Characterizations of the Cumulative Offer Process

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Abstract

In the matching with contracts setting, we provide two new axiomatic characterizations of the "cumulative offer process" (COP) in the domain of hospital choices satisfying "unilateral substitutes" and "irrelevance of rejected contracts." We say that a mechanism is truncation-proof if no doctor can ever benefit from truncating his preferences. Our first result shows that the COP is the unique stable and truncation-proof mechanism. Next, we say that a mechanism is invariant to lower tail preferences change if any doctor's assignment does not depend on his preferences over worse contracts. Our second characterization shows that a mechanism is stable and invariant to lower tail preferences change if and only if it is the COP.

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Keywords: the cumulative offer process, truncation, invariance, characterization, unilateral substitutes, irrelevance of rejected contracts.

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1 Introduction

Hatfield and Milgrom (2005) introduce a matching with contract framework which admits the Gale and Shapley (1962)'s standard matching and Kelso and Crawford (1982)'s labor market models as its special cases.[1] They adopt the substitutes condition in the conventional matching literature (e.g., see Roth and Sotomayor (1990)) and introduce a "*Cumulative Offer Process*" (henceforth, COP), which is a generalization of the doctor-proposing deferred acceptance algorithm (henceforth, DA) of Gale and Shapley (1962). Hatfield and Milgrom (2005) then show that the COP produces stable allocations whenever contracts are substitutes. Indeed, it produces the doctor-optimal stable allocation–the unanimously preferred stable allocation by doctors to any other stable allocation. Moreover, they show that the COP becomes strategy-proof[2] with an additional "*Law of Aggregate Demand*" condition (LAD).

Since then, the COP has been the main mechanism in the matching with contract literature and received attention from researchers. Hatfield and Kojima (2010) introduce two weaker conditions than substitutes (from weaker to stronger): "bilateral substitutes" (BS) and "unilateral substitutes" (US) and show that the COP is stable under the former, and it produces the doctor-optimal stable allocation under the latter. They also show its immunity against preference manipulations under US and the LAD. All of these results are obtained for the case where hospitals have choices, rather than preferences, as primitives with an additional "irrelevance of rejected contracts" (IRC) condition by Aygün and Sönmez (2013). There are other studies as to the COP, and some of them are to be visited in the next Related Literature section.

In this paper, we provide two axiomatic characterizations of the COP in the domain of choice functions which are US satisfying the IRC. While there are various axiomatizations of its predecessor DA in the standard matching framework for various domains, to the best

[1] Echhenique (2012) shows that under the substitutes condition of Hatfield and Milgrom (2005), matching with contracts problems can be embedded into Kelso and Crawford (1982)'s setting.

[2] That is, no doctor ever has incentive to misreport his preferences.

of our knowledge, there are only two studies providing characterizations of the COP in the current matching with contract setting. First, Hatfield and Kojima (2010) show that under US and the IRC, the COP is the most favorable stable mechanism for the doctor side, that is, it is the doctor-optimal stable mechanism. In another recent study, Hatfield et al. (2015) provide three conditions giving the maximal hospital choices domain to have a stable and strategy-proof mechanism, and within that domain, the COP is the unique such mechanism. Their tree conditions are weaker than the combination of US and the LAD. As we do not impose the latter, their results do not imply ours.

Our first characterization is as to the COP's strategic properties. A preference list is said to be a *truncation* of another preference list if the relative rankings of the contracts remain the same while the set of contracts which are better than being unassigned shrinks under the truncation. Truncation strategies are well-studied in the literature. They are shown to be easy strategies to be used in certain sense by Roth and Rothblum (1999). Moreover, Mongell and Roth (1991) empirically document that agents have used them in real-life matching problems. Hence, truncation strategies are important for both theoretical and practical purposes; thereby it is desirable for a mechanism to be at least immune to truncations if it is not strategy-proof. Due to Hatfield and Milgrom (2005), we know that the COP is not strategy-proof under the substitutes and the IRC conditions.[3] Yet, our first theorem shows that under the weaker US and the IRC conditions, the COP is immune to truncations; and indeed, it is the unique such stable mechanism.

Our second axiomatization deals with certain invariance property. One could argue that doctors' assignments should not depend on their preferences over worse alternatives. Our invariance axiom formalizes this, and we say that a mechanism is *invariant to lower tail preferences change* if no doctor's assignment depends on his preferences over worse contracts. We show that under US and the IRC, the COP is the unique stable and invariant to lower tail preferences change mechanism.

[3]They assume that hospitals have underlying preferences, inducing their choices. Therefore, the IRC automatically comes.

One thing to emphasize is that the COP is shown to coincide with the DA under US and the IRC (see Hatfield and Kojima (2010) and Aygün and Sönmez (2012)). Hence, this paper effectively axiomatizes the DA; however we prefer referring to the mechanism as the COP in order to stick to the matching with contracts literature. In relation to this, another way of stating the paper's contribution is that it provides characterizations of the DA in the largest choice domain up to date in the literature for which axiomatization of the DA is offered.[4]

2 Related Literature

Beside the theoretical appeal of the matching with contract framework, recent surge in the literature shows its practical relevance. Sönmez and Switzer (2013) and Sönmez (2013) formulate cadet-branch matching in the U.S. Army as a matching with contracts problem. Both papers show that the currently used mechanisms fail to admit desirable properties, and they propose to replace them by the COP. Kominers and Sönmez (2013) allow school priorities to change across seats within schools in school choice. They study the problem in the matching with contracts framework and offer the COP to be used and show that their results also have applications to airline seat upgrades and affirmative action problems. In other recent papers, Aygün and Bo (2014) and Aygün and Turhan (2014) consider affirmative action policies in Brazil and Indian colleges and document the weaknesses of the current mechanisms. They both study the problem in the matching with contract setting and propose the COP against the currently used mechanisms.

Given that the COP has proved to be important for both theoretical and practical purposes, various properties of it have been studied in the literature. Hatfield and Kojima (2010) show that under US, no previously rejected contract is accepted in a later step in

[4] As pointed out earlier, Hatfield et al. (2015)'s conditions are weaker than the combination of US and the LAD. As we do not impose the latter, their domain does not include ours. This is also obvious from the results. They show that the COP is strategy-proof in their domain, yet it fails to be so in the current US and IRC domain (see Hatfield and Milgrom (2005)).

the course of the COP. This result is extended to the case where hospitals have choices, instead of preferences, with the additional IRC by Aygün and Sönmez (2012). Hatfield and Milgrom (2005) introduce a version of the COP in which doctors make offers simultaneously, whereas in Hatfield and Kojima (2010)'s version, doctors make offers sequentially. Hirata and Kasuya (2014) prove that under BS and the IRC, these two versions coincide with each other regardless of the order in which doctors make offer in the sequential version. Afacan (2014) demonstrates that the COP is both resource and extension monotonic under US and the IRC; and with the additional LAD, it also respects improvements.

There are many axiomatic characterization papers on well-known and in use matching mechanisms, including Boston, Top Trading Cycles (attributed to David Gale), and the DA. Some of them are Kojima and Ünver (2014), Afacan (2013), Abdulkadiroğlu and Che (2010), Dur (2015), Morrill (2013b), Kojima and Manea (2010), Morrill (2013a), and Alcalde and Barbera (1994).

3 The Model and Results

There are finite sets D and H of doctors and hospitals, and a finite set X of contracts. Each *contract* $x \in X$ is associated with one doctor $x_D \in D$ and one hospital $x_H \in H$. Each doctor can sign at most one contract. The *null contract*, denoted by $\emptyset$, means that the doctor has no contract. For $X' \subseteq X$, let $X'_D = \{d \in D : \exists\, x \in X' \text{ with } x_D = d\}$.

Each doctor $d \in D$ has a strict *preference relation* P_d over $\{x \in X : x_D = d\} \cup \{\emptyset\}$. Given any two contracts x', x where $x'_D = x_D = d$, we write $x' R_d x$ only if $x' P_d x$ or $x' = x$. A contract x is *acceptable* to doctor d if $x P_d \emptyset$. It is otherwise unacceptable. The chosen contract of doctor d from $X' \subseteq X$ is given as

$$C_d(X') = \max_{P_d}[\{x \in X' : x_D = d\} \cup \{\emptyset\}].$$

We write $C_D(X') = \bigcup_{d \in D} C_d(X')$ for the set of contracts chosen from X' by some doctor. Let $P = (P_d)_{d \in D}$ be the preference profile of the doctors.

Each hospital h has a *choice function* $C_h : 2^X \to 2^X$ defined as follows: for any $X' \subseteq X$,

$$C_h(X') \in \{X'' \subseteq X' : (\text{for each } x \in X'',\ x_H = h) \text{ and } (\text{for any } x', x'' \in X'',\ x'_D \neq x''_D)\}.$$

We write $C_H(X') = \bigcup_{h \in H} C_h(X')$ for the set of contracts chosen from X' by some hospital. The choice function profile of hospitals is $C = (C_h)_{h \in H}$. In the rest of the paper, we fix D, H, and C; thereby we suppress them from the notation and just write P to denote the problem.

A set of contracts $X' \subseteq X$ is an *allocation* if $x, x' \in X'$ and $x \neq x'$ imply $x_D \neq x'_D$. We extend the preferences of doctors over the set of allocations in a natural way as follows: for any given two allocations X' and X'', $X' P_d X''$ if and only if $\{x' \in X' : \ x'_D = d\}\ P_d \{x'' \in X'' : \ x''_D = d\}$.

Definition 1. *An allocation X' is stable if*

(1) $C_D(X') = C_H(X') = X'$ *and*

(2) *there exist no hospital h and set of contracts $X'' \neq C_h(X')$ such that $X'' = C_h(X' \cup X'') \subseteq C_D(X' \cup X'')$.*

A *mechanism* ψ is a function producing an allocation $\psi(P)$ for any problem P. Mechanism ψ is stable if $\psi(P)$ is stable for every problem P.

Hatfield and Milgrom (2005) generalize Gale and Shapley (1962)'s celebrated DA to the matching with contracts problem by introducing the following COP.

Step 1: One arbitrarily chosen doctor d offers her first choice contract x_1. The offer receiving hospital h holds the contract if $x_1 = C_h(\{x_1\})$ and rejects it otherwise. Let $A_h(1) = \{x_1\}$ and $A_{h'}(1) = \emptyset$ for all $h' \neq h$.

In general,

Step t: One arbitrarily chosen doctor currently having no contract held by any hospital offers her preferred contract x_t from among those that have not been rejected in the previous steps. The offer receiving hospital h holds x_t if $x_t \in C_h(A_h(t-1) \cup \{x_t\})$ and rejects it otherwise. Let $A_h(t) = A_h(t-1) \cup \{x_t\}$ and $A_{h'}(t) = A_{h'}(t-1)$ for all $h' \neq h$.

The algorithm terminates when every doctor is matched to a hospital or every unmatched doctor has all acceptable contracts rejected. As there are finite contracts, the algorithm terminates in some finite step T. The final outcome is $\bigcup_{h \in H} C_h(A_h(T))$.

The COP may not even produce an allocation without any structure on hospital choices. The following two conditions have proved to be useful.

Definition 2 (Hatfield and Kojima (2010))**.** *Contracts are unilateral substitutes (US) for hospital h if there are no set of contracts $Y \subset X$ and another pair of contracts $x, z \in X \setminus Y$ such that*

$$z \notin C_h(Y \cup \{z\}),\ z_D \notin Y_D,\ \text{and}\ z \in C_h(Y \cup \{x, z\}).$$

In words, US ensures that if a rejected contract z of doctor z_D from Y starts to be chosen whenever a new contract x becomes available, then that doctor has to have another contract in Y.

Definition 3 (Aygün and Sönmez (2013))**.** *Contracts satisfy the irrelevance of rejected contracts (IRC) for hospital h if for any $Y \subset X$ and $z \in X \setminus Y$,*

$$z \notin C_h(Y \cup \{z\}) \Rightarrow\ C_h(Y) = C_h(Y \cup \{z\}).$$

The IRC requires that the removal of rejected contracts has no effect on the chosen sets.[5]

Hatfield and Kojima (2010) and Aygün and Sönmez (2012) show that the COP produces stable allocation even under the weaker BS[6] and the IRC conditions; thereby the COP is a stable mechanism under US and the IRC. In what follows, we provide two axiomatizations of the COP under US and the IRC.

[5]In the many-to-many matching context (without contracts), Blair (1988) and Alkan (2002) use this condition. The latter refers to it as "consistency."

[6]See Footnote 11 for its definition.

For a given doctor d with preferences P_d, let $Ac(P_d) = \{x \in X : x_D = d \text{ and } xP_d\emptyset\}$. That is, it is the set of contracts doctor d finds acceptable. A preference list P'_d is a *truncation* of P_d if $Ac(P'_d) \subset Ac(P_d)$, and for any $x, x' \in X$ with $x_D = x'_D = d$, xP_dx' if and only if xP'_dx'.

Definition 4. *A mechanism ψ is truncation-proof if there are no problem P, doctor $d \in D$, and a truncation P'_d such that $\psi(P'_d, P_{-d})P_d\psi(P)$.*[7]

Roth and Rothblum (1999) demonstrate that truncation strategies are easy to employ in the sense that agents need less information about others' preferences to profitably employ them. On the practical ground, on the other hand, Mongell and Roth (1991) show that truncation strategies have been employed in real-life matching problems. Therefore, it is desirable for a mechanism to be truncation-proof in both theory and practice. We already know that the COP is not strategy-proof under US and the IRC. However, our first result below shows that it is at least non-manipulable by truncation strategies; and furthermore, it is the unique such rule among stable mechanisms.

Theorem 1. *Under US and the IRC, a mechanism is stable and truncation-proof if and only if it is the COP.*

Proof. See Appendix.

□

We now introduce another axiom, which is new in the literature even though close variants of it have been introduced in other settings. It restricts how a mechanism responds to certain changes in preferences. For a given doctor d with his preferences P_d and a contract of himself x, let $U(P_d, x) = \{x' \in X : x'_D = d \text{ and } x'R_dx\}$. That is, it is the set of all contracts which are no worse than x. Moreover, let $P_{d_{|U(P_d,x)}}$ be the restriction of P_d to $U(P_d, x)$, that is, it is the part of P_d over $U(P_d, x)$.

Definition 5. *A mechanism ψ is invariant to lower tail preferences change if for any problem P, any doctor $d \in D$, and any P'_d such that $P_{d_{|U(P_d,\psi_d(P))}} = P'_{d_{|U(P_d,\psi_d(P))}}$, $\psi_d(P) = \psi_d(P'_d, P_{-d})$.*

[7] P_{-d} is the preference profile of all doctors but doctor d.

Less formally, it imposes that no doctor's assignment depends on his preferences over less preferred contracts. Different variants of this axiom have been introduced in other contexts.[8]

Theorem 2. *Under US and the IRC, a mechanism is stable and invariant to lower tail preferences change if and only if it is the COP.*

Proof. See Appendix. □

Remark 1. It is easy to see that stability are separately independent of truncation-proofness and invariance to lower tail preferences change. Moreover, truncation-proofness and invariance to lower tail preferences change are independent of each other as well. To see this, consider a problem instance where there exist one doctor d and one hospital h. Suppose that there are three different contracts: x, x', and x''. Let $P : x, x', x'', \emptyset$ and $P' : x, \emptyset, x', x''$.[9] Hospital h has preferences as well and let $P_h :\ x, x', x'', \emptyset$.[10] Consider a mechanism ψ such that $\psi(P) = \{x\}$ and $\psi(P') = \{\emptyset\}$, and it coincides with the COP at other instances. It is truncation-proof; however it is not invariant to lower tail preferences change.

For the converse, let $P'' : x, x', \emptyset, x''$ and consider mechanism ϕ such that $\phi(P) = \phi(P'') = \{x''\}$, $\phi(P') = \{x'\}$, and $\phi(\hat{P}) = \{x'\}$ for any other $\hat{P}$. It is invariant to lower tail preferences change, yet it is not truncation-proof.

Remark 2. Our characterizations do not carry over to the larger domain of BS and the IRC.[11] Specifically, the COP loses invariance to lower tail preferences changes. On the other hand, while it is still truncation-proof,[12] it is not the unique such mechanism among

[8]In the random matching context, some stronger variants of this axiom have been used in different papers, including Hashimoto and Hirata (2011), Hashimoto et al. (2014), Bogomolnaia and Heo (2011), and Heo and Yilmaz (2015).

[9]The earlier a contract appears in a preference list, the more preferred it is. For instance, under P, x is the top contract, then x', and so on. The same way of writing applies to hospital preferences as well.

[10]Note that hospital choices satisfy both US and the IRC.

[11]Contracts are BS if there are no set of contracts $Y \subset X$ and another pair of contracts $x, z \in X \setminus Y$ such that $z \notin C_h(Y \cup \{z\})$, $z_D, x_D \notin Y_D$, and $z \in C_h(Y \cup \{x, z\})$.

[12]To see this, if a doctor truncates his preferences such that the last offer he makes in the COP under the true preference profile is still acceptable, then the outcome would not change. Otherwise, he becomes unassigned in some step. In this case, from the proof of Theorem 1 of Hatfield and Kojima (2010), we know that no contract of him is accepted after that step; thereby becomes unassigned at the end of the COP. Hence, the COP is truncation-proof under BS and the IRC.

stable solutions. To see these, let $D = \{d_1, d_2\}$ and $H = \{h_1\}$. Consider the following preference profile (assuming that the hospital's choices are generated by its preferences):

$$P_{d_1}: \ x, x', \emptyset; \ P_{d_2}: \ y, y', \emptyset; \ P_{h_1}: \ \{x, y'\} \succ x' \succ y \succ x \succ \ ..anything.. \ \succ \emptyset.$$

It is easy to verify that the contracts are BS (not US), satisfying the IRC.[13] The COP outcome at the above problem is $\{x, y'\}$. Let us now consider $P'_{d_1}: \ x, \emptyset, x'$. Then, $COP(P'_{d_1}, P_{d_2}) = \{y\}$. Therefore, it is not invariant to lower tail preferences change under BS and the IRC.

For the other case, consider the same set of doctors and hospital along with the same doctor preferences. Let the hospital preferences be as follows:

$$P'_{h_1}: \ \{x', y'\} \succ \{x, y'\} \succ x' \succ y \succ \{x, y\} \succ ..anything.. \succ \emptyset.$$

It is easy to verify that the contracts are BS (not US), satisfying the IRC. The COP outcome above is $\{x', y'\}$. Consider another stable mechanism ϕ where it produces $\{x, y'\}$ at the above problem, and it coincides with the COP at other problem instances. It is truncation-proof as well as stable.

Appendix

Proof of Theorem 1. **"If" Part.** Under US and the IRC, due to Hatfield and Kojima (2010) and Aygün and Sönmez (2012), we already know that the COP is stable. We also know that under US and the IRC, no previously rejected contract is accepted in a later step in the course of the COP. In other words, the COP coincides with the DA. Hence, if a doctor truncates his preferences, he would either receive the same contract as at the true preference profile or be unassigned. In either case, he would not benefit from truncating, showing that the COP is truncation-proof under US and the IRC.

[13] Because the hospital's choices are generated by its preferences, the contracts automatically satisfy the IRC.

"Only If" Part. Let ψ be a stable mechanism which is truncation-proof. Assume for a contradiction that $\psi \neq COP$, and let P be such that $\psi(P) \neq COP(P)$. For ease of notation, let $\psi(P) = X'$ and $COP(P) = X''$. For any allocation X and doctor d, let x_d be doctor d's contract at X.

Let $S^0 = \{d \in D: \ X'' P_d X'\}$. From Hatfield and Kojima (2010), we know that X'' is at least as good as X' for any doctor. This, along with $X' \neq X''$, implies that S^0 is a non-empty set. Let $d_1 \in S$ and consider P'_{d_1} which is the truncation of P_{d_1} such that $x \notin Ac(P'_{d_1})$ if and only if $x_D = d_1$ and $x''_{d_1} P_{d_1} x$. Let $P^1 = (P'_{d_1}, P_{-d_1})$.

As the COP coincides with the DA under US and the IRC, we have $COP(P^1) = X''$. On the other hand, due to the truncation-proofness of ψ, we have $\psi_{d_1}(P^1) = \emptyset$.

Let $S^1 = \{d \in D: \ X'' P_d \psi(P^1)\} \setminus \{d_1\}$. We now claim that S^1 is non-empty. Assume for a contradiction that it is empty. This means that for any doctor $d \in D \setminus \{d_1\}$, we have $x''_d = \psi_d(P^1)$. This is because X'' cannot be worse than $\psi(P^1)$ for any doctor under US and the IRC (Hatfield and Kojima (2010) and Aygün and Sönmez (2012)). Moreover, we have $\psi_{d_1}(P^1) = \emptyset$. These, however, contradict the stability of $\psi(P^1)$, as doctor d_1 would block it with his assigned hospital at X'' through signing contract x''_{d_1}. This shows that $S^1 \neq \emptyset$.

Let $d_2 \in S^1$ and, similar to above, consider P'_{d_2} which is the truncation of P_{d_2} such that $x \notin Ac(P'_{d_2})$ if and only if $x_D = d_2$ and $x''_{d_2} P_{d_2} x$. Define $P^2 = (P'_{d_2}, P^1_{-d_2})$. By the same reason as above, $COP(P^2) = X''$ and $\psi_{d_2}(P^2) = \emptyset$. Moreover, by the definition of P^2 and the fact that COP is the doctor-optimal stable mechanism, either $\psi_{d_1}(P^2) = \emptyset$ or $\psi_{d_1}(P^2) = x''_{d_1}$.

Let us now define $S^2 = \{d \in D: \ X'' P_d \psi(P^2)\} \setminus \{d_1, d_2\}$. Similar to above, we claim that S^2 is non-empty. Assume for a contradiction that it is empty. This means that for any doctor $d \in D \setminus \{d_1, d_2\}$, $x''_d = \psi_d(P^2)$. On the other hand, we have $\psi_{d_2}(P^2) = \emptyset$. In either case of $\psi_{d_1}(P^2) = \emptyset$ or $\psi_{d_1}(P^2) = x''_{d_1}$, doctor d_2 would block $\psi(P^2)$ with his assigned hospital at X'' through signing contract x''_{d_2}, contradicting the stability of $\psi(P^2)$.[14] Hence,

[14] If $\psi_{d_1}(P^2) = x''_{d_1}$, then it is direct to see that blocking. On the other hand, if $\psi_{d_1}(P^2) = \emptyset$, then let $(x''_{d_2})_H = h$. If $(x''_{d_1})_H \neq h$, then again it is straightforward to see the blocking. Suppose that $(x''_{d_1})_H = h$. Then, by the stability of X'', $x''_{d_2} \in C_h(X'')$. By US, it also has to be that $x''_{d_2} \in C_h(X'' \setminus \{x''_{d_1}\})$. Hence, the blocking is present in this case as well.

S^2 is non-empty.

If we keep applying the same arguments above, each iteration would give us a different doctor. This, however, contradicts the finiteness of the doctor set, contradicting our starting supposition that $\psi \neq COP$. This finishes the proof.

□

Proof of Theorem 2. **"If" Part.** From Hatfield and Kojima (2010) and Aygün and Sönmez (2012), we know that under US and the IRC, no previously rejected contract is accepted in a later step in the COP. This easily implies that the COP is invariant to lower tail preferences change.

"Only If" Part. We now claim that stability and invariance to lower tail preferences change together implies truncation-proofness.[15] Let ψ be a stable mechanism which is invariant to lower tail preferences change. Assume for a contradiction that it is not truncation-proof. That is, there exist a problem instance P, doctor d, and a truncation P'_d of P_d such that $\psi_d(P) = x$ and $\psi_d(P'_d, P_{-d}) = x'$ with $x' P_d x$. Due to the stability of ψ and the manipulation via truncation P'_d, there exists a contract $x'' \neq x'$ with $x''_D = d$ and $x' P_d x'' P_d \emptyset$ (it may be that $x = x''$).

Let P''_d be the truncation such that $\tilde{x} \notin Ac(P''_d)$ if and only if $\tilde{x}_D = d$ and $x' P_d \tilde{x}$. Then, due to the invariance to lower tail preferences change property of ψ, we have $\psi_d(P''_d, P_{-d}) = x'$. This, along with $\psi_d(P) = x$, contradicts ψ being invariant to lower tail preferences change.

The above observation, along with Theorem 1, finishes the proof.

□

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[15]From Remark 1, we know that invariance to lower tail preferences change alone does not imply truncation-proofness.

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