.2	282.46	292.39
					L	0.0552	0	0.68	0.61	1.085	0	2.68	2.52	2.905	218.5	281.85	281.1	293.13
			0.0015	0.95	E	0.1504	0	0.19	0.15	0.66	0	1.07	0.95	1.63	266.67	277.05	276.61	293.83
					L	0.1548	0	0.035	0.02	0.601	0	0.34	0.26	1.56	262.42	272.42	272.01	295.63
				0.85	E	0.1504	0	0.014	0.007	1.02	0	0.28	0.13	2.55	238.38	255.91	255.19	292.58
					L	0.1548	0	0	0	0.97	0	0	0	2.5	234.73	251.17	250.37	294.05
				0.75	E	0.1504	0	0.018	0.01	1.48	0	0.018	0.189	3.59	210.5	261.31	260.59	292.76
					L	0.1548	0	0.003	0.004	1.28	0	0.09	0.126	3.26	207.21	257.64	256.85	293.42
			0.00035	0.95	E	0.3086	0	0	0	0.75	0	0	0	1.825	233.38	234.44	234.43	293.84
					L	0.3169	0	0	0	0.6	0	0	0	1.51	223.74	224.34	224.34	295.46
				0.85	E	0.3086	0	0	0	1.11	0	0	0	2.81	208.69	214.42	213.73	292.54
					L	0.3169	0	0	0	0.98	0	0	0	2.568	200.22	205.86	205.13	293.16
				0.75	E	0.3086	0	0	0	0.41	0	0	0	2.04	184.09	222.16	222.16	278.74
					L	0.3169	0	0	0	0.26	0	0	0	1.5	176.63	216.53	216.53	275.59

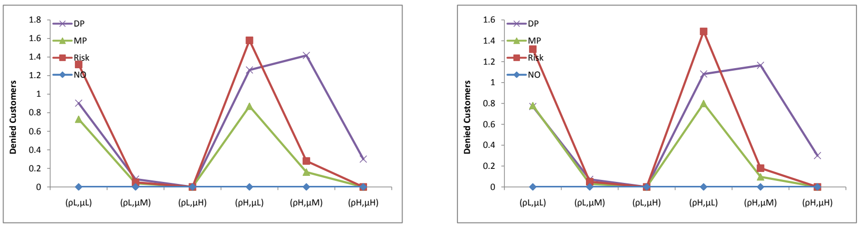
Table 4.13: Sample Mean of Denied and Show-Up Customers ($P = 150$)

		Number of Replications								
Number of Denied Boarding		0	1	2	3	4	5	6	7	8
Policy	EMSR Risk	974	8	6	5	2	4	1	0	0
	EMSR MP	987	4	4	3	1	1	0	0	0
	DP	835	40	33	36	20	16	7	6	7

Table 4.14: Histogram of Denied Customers
($P = 300,\ m = 2,\ \beta_H^s = 0.95,\ \mu_M = 0.0015,\ \rho_L = 1.4,\ t_* = E$).

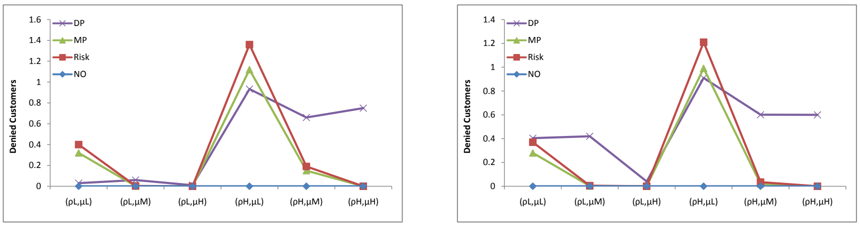
		Number of Replications																	
Number of Denied Boarding		0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17
Policy	EMSR Risk	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	EMSR MP	1000	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
	DP	809	19	26	19	23	22	16	13	9	4	8	9	7	2	2	3	3	2

Table 4.15: Histogram of Denied Customers
($P = 300,\ m = 2,\ \beta_M^s = 0.85,\ \mu_H = 0.0035,\ \rho_H = 1.8,\ t_* = E$).



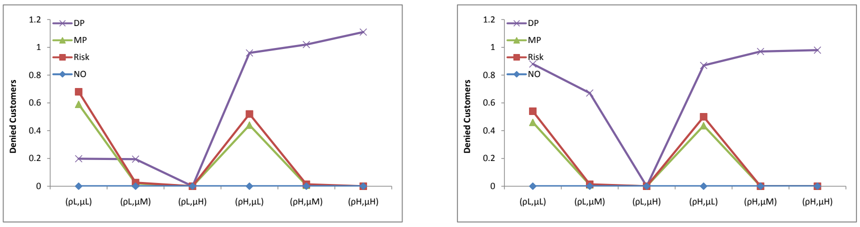
(a) $t_* = E$ (b) $t_* = L$

Figure 4.29: Sample Mean of Denied Customers ($P = 300,\ m = 2,\ \beta_L^s = 0.75$).



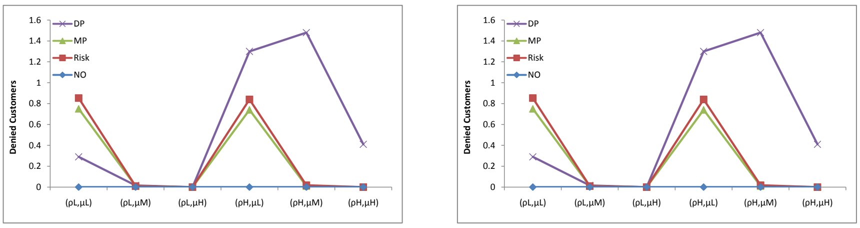
(a) $t_* = E$ (b) $t_* = L$

Figure 4.30: Sample Mean of Denied Customers ($P = 300,\ \ m = 4,\ \ \beta_H^s = 0.95$).



(a) $t_* = E$ (b) $t_* = L$

Figure 4.31: Sample Mean of Denied Customers ($P = 300,\ \ m = 4,\ \ \beta_M^s$=0.85).



(a) $t_* = E$ (b) $t_* = L$

Figure 4.32: Sample Mean of Denied Customers ($P = 300,\ \ m = 4,\ \ \beta_L^s = 0.75$).

cancelation rate and show-up probability implicitly have an effect on load factor. When cancelation rate is relatively high and show-up probability is low, load factor implicitly diminishes dramatically. Both for small and medium sized plane capacities, scenarios with small load factor and show-up probability and high cancelation, average net revenues come from EMSR MP, EMSR Risk and DP policies are approximately same. The reason behind this consequences is all polices just fill the planes without rejecting anyone. For such scenarios, as we expect, seat allocation policies lose their power for generating extra revenues.

We conduct the computations experiments on computer with 2.3 GHz Pentium Dual Core and 2 GB of RAM. The visual studio 2010 C++ is used for coding under Windows 7 operating system. Computing EMSR policies take less than 0.1 second. The computations V function for DP policy takes on average 80 and 300 seconds respectively $P = 150$ and $P = 300$. Due to our formulation basing on successive iterations, we can also report average iteration numbers. For small-sized planes which capacities are 150, average iteration number 200 and for medium-sized planes, iterations number increases until 500. Considering the expected net revenue gain from the DP policy, few minutes can be ignored in long term and total gain.

4.3 Counterintuitive Examples

In Chapter 2, we show some properties about the V function. According to Lemma 10, $s \mapsto V(t, s)$ is concave and non-increasing function for each $t \leq T$. However, there is not any clue that, these properties also hold for the mapping $t \mapsto V(t, s)$ for each $s \leq \bar{P}$. Intuitively, expecting a $V(t, s)$ monotone increasing in t seems reasonable, due to more time to departure has more possibility that generating more revenue comparing to less time. However, the intuition is valid for only some scenarios. Example 17 presents one feasible case which the function $V(t, s)$ is not increasing monotone in t.

We have also defined the booking limit in relation 2.45 in Chapter 2. In example 18, reader find the another counterintuitive result for booking limits.

Example 17 *In this example, we show that a counterintuitive example for monotonicity of mapping* $t \mapsto V(t,s)$*. We consider two fare classes. The fare of economic class is* 50 *and business class fare is* 200*. If anybody cancel his/her reservation during the booking period with rate* $\mu = 0.019$*, he/she is refunded by* 25 *independently from the fare class. The airline company pays* 300 *for each denied boarding customer. At departure time, each reservation may not show-up with probability* $\beta^s = 0.95$*. For plane capacity* 150 *and total demand equals to* 90*, the behavior of value function* $V(t,s)$ *in t depicts in the Figure for two different value of s.*

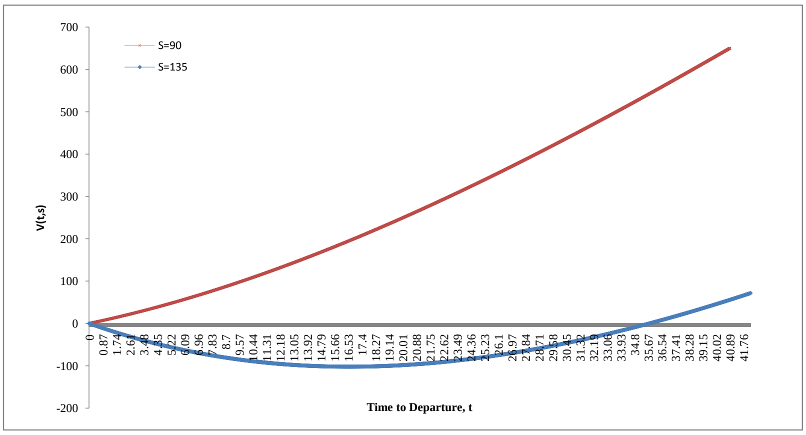


Figure 4.33: $V(t,s)$ versus t for different values of s given in Example 17

Note that the function $V(t,s)$ *is not monotonically increasing in t. The similar result has been also shown in the study of Subramanian et al. [31] in discrete in discrete time model. They put the example to show counterexample for the study of Lee and Hersh [18]. The main reason of this counterintuitive behavior comes from the high probability of cancelation. It can see from the Figure 4.33, when reserved seat is more, the cancelation affect the total net revenue.*

Example 18 *This example aims to show non-monotone behavior of booking class in t. We consider plane capacity is again* 150 *but in this example four fare classes request a seat. The fares are given respectively:* $r_1 = 50$, $r_2 = 100$, $r_3 = 150$, $r_4 = 200$. *Booking period equals* 200, *show-up probability* $\beta^s = 0.95$ *and cancelation rate* μ *is* 0.015. *Figure 4.34 plots the booking limits of four fare class. As seen, booking limits of fare class non-monotone in t.*

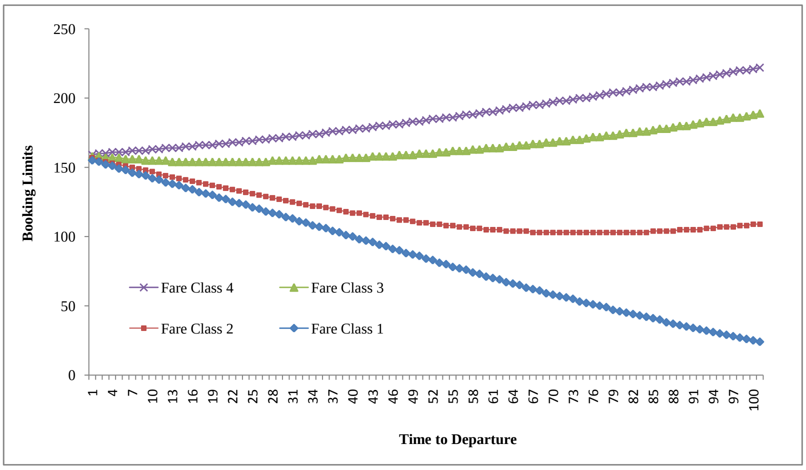


Figure 4.34: Booking Limits versus t given in Example 18

Chapter 5

Concluding Remarks

In this study, we have modeled a singe-leg airline revenue management problem with time-dependent multiple fare class arrivals. Each reserved seat can be canceled and also a reserved customer may not show-up at the departure time. Moreover, the model allows the overbooking in order to decrease the number of empty seats at boarding. We assume that booking requests of each fare class follow an independent non-homogenous Poisson process. We also assume that arrivals are independent from current reserved seats whereas cancelation process is dependent.

We formulate the problem as continuous time which is intuitive. We clearly construct value(objective) function V. In order to compute the value function, we define a sequence of functions. The elements of this sequence are successively computed via the dynamic programming operator $\mathcal{L}$. Thanks to the operator and dynamic programming principles, we can approximate the value function after a finite number of iterations for a given error $\epsilon > 0$. Markovian structure of optimal policy allow us to derive Hamilton Jacobi Bellman equation which is later used in numerical implementation in Chapter 4.

In our computational section, we report detailed simulation setup. Numerical results indicate that our continuous time dynamic model performs better than well-known EMSR based heuristics. The relative improvement, in the mean net revenue is observed to be in the range %1-%12 depending on the problem parameters. Moreover, we have presented two counterintuitive examples in which optimal booking policy and value functions are not monotone in t.

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Appendices

A Auxiliary results

Remark 19 Let $\Lambda(t) = \int_0^t \lambda(s)ds$. Then we have

$$g_j(t) := \sum_{\ell=j}^{\infty} \frac{[\Lambda(t)]^\ell}{\ell!} = e^{\Lambda(t)} \int_0^t \lambda(u) \frac{[\Lambda(u)]^{j-1}}{(j-1)!} e^{-\Lambda(u)} du, \qquad \text{for } t \geq 0 \text{ and } j \geq 1.$$

Proof. The proof follows after noting that

$$\frac{dg_j(t)}{dt} = \lambda(t) \frac{[\Lambda(t)]^{j-1}}{(j-1)!} + \lambda(t) g_j(t)$$

with the boundary condition $g_j(0) = 0$. □

Remark 20 We have

$$\mathbb{E} \sum_{j=n+1}^{\infty} 1_{\{T_j \leq t\}} \leq \frac{[\Lambda(T)]^{n+1}}{(n-1)!}, \qquad \text{for all } t \leq T \text{ and } n \geq 1.$$

Proof. Note that $\mathbb{E} \sum_{j=n+1}^{\infty} 1_{\{T_j \leq t\}} =$

$$\begin{aligned} \sum_{j=n+1}^{\infty} \mathbb{P}\{N_t \geq j\} \leq & \sum_{j=n+1}^{\infty} \mathbb{P}\{N_T \geq j\} \\ & = \sum_{j=n+1}^{\infty} \sum_{\ell=j}^{\infty} \mathbb{P}\{N_T = \ell\} = \sum_{j=n+1}^{\infty} \sum_{\ell=j}^{\infty} e^{-\Lambda(T)} \frac{[\Lambda(T)]^\ell}{\ell!}. \end{aligned}$$

Using Remark 19 above, we get the following steps

$$\begin{aligned}
&\mathbb{E}\sum_{j=n+1}^{\infty} 1_{\{T_j \le t\}} \le \sum_{j=n+1}^{\infty} e^{-\Lambda(T)} g_j(T) = \sum_{j=n+1}^{\infty} e^{-\Lambda(T)} e^{\Lambda(T)} \int_0^T \lambda(u) \frac{[\Lambda(u)]^{j-1}}{(j-1)!} e^{-\Lambda(u)} du \\
&= \int_0^T \lambda(u) \left[\sum_{j=n}^{\infty} \frac{[\Lambda(u)]^{j}}{(j)!} \right] e^{-\Lambda(u)} du \equiv \int_0^T \lambda(u) \left[g_n(u) \right] e^{-\Lambda(u)} du \\
&= \int_0^T \lambda(u) \left[\int_0^u \lambda(w) \frac{[\Lambda(w)]^{n-1}}{(n-1)!} e^{-\Lambda(w)} dw \right] du.
\end{aligned}$$

Since $\Lambda(\cdot)$ is increasing and $e^{-\Lambda(w)} \le 1$, we have

$$\begin{aligned}
&\int_0^T \lambda(u) \left[\int_0^u \lambda(w) \frac{[\Lambda(w)]^{n-1}}{(n-1)!} e^{-\Lambda(w)} dw \right] du \\
&\le \frac{[\Lambda(T)]^{n-1}}{(n-1)!} \int_0^T \lambda(u) \int_0^u \lambda(w) \, dw \, du \le \frac{[\Lambda(T)]^{n+1}}{(n-1)!},
\end{aligned}$$

and this concludes the proof. □

B Other proofs

Proof of Remark 5. Note that the expressions in (2.22) and (2.23) are continuous in t for fixed s. To show the continuity of the last term in (2.20) we note that

$$\mathbb{E}\Big[1_{\{T_1\le t\}}\cdot \mathcal{M}_{r(L_1)}[f](t-T_1,S_{T_1-})\Big]$$

$$= E\Bigg[1_{\{T_1\le t\}}\sum_{i=0}^{s}\sum_{j=1}^{m}\binom{s}{i}(e^{-\mu T_1})^i(1-e^{-\mu T_1})^{s-i}q(T-t+T_1,j)$$

$$\max\big\{r(j)+f(t-T_1,i+1),f(t-T_1,i)\big\}\Bigg]$$

$$=\sum_{i=0}^{s}\sum_{j=1}^{m}\binom{s}{i}\int_0^t \lambda^{(t)}(u)\,e^{-\Lambda^{(t)}(u)}\,(e^{-\mu u})^i(1-e^{-\mu u})^{s-i}q(T-t+u,j)$$

$$\max\{r(j)+f(t-u,i+1),f(t-u,i)\}du.$$

After a change of variable, the expression becomes

$$=\sum_{i=0}^{s}\sum_{j=1}^{m}\binom{s}{i}e^{-\Lambda(T-t)}e^{-\mu ti}\int_0^t \lambda(T-w)\,e^{\Lambda(T-w)}\,(e^{\mu w})^i(1-e^{-\mu t}e^{-\mu w})^{s-i}q(T-w,j)$$

$$\max\{r(j)+f(w,i+1),f(w,i)\}dw.$$

For fixed $i\le s$ and $j\le m$, the integral has the form $\int_0^t g_1(t,w)g_2(w)dw$ where $g_1(\cdot,\cdot)$ is a bounded where $g_1(t,w)=\lambda(T-w)\,e^{\Lambda(T-w)}\,(e^{\mu w})^i(1-e^{-\mu t}e^{-\mu w})^{s-i}q(T-w,j)$ and $g_2(w)=\max\{r(j)+f(w,i+1),f(w,i)\}$ both being bounded functions on $[0,T]\times[0,T]$ and $[0,T]$ respectively. As a result, each integral and therefore the double sum above is continuous in t, for fixed s. □

Proof of Remark 6. For fixed $(t,s)\in\triangle$, we can decompose the difference as

$$\mathcal{L}[f](t,s)-\mathcal{L}[f](t,s+1)=I_1(t,s)+I_2(t,s)+I_3(t,s),$$

where

$$
\begin{aligned}
I_1(t,s) &= -\kappa\,\mathbb{E}^{(t,s)}\left[C_{t\wedge T_1}\right] + \kappa\,\mathbb{E}^{(t,s+1)}\left[C_{t\wedge T_1}\right] \\
I_2(t,s) &= -\gamma\mathbb{E}^{(t,s)}\left[1_{\{T_1>t\}}\left(\sum_{i=1}^{S_t} B_i - P\right)^+\right] + \gamma\mathbb{E}^{(t,s+1)}\left[1_{\{T_1>t\}}\left(\sum_{i=1}^{S_t} B_i - P\right)^+\right] \\
I_3(t,s) &= \mathbb{E}^{(t,s)}\left[1_{\{T_1\le t\}}\cdot\mathcal{M}_{r(L_1)}[f](t-T_1,S_{T_1-})\right] \\
&\qquad\qquad - \mathbb{E}^{(t,s+1)}\left[1_{\{T_1\le t\}}\cdot\mathcal{M}_{r(L_1)}[f](t-T_1,S_{T_1-})\right].
\end{aligned}
$$

It is easy to see that

$$
I_1(t,s) = -\kappa s\,\mathbb{E}^{(t,\cdot)}\left[1-e^{\mu(t\wedge T_1)}\right] + \kappa(s+1)\,\mathbb{E}^{(t,\cdot)}\left[1-e^{\mu(t\wedge T_1)}\right] = \kappa\,\mathbb{E}^{(t,\cdot)}\left[1-e^{\mu(t\wedge T_1)}\right]. \tag{B.1}
$$

Also note that $I_2(t,s)$ is non-negative and can be rewritten as

$$
I_2(t,s) = -\gamma\mathbb{E}^{(t,s)}\left[1_{\{T_1>t\}}\left(\sum_{i=1}^{S_t} B_i - P\right)^+\right] + \gamma\mathbb{E}^{(t,s)}\left[1_{\{T_1>t\}}\left(\sum_{i=1}^{S_t} B_i + Z - P\right)^+\right]
$$

in terms of an independent Bernoulli random variable Z with success probability $pe^{-\mu t}$. Then we have

$$
\begin{aligned}
0 \le I_2(t,s) &= \gamma\,\mathbb{E}^{(t,s)}1_{\{T_1>t\}}1_{\{Z=1\}}\left[\left(\sum_{i=1}^{S_t} B_i + 1 - P\right)^+ - \left(\sum_{i=1}^{S_t} B_i - P\right)^+\right] \\
&\le \gamma\,\mathbb{E}^{(t,s)}1_{\{T_1>t\}}1_{\{Z=1\}}1 = \gamma p e^{-\mu t}\,\mathbb{E}^{(t,s)}1_{\{T_1>t\}}.
\end{aligned} \tag{B.2}
$$

As for the last term we observe that

$$
I_3(t,s) = \mathbb{E}^{(t,s)}1_{\{T_1\le t\}}\cdot\left[\mathcal{M}_{r(L_1)}[f](t-T_1,S_{T_1-}) - \mathcal{M}_{r(L_1)}[f](t-T_1,S_{T_1-}+W)\right]
$$

for a *conditionally* (conditioned on T_1) Bernoulli random variable W with success probability $e^{-\mu T_1}$. This observation further gives

$$I_3(t,s) = \mathbb{E}^{(t,s)} 1_{\{T_1 \le t\}} 1_{\{W=1\}} \cdot \left[\mathcal{M}_{r(L_1)}[f](t-T_1, S_{T_1-}) - \mathcal{M}_{r(L_1)}[f](t-T_1, S_{T_1-}+1)\right]$$
$$= \mathbb{E}^{(t,s)} 1_{\{T_1 \le t\}} e^{-\mu T_1} \cdot \left[\mathcal{M}_{r(L_1)}[f](t-T_1, S_{T_1-}) - \mathcal{M}_{r(L_1)}[f](t-T_1, S_{T_1-}+1)\right].$$

Note that the operator $\mathcal{M}$ preserves the upper and lower bounds given in Remark 6. Therefore, we have

$$\mathbb{E}^{(t,s)} 1_{\{T_1 \le t\}} e^{-\mu T_1} \kappa (1 - e^{-\mu(t-T_1)}) \le I_3(t,s) \le \mathbb{E}^{(t,s)} 1_{\{T_1 \le t\}} e^{-\mu T_1} \left[\kappa(1 - e^{-\mu(t-T_1)}) + \gamma p e^{-\mu(t-T_1)}\right]. \tag{B.3}$$

Finally, when we combine the identity in (B.1) with the upper and lower bounds in (B.2) and (B.3), straightforward algebra yields

$$\kappa(1 - e^{-\mu t}) \le I_1(t,s) + I_2(t,s) + I_3(t,s) \le \kappa(1 - e^{-\mu t}) + \gamma p e^{-\mu t}.$$

□

Proof of Lemma 7. By construction, control processes $\widetilde{\mathcal{A}}^{(n)}$'s and associated random variables $\widetilde{A}_i^{(n)}$'s all depend on (t,s). Here, we suppress this dependence for notational convenience only.

The lemma clearly holds for $n = 0$ (with $\widetilde{D}_u^{(0)} = 0$). For $n \ge 1$, we will prove (2.26) in two steps: 1) first, we show $U_n(\cdot,\cdot) \ge V_n(\cdot,\cdot)$ for each $n \ge 1$, and then 2) we establish $U_n(\cdot,\cdot) = G^{(\widetilde{\mathcal{A}}^{(n)})}(t,s)$, for each $n \ge 1$ again. Since $\widetilde{\mathcal{A}}^{(n)} \in \mathcal{D}_n$, the inequalities yield $V_n(\cdot,\cdot) \le U_n(\cdot,\cdot) = G^{(\widetilde{\mathcal{A}}^{(n)})}(t,s) \le V_n(\cdot,\cdot)$.

Step 1: To prove the inequality $U_n(t,s) \ge V_n(t,s)$, we fix an arbitrary $D \in \mathcal{D}_n$ with the decisions $A_1, A_2, \dots A_n$ and we show

$$G^{(\mathcal{A})}(t,s) \le \mathbb{E}^{(t,s)}\Bigg[\sum_{i=1}^{N_{t\wedge T_{n-k+1}}} A_i\, r(L_i) - \kappa C_{t\wedge T_{n-k+1}} - 1_{\{t<T_{n-k+1}\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ \\ + 1_{\{T_{n-k+1}\le t\}} U_{k-1}(t - T_{n-k+1}, S_{T_{n-k+1}})\Bigg] \quad \text{(B.4)}$$

inductively for $k = 1, \dots, n+1$. When take $k = n+1$ in (B.4), we get $G^{(\mathcal{A})}(t,s) \le U_n(t,s)$, which further implies that $V_n(t,s) \le U_n(t,s)$ since D above was an arbitrary policy in $\mathcal{D}_n$.

For $k = 1$, the inequality in (B.4) is immediate as it becomes an equality thanks to the strong Markov property. Assume now that the inequality holds for some $1 \le k < n+1$, and prove it for $k+1$. Note that we can write the right hand side in (B.4), call R_k, as $R_{k-1} = R^{(1)}_{k-1} + R^{(2)}_{k-1}$, where

$$R^{(1)}_{k-1} := \mathbb{E}^{(t,s)}\left[\sum_{i=1}^{N_{t\wedge T_{n-k}}} A_i\, r(L_i) - \kappa C_{t\wedge T_{n-k}} - 1_{\{t<T_{n-k}\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+\right] \quad \text{(B.5)}$$

and

$$R^{(2)}_{k-1} := \mathbb{E}^{(t,s)} 1_{\{t\ge T_{n-k}\}}\Bigg[-\kappa(C_{t\wedge T_{n-k+1}} - C_{T_{n-k}}) - 1_{\{t<T_{n-k+1}\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ \\ + 1_{\{t\ge T_{n-k+1}\}}\big(A_{n-k+1}\cdot r(L_{n-k+1}) + U_{k-1}(t - T_{n-k+1}, S_{T_{n-k+1}})\big)\Bigg]. \quad \text{(B.6)}$$

Note that we have

$$A_{n-k+1}\cdot r(L_{n-k+1}) + U_{k-1}(t - T_{n-k+1}, S_{T_{n-k+1}}) \\ \le \mathcal{M}_{r(L_{n-k+1})}[U_{k-1}]\big(t - T_{n-k+1}, S_{T_{n-k+1}-}\big).$$

This yields

$$
\begin{aligned}
R^{(2)}_{k-1} \leq \mathbb{E}^{(t,s)} 1_{\{t \geq T_{n-k}\}} \Bigg[& -\kappa (C_{t\wedge T_{n-k+1}} - C_{T_{n-k}}) - 1_{\{t<T_{n-k+1}\}} \gamma \left(\sum_{i=1}^{S_t} B_i - P \right)^+ \\
& + 1_{\{t \geq T_{n-k+1}\}} \mathcal{M}_{r(L_{n-k+1})}[U_{k-1}]\big(t - T_{n-k+1}, S_{T_{n-k+1}-}\big) \Bigg] \\
= \mathbb{E}^{(t,s)} 1_{\{t \geq T_{n-k}\}} \mathbb{E}^{(t,s)} \Bigg[& -\kappa (C_{t\wedge T_{n-k+1}} - C_{T_{n-k}}) - 1_{\{t<T_{n-k+1}\}} \gamma \left(\sum_{i=1}^{S_t} B_i - P \right)^+ \\
& + 1_{\{t \geq T_{n-k+1}\}} \mathcal{M}_{r(L_{n-k+1})}[U_{k-1}]\big(t - T_{n-k+1}, S_{T_{n-k+1}-}\big) \Bigg] \Bigg| \mathcal{F}_{T_{n-k}} \Bigg] \\
= \mathbb{E}^{(t,s)} 1_{\{t \geq T_{n-k}\}} & \mathcal{L}[U_{k-1}](t - T_{n-k}, S_{T_{n-k}}) = \mathbb{E}^{(t,s)} 1_{\{t \geq T_{n-k}\}} U_k(t - T_{n-k}, S_{T_{n-k}})
\end{aligned}
$$

where $\mathcal{F}_{T_{n-k}}$ denotes the information generated by arrivals and cancellations by time T_{n-k}. Using this upper bound on $R^{(2)}_{k-1}$, we get

$$
\begin{aligned}
R_{k-1} \leq \mathbb{E}^{(t,s)} \Big[& \sum_{i=1}^{N_{t\wedge T_{n-k}}} A_i \, r(L_i) - \kappa C_{t\wedge T_{n-k}} - 1_{\{t<T_{n-k}\}} \gamma \left(\sum_{i=1}^{S_t} B_i - P \right)^+ \\
& + 1_{\{t \geq T_{n-k}\}} U_k(t - T_{n-k}, S_{T_{n-k}}) \Big] \equiv R_k.
\end{aligned}
$$

Hence, the inequality also holds for $k+1$. Then, by induction it holds for all $1 \leq k \leq n+1$, and we have $V_n(t,s) \leq U_n(t,s)$.

Step 2: Let us now show $U_n(t,s) = G^{(\widetilde{A}^{(n)})}(t,s)$, for each $n \geq 1$. For $n = 1$, we first recall that

$$
G^{(\widetilde{D}^{(0)})}(t,s) = U_0(t,s) = V_0(t,s) = \mathbb{E}^{(t,s)} \left[-\kappa C_t - \gamma \left(\sum_{i=1}^{s-C_t} B_i - P \right)^+ \right],
$$

where $\widetilde{D}^{(0)}$ is the 'null' policy where all A_i's are zero. Next, by Remark 3 we have

$$U_1(t,s) =$$

$$\mathbb{E}^{(t,s)}\Bigg\{ -\kappa C_{t\wedge T_1} - 1_{\{T_1>t\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ + 1_{\{T_1\le t\}}\Big[\widetilde{A}_1^{(1)}\cdot r(L_1) + U_0(t-T_1, S_{T_1-}+\widetilde{A}_1^{(1)})\Big]\Bigg\}$$

$$\mathbb{E}^{(t,s)}\Bigg\{ -\kappa C_{t\wedge T_1} - 1_{\{T_1>t\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ + 1_{\{T_1\le t\}}\Big[\widetilde{A}_1^{(1)}\cdot r(L_1) + G^{\widetilde{D}^{(0)}}(t-T_1, S_{T_1-}+\widetilde{A}_1^{(1)})\Big]\Bigg\}$$

$$= G^{\widetilde{D}^{(1)}}(t,s),$$

thanks to strong Markov property. Next, let us assume by induction that $U_n(t,s) = G^{\widetilde{\mathcal{A}}^{(n)}}(t,s)$ for some $n\ge 1$. Note that $G^{\widetilde{D}^{(n+1)}}(\cdot,\cdot)$ satisfies

$$G^{\widetilde{D}^{(n+1)}}(t,s) = \mathbb{E}^{(t,s)}\Bigg\{ -\kappa C_{t\wedge T_1} - 1_{\{T_1>t\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ + 1_{\{T_1\le t\}}\Big[\widetilde{A}_1^{(n+1)}\cdot r(L_1) + G^{\widetilde{\mathcal{A}}^{(n)}}\Big(t-T_1, S_{T_1-}+\widetilde{A}_1^{(n+1)}\Big)\Big]\Bigg\}$$

again due to strong Markov property. Then by induction hypothesis we have $G^{\widetilde{\mathcal{A}}^{(n)}}(\cdot,\cdot) = U_n(\cdot,\cdot)$, and this yields

$$G^{\widetilde{D}^{(n+1)}}(t,s) = \mathbb{E}^{(t,s)}\Bigg\{ -\kappa C_{t\wedge T_1} - 1_{\{T_1>t\}}\gamma\left(\sum_{i=1}^{S_t} B_i - P\right)^+ + 1_{\{T_1\le t\}}\Big[\widetilde{A}_1^{(n+1)}\cdot r(L_1) + U_n\Big(t-T_1, S_{T_1-}+\widetilde{A}_1^{(n+1)}\Big)\Big]\Bigg\} = U_{n+1}(t,s),$$

thanks to Remark 3. Hence, the identity $G^{\widetilde{\mathcal{A}}^{(n)}}(\cdot,\cdot) = U_n(\cdot,\cdot)$ holds for all $n\ge 1$, and this completes the proof. $\square$

C Arrival Intensity Functions

In this appendix, the selection procedure of arrival intensity functions is described in detail.

C.1 Linear Case

Assume that the arrival intensity functions λ_1 and λ_2 are given by

$$\lambda_1(t) = \sigma_1 a_1(t), \lambda_2(t) = \alpha\sigma_1 a_2(t) \tag{C.7}$$

In this subsection the functions $a_i : [0, T] \to \mathbb{R}, i = 1, 2$ have the parametric form

$$a_1(s) = a_{11} - a_{12}s, a_2(s) = a_{21} + a_{22}s.$$

Since these parametric functions satisfy relations (4.5), (4.6) and (4.7), the feasible region of parameters $\mathbf{a} = (a_{11,}a_{12}, a_{21}, a_{22})$ is given by the set of linear (in)equalities.

$$\begin{array}{ccccccc}
a_{11} & -a_{12}T & & & \geq & 0 \\
a_{11}T & -\frac{1}{2}a_{12}T^2 & & & = & 1 \\
 & & a_{21} & +a_{22}T & \geq & 0 \\
 & & a_{21}T & +\frac{1}{2}a_{22}T^2 & = & 1 \\
a_{11} & & -\alpha a_{21} & & \geq & 0 \\
-a_{11} & +a_{12}T & +\alpha a_{21} & +\alpha a_{22}T & \geq & 0 \\
a_{11} & a_{12} & a_{21} & a_{22} & \geq & 0
\end{array} \tag{C.8}$$

If we would like the overall arrival intensity $\lambda(t) = \lambda_1(t) + \lambda_2(t)$ to be non-increasing, we need to add the linear restriction

$$-a_{12} + \alpha a_{22} \leq 0.$$

If the set of linear inequalities in (C.8) is denoted by the polytope P then one can show the following result.

Lemma 21 *If* $\mathbf{a}$ *belongs to* P *and* $\alpha < 1$ *then it follows that* $a_{12} + \alpha a_{22} > 0$.

Proof. Since $\mathbf{a}$ belongs to P we know by the nonnegativity restrictions that $a_{12}+\alpha a_{22} \geq 0$. Suppose now by contradiction that $a_{12} + \alpha a_{22} = 0$. By the nonnegativity of the decision variables this yields $a_{12} = a_{22} = 0$. Since $a_1(0) \geq \alpha a_2(0)$ and $a_1(T) \leq \alpha a_2(T)$ we therefore obtain $a_{11} \geq \alpha a_{21}$ and $a_{11} \leq \alpha a_{21}$ and so $a_{11} = \alpha a_{21}$. This shows $a_1(s) = a_{11} = \alpha a_{21} = \alpha a_2(s)$ and so by the normalization restrictions

$$1 = \int_0^T a_1(s)ds = \alpha \int_0^T a_2(s)ds = \alpha.$$

Since $\alpha < 1$ this yields a contradiction and so $a_{12} + \alpha a_{22} > 0$ for every $\mathbf{a} \in P$. ☐

Moreover it follows by Lemma 15 for $\alpha a_{22} + a_{12} > 0$ that

$$p_1(t^*) = p_2(t^*) \Longleftrightarrow t^* = \frac{a_{11} - \alpha a_{21}}{\alpha a_{22} + a_{12}}.$$

Therefore by Lemma 21 we need to solve the optimization problems

$$\min \left\{ \frac{a_{11} - \alpha a_{21}}{\alpha a_{22} + a_{12}} : \mathbf{a} \in P \right\} \tag{P}$$

and

$$\max \left\{ \frac{a_{11} - \alpha a_{21}}{\alpha a_{22} + a_{12}} : \mathbf{a} \in P \right\}. \tag{Q}$$

It is now possible to show the following result.

Lemma 22 *If follows for any* $0 < \alpha < 1$ *that the minimum values of the intersection point* t^* *in optimization problem P is given by* $(\alpha + 1)T$ *and this minimum is achieved by the linear functions*

$$a_1(s) = 2T^{-1} - 2T^{-2}s, a_2(s) = 2T^{-2}s.$$

Also the maximum value of the intersection point t^* *in optimization problem Q is given by* T *and this is achieved by the functions*

$$a_2(s) = T^{-1}, a_1(s) = (2 - \alpha)T^{-1} - 2(1 - \alpha)T^{-2}s$$

or

$$a_2(s) = 2T^{-2}s, a_1(s) = (2 - 2\alpha)T^{-1} - (2 - 4\alpha)T^{-2}s.$$

To reduce the number of decision variables in the polytope P we replace the set of equations

$$a_{21}T + \frac{1}{2}a_{22}T^2 = 1, \;\; a_{21} \geq 0, \;\; a_{22} \geq 0$$

with the equivalent set of equations

$$a_{21} = T^{-1} - \frac{1}{2}a_{22}T, \;\; a_{22}T^2 \leq 2, \;\; a_{22} \geq 0. \tag{C.9}$$

Similar by we replace the set of equations

$$a_{11}T - \frac{1}{2}a_{12}T^2 = 1, \;\; a_{11} \geq 0, \;\; a_{12} \geq 0$$

with the equivalent set

$$a_{11} = T^{-1} + \frac{1}{2}a_{12}T, \;\; a_{12} \geq 0, \;\; a_{12}T^2 \geq -2. \tag{C.10}$$

Substituting this we obtain a linear fractional programming problem in the nonnegative decision variables a_{12}, a_{22}. In this case the reduced feasible region is replaced by

$$\begin{array}{llcl} a_{12}T^2 & & \leq & 2 \\ a_{12}T^2 & & \geq & -2 \\ & a_{22}T^2 & \leq & 2 \\ a_{12}T^2 & +\alpha a_{22}T^2 & \geq & 2(1-\alpha) \\ a_{12} & a_{22} & \geq & 0. \end{array}$$

Clearly this is the same feasible region as the polytope P_r given by

$$\begin{array}{llcl} a_{12}T^2 & & \leq & 2 \\ & a_{22}T^2 & \leq & 2 \\ a_{12}T^2 & +\alpha a_{22}T^2 & \geq & 2(1-\alpha) \\ a_{12} & a_{22} & \geq & 0. \end{array}$$

Also by relation (C.9) and (C.10) the objective function

$$(a_{11}, a_{12}, a_{21}, a_{22}) \mapsto \frac{a_{11} - \alpha a_{21}}{\alpha a_{22} + a_{12}}$$

is replaced by the linear fractional function

$$(a_{12}, a_{22}) \mapsto \frac{(1-\alpha)T^{-1} + \frac{T}{2}a_{12} + \frac{\alpha T}{2}a_{22}}{\alpha a_{22} + a_{12}}.$$

Hence we are dealing with a reduced 2-dimensional linear fractional programming problem and this is given by

$$\min\left\{\frac{(1-\alpha)T^{-1}+\frac{1}{2}a_{12}T+\frac{1}{2}\alpha a_{22}T}{\alpha a_{22}+a_{12}}:(a_{12},a_{22})\in P_r\right\}$$

and

$$\max\left\{\frac{(1-\alpha)T^{-1}+\frac{T}{2}a_{12}+\frac{\alpha T}{2}a_{22}}{\alpha a_{22}+a_{12}}:(a_{12},a_{22})\in P_r\right\}.$$

Since a fractional linear function is both quasiconcave and quasiconvex it follows that for both problems the optimal solution is obtained at a vertex of the polytope P_r. Since the polytope is a subset of $\mathbb{R}^2_+$ we can analytically compute the vertices and so we can easily evaluate the optimal values of the two optimization problems.

C.2 Quadratic Case

Let the functions $a_i:[0,T]\to\mathbb{R}_+, i=1,2$ satisfy the parametric representation

$$a_i(s)=a_{i1}+a_{i2}s+a_{i3}s^2=a_i(0)+a_i'(0)s+a_i''(0)s^2.$$

Clearly, a_1 satisfies (4.5) if and only if

$$\begin{array}{llllll} a_1(0) & +a_1'(0)T & +a_1''(0)T^2 & \geq & 0 \\ a_1(0) & & & \geq & 0 \\ & a_1'(0) & & \leq & 0 \\ & a_1'(0) & +2a_1''(0)T & \leq & 0 \\ a_1(0)T & +\frac{1}{2}a_1'(0)T^2 & +\frac{1}{3}a_2''(0)T^3 & = & 1. \end{array} \tag{C.11}$$

Also, a_2 satisfies relation (4.6) if and only if

$$\begin{array}{lllcl} a_2(0) & +a_2'(0)T & +a_2''(0)T^2 & \geq & 0 \\ a_2(0) & & & \geq & 0 \\ & a_2'(0) & & \geq & 0 \\ & a_2'(0) & +2a_2''(0)T & \geq & 0 \\ a_2(0)T & +\frac{1}{2}a_2'(0)T^2 & +\frac{1}{3}a_2^{''}(0)T^3 & = & 1. \end{array} \quad \text{(C.12)}$$

Also if the functions a_1 and a_2 satisfy condition (4.7) this is the same as

$$\begin{array}{lllllllcl} a_1(0) & & & -\alpha a_2(0) & & & \geq & 0 \\ \alpha a_2(0) & +\alpha a_2'(0)T & +\alpha a_2''(0)T^2 & -a_1(0) & -a_1'(0)T & -a_2''(0)T^2 & \geq & 0. \end{array} \quad \text{(C.13)}$$

Hence by relations (C.11), (C.12) and (C.13) the feasible polytope P is given by

$$
\begin{array}{llllllll}
a_1(0) & +a_1'(0)T & +a_1''(0)T^2 & & & & \geq & 0 \\
 & a_1'(0) & +2a_1''(0)T & & & & \leq & 0 \\
a_1(0)T & +\frac{1}{2}a_1'(0)T^2 & +\frac{1}{3}a_1''(0)T^3 & & & & = & 1 \\
 & & & a_2(0) & +a_2'(0)T & +a_2''(0)T^2 & \geq & 0 \\
 & & & & a_2'(0) & +2a_2''(0)T^2 & \geq & 0 \\
 & & & a_2(0)T & +\frac{1}{2}a_2'(0)T^2 & +\frac{1}{3}a_2''(0)T^3 & = & 1 \\
a_1(0) & & & -\alpha a_2(0) & & & \geq & 0 \\
-a_1(0) & -a_1'(0)T & -a_1''(0)T^2 & +\alpha a_2(0) & +\alpha a_2'(0)T & +\alpha a_2''(0)T^2 & \geq & 0 \\
a_1(0) & & & & & & \geq & 0 \\
 & a_1'(0) & & & & & \leq & 0 \\
 & & & a_2(0) & & & \geq & 0 \\
 & & & & a_2'(0) & & \geq & 0.
\end{array}
$$

Introduce now for $\mathbf{a} = (a_1(0), ..., a_2''(0))$ belonging to P the function $h : P \times \mathbb{R} \to \mathbb{R}$ given by

$$h(\mathbf{a},t) := a_1(t) - \alpha a_2(t)$$

Observe for every t the function $\mathbf{a} \mapsto h(\mathbf{a},t)$ is a linear function. Also we know by Lemma 15 that $p_1(t) \geq p_2(t)$ if and only if $h(\mathbf{a},t) \geq 0$. The following result is easy to verify

Lemma 23 *For $\alpha < 1$ and the vector $\mathbf{a} = (a_1(0), ..., a_2''(0))$ belonging to P it follows that the function $t \mapsto h(\mathbf{a},t)$ is quadratic and strictly decreasing and the system of equations $h(\mathbf{a}, t) = 0, 0 \leq t \leq T$ has exactly one positive solution $t(\mathbf{a})$.*

Proof. Since the vector $\mathbf{a} = (a_1(0), ...a_2''(0))$ belongs to P we know that the continuous quadratic function $t \mapsto h(\mathbf{a}, t)$ satisfies $h(\mathbf{a}, 0) \geq 0$ and $h(\mathbf{a}, T) \leq 0$. Hence there exists some $0 \leq t^* \leq T$ satisfying $h(\mathbf{a}, t^*) = 0$. Also since $\mathbf{a} \in P$ the function a_1 is decreasing and a_2 is increasing implying $t \mapsto h(\mathbf{a}, t)$ is decreasing Moreover. If $h(\mathbf{a}, t_1) = h(\mathbf{a}, t_2)$ for some $0 \leq t_1 < t_2 \leq T$ then by the monotonicity of h we obtain for every $0 < t_1 \leq t < t_2$ that $h(\mathbf{a}, t) = h(\mathbf{a}, t_1)$. This shows $h'_{\mathbf{a}}(t) = h''_{\mathbf{a}}(t) = 0$ for every $t_1 < t < t_2$ and since h is quadratic we obtain $a_1(s) = \alpha a_2(s)$ for every $0 \leq s \leq T$. This implies

$$1 = \int_0^T a_1(s)ds = \alpha \int_0^T a_2(s)ds = \alpha$$

and we obtain a contradiction with $\alpha < 1$. Hence $h_{\mathbf{a}}$ is strictly decreasing. This immediately implies that the system of equations $h(\mathbf{a}, t) = 0, 0 \leq t \leq T$ has exactly one solution $t(\mathbf{a})$. If $t(\mathbf{a}) = 0$ it follows that $a_1(t) < \alpha a_2(t)$ for every $0 < t \leq T$ and this shows

$$1 = \int_0^T a_1(s)ds < \alpha \int_0^T a_2(s)ds = \alpha$$

yielding a contradiction and so $t(\mathbf{a})$ is positive. □

The next lemma is also easy to verify.

Lemma 24 *The function* $t : P \to [0, T]$ *is continuous and quasiconcave.*

Proof. We first show that the function $\mathbf{a} \to t(\mathbf{a})$ is continuous. Let $(\mathbf{a}_n)_{n \in \mathbb{N}}$ be a converging sequence in P with limit $\mathbf{a}_\infty$. If the sequence $t(\mathbf{a}_n)$ does not convergence to $t(\mathbf{a})$ then there exists some subsequence $(n_k)_{k \in \mathbb{N}}$ and $m \in \mathbb{N}$ satisfying

$$\mid t(\mathbf{a}_{n_k}) - t(\mathbf{a}_\infty) \mid > m^{-1} \tag{C.14}$$

for every k. Since $t(\mathbf{a}_{n_k})$ belongs to the compact set $[0, T]$ there exists by the Heine-Borel Theorem some converging subsequence $I \subseteq \{n_k\}_{k \in \mathbb{N}}$ satisfying $\lim_{i \in I} t(\mathbf{a}_i) = c$ and by the continuity of the function $h : P \times \mathbb{R} \to \mathbb{R}$ this shows

$$0 = \lim_{i \in I \uparrow \infty} h(\mathbf{a}_i, t(\mathbf{a}_i)) = h(\mathbf{a}_\infty, c).$$

By Lemma 23 it must follow that $c = t(\mathbf{a}_\infty)$ and this yields a contradiction with relation (C.14). To show the quasi-concavity assume by contradiction that there exist some $0 < \lambda < 1$ and $\mathbf{a}_1.\mathbf{a}_2 \in P$ satisfying

$$t(\lambda\mathbf{a}_1 + (1-\lambda)\mathbf{a}_2) < \min(t(\mathbf{a}_1).t(\mathbf{a}_2)). \tag{C.15}$$

First observe

$$\begin{aligned} 0 &= h(\lambda\mathbf{a}_1 + (1-\lambda)\mathbf{a}_2, t(\lambda\mathbf{a}_1 + (1-\lambda)\mathbf{a}_2) \\ &= \lambda h(\mathbf{a}_1, t(\lambda\mathbf{a}_1 + (1-\lambda)\mathbf{a}_2) + (1-\lambda)h(\mathbf{a}_2, t(\lambda\mathbf{a}_1 + (1-\lambda)\mathbf{a}_2). \end{aligned} \tag{C.16}$$

Since by Lemma 23 the function $t \mapsto h(\mathbf{a}, t)$ is strictly decreasing for each $\mathbf{a} \in P$ we obtain by relation (C.15) that

$$h(\mathbf{a}_1, t(\lambda\mathbf{a}_1 + (1-\lambda)\mathbf{a}_2) > h(\mathbf{a}_1, t(\mathbf{a}_1)) = 0$$

and

$$h(\mathbf{a}_2, t(\lambda\mathbf{a}_1 + (1-\lambda)\mathbf{a}_2) > h(\mathbf{a}_2, t(\mathbf{a}_2)) = 0.$$

Applying now relation (C.16) we obtain a contradiction and this yields the desired result. □

Hence we now have to solve the optimization problem

$$\min\{t(\mathbf{a}) : \mathbf{a} \in P\}.$$

By Lemma 24 we know that an optimal solution is obtained at a vertex. Before deriving an explicit formula for $t(\mathbf{a})$ we introduce for $\mathbf{a} \in P$ satisfying $a_1''(0) - \alpha a_2''(0) \neq 0$ the functions

$$b(\mathbf{a}) = \frac{a_1'(0) - \alpha a_2'(0)}{a_1''(0) - \alpha a_2''(0)}. \tag{C.17}$$

and

$$c(\mathbf{a}) = \frac{a_1(0) - \alpha a_2(0)}{a_1''(0) - \alpha a_2''(0)} \tag{C.18}$$

Looking at the system $h(\mathbf{a}, t) = 0, 0 \leq t \leq T$ with

$$h(\mathbf{a}, t) = a_1(0) - \alpha a_2(0) + (a_1'(0) - \alpha a_2'(0))t + (a_1''(0) - \alpha a_2''(0))t^2$$

it follows for $a_1''(0) - \alpha a_2''(0) \neq 0$ that $h(\mathbf{a}, t) = 0, 0 \leq t \leq T$ is the same system as

$$c(\mathbf{a}) + b(\mathbf{a})t + t^2 = 0, 0 \leq t \leq T. \tag{C.19}$$

The two only candidates which might solve this system are given by

$$t_+(\mathbf{a}) := -\frac{1}{2}b(\mathbf{a}) + \frac{1}{2}\sqrt[2]{b(\mathbf{a})^2 - 4c(\mathbf{a})} \tag{C.20}$$

and

$$t_-(\mathbf{a}) = -\frac{1}{2}b(\mathbf{a}) - \frac{1}{2}\sqrt[2]{b(\mathbf{a})^2 - 4c(\mathbf{a})}. \tag{C.21}$$

Since by Lemma 23 the system $h(\mathbf{a}, t) = 0, 0 \leq t \leq T$ has a solution it must follow

$$b(\mathbf{a})^2 - 4c(\mathbf{a}) \geq 0. \tag{C.22}$$

Also both roots $t_+(\mathbf{a})$ and $t_-(\mathbf{a})$ are real valued and satisfy

$$t_-(\mathbf{a}) \leq t_+(\mathbf{a}). \tag{C.23}$$

Onr can now show the following result.

Lemma 25 *It follows for every* $\mathbf{a} \in P$ *that*

$$t(\mathbf{a}) = \begin{cases} t_+(\mathbf{a}) & \textit{if } a_1''(0) - \alpha a_2''(0) < 0 \\ \frac{\alpha a_2(0) - a_1(0)}{a_1'(0) - \alpha a_2'(0)} & \textit{if } a_1''(0) - \alpha a_2''(0) = 0 \\ t_-(\mathbf{a}) & \textit{if } a_1''(0) - \alpha a_2''(0) > 0. \end{cases}$$

Proof. Clearly for every $\mathbf{a} \in P$ satisfying $a_1''(0) - \alpha a_2''(0) < 0$ it follows using $a_1'(0) - \alpha a_2'(0) \leq 0$ and $a_1(0) - \alpha a_2(0) \geq 0$ that

$$c(\mathbf{a}) \leq 0, \;\; b(\mathbf{a}) \geq 0. \tag{C.24}$$

Also the condition $a_1''(0) - \alpha a_2''(0) < 0$ implies the restriction

$$b(\mathbf{a})^2 - 4c(\mathbf{a}) > 0.$$

To show this we assume by contradiction that this inequality does not hold. Hence by relation (C.22) we obtain

$$b(\mathbf{a})^2 - 4c(\mathbf{a}) = 0.$$

By relation (C.24) this yields

$$0 \leq b(\mathbf{a})^2 = 4c(\mathbf{a}) \leq 0$$

and so $b(\mathbf{a}) = c(\mathbf{a}) = 0$. Now the only solution of the system $h(\mathbf{a}, t) = 0, 0 \leq t \leq T$ is given by $t = 0$ and we obtain a contradiction with Lemma 23. Therefore it must hold for $a_1''(0) - \alpha a_2''(0) < 0$ that $b(\mathbf{a})^2 - 4c(\mathbf{a}) > 0$ and applying relation (C.21) and (C.24) yields $t_-(\mathbf{a}) < 0$. Since the only possible solutions of $h(\mathbf{a}, t) = 0, 0 \leq t \leq T$ are either $t_+(\mathbf{a})$ or $t_-(\mathbf{a})$ this implies $t(\mathbf{a}) = t_+(\mathbf{a})$. If $a_1''(0) - \alpha a_2''(0) > 0$ we know using $a_1(0) - \alpha a_2(0) \geq 0$ and $a_1'(0) - \alpha a_2'(0) \leq 0$ that

$$c(\mathbf{a}) \geq 0, \;\; b(\mathbf{a}) \leq 0. \tag{C.25}$$

This shows

$$0 \leq \sqrt[2]{b(\mathbf{a})^2 - 4c(\mathbf{a})} \leq \mid b(\mathbf{a}) \mid .$$

and hence using also relation (C.25) we obtain

$$t_{-}(\mathbf{a}) \geq -\frac{1}{2}b(\mathbf{a}) - \frac{1}{2} \mid b(\mathbf{a}) \mid = 0.$$

Applying now relation (C.23) and Lemma 23 yields $t(\mathbf{a}) = t_{-}(\mathbf{a})$. The case $a_1''(0) - \alpha a_2''(0) = 0$ is obvious, so its proof is omitted. □