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Stable schedule matching under revealed preference

Ahmet Alkan[a,*] and David Gale[b]

[a] *Sabanci University, Tuzla, Istanbul 81474, Turkey*
[b] *University of California, Berkeley, CA 94720, USA*

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Abstract

Baiou and Balinski (Math. Oper. Res., 27 (2002) 485) studied *schedule matching* where one determines the partnerships that form and how much time they spend together, under the assumption that each agent has a ranking on all potential partners. Here we study schedule matching under more general preferences that extend the *substitutable* preferences in Roth (Econometrica 52 (1984) 47) by an extension of the *revealed* preference approach in Alkan (Econom. Theory 19 (2002) 737). We give a generalization of the Gale–Shapley algorithm and show that some familiar properties of ordinary stable matchings continue to hold. Our main result is that, when preferences satisfy an additional property called size monotonicity, stable matchings are a lattice under the joint preferences of all agents on each side and have other interesting structural properties.

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1. Introduction

The formulation of the *Stable Matching Problem* [11] was originally motivated by the real world problem of college admissions. It was an attempt to find a rational criterion for matching students with colleges which respected the preferences of both groups. The original approach was to first consider a special case, the so-called

*Corresponding author.
E-mail addresses: alkan@sabanciuniv.edu (A. Alkan), gale@math.berkeley.edu (D. Gale).

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Stable Marriage Problem in which each college could accept only one student. The general case was then reduced to the marriage case by assuming that each college had a complete preference ordering on the set of students it was willing to admit as well as a quota giving an upper bound to the number of students that could be admitted. The model has applications in other situations. A particularly natural application is the problem of hiring of workers by firms. In general we refer to such a model as a *market* and the participants on the two sides as *agents*.

The present paper presents a broad generalization of the original model incorporating extensions in several directions.

(1) The market is *symmetric* in the sense that all agents may form multiple partnerships (with agents on the other side of the market.)

(2) Preferences of agents over sets of possible partners are given by choice functions that are more general than those given by complete orderings of individuals. This is especially relevant for the college market where colleges are typically interested in the overall composition of an entering class, particularly these days as regards diversity.[1] A simple example will illustrate the point.

College A can admit two students. The applicants are two men m and m' and two women w and w'.

A's first choice is the pair mw but if m (w) is not available the choice is $m'w(mw')$. One sees at once that these choices are not possible from any strict ordering of the students. For example if the ordering was $m > w > m' > w'$ then it would mean that mm' was preferred to the diversified pair $m'w'$.

Indeed, as regards diversity, in the algorithm which solves the original college admissions problem, there is nothing to prevent a college from ending up with a class which is either ninety percent male or female.

The remedy for this via choice functions simply formalizes what happens approximately in actual negotiations between colleges and students or firms and workers. Each agent is assumed to have a *choice function* C which, given a set P of agents on the other side of the market, picks out the most preferred subset $S = C(P)$ contained in P. S is then said to be *revealed preferred* to all other subsets of P. The case where colleges rank-order applicants is then a special case in which $C(P)$ consists of the q highest ranked applicants in P, but if, for example, the goal was gender balance one could choose, roughly, the highest ranked $q/2$ applicants of each sex or if, say there was an insufficient number of male applicants then choose all the men and fill the quota with the highest ranked women.

Choice functions have been a standard tool in the matching literature since Roth [15] which followed the seminal work of Kelso and Crawford [13] in broadening the matching model and allowing more general preferences. (In fact the symmetric multiple partnership model goes back to [15]). The revealed preference ordering was

[1] We quote Mr. Bollinger, the president of the University of Michigan at Ann Arbor: "Admissions is not and should not be a linear process of lining up applicants according to their grades and test scores and then drawing a line through the list. It shows the importance of seeing racial and ethnic diversity in a broader context of diversity, which is geographic and international and socio-economic and athletic and all the various forms of differences, complementary differences, that we draw on to compose classes year after year."

first utilized by Blair [7] under somewhat different terminology, and the approach was further developed by Alkan [1,2] which we adopt and extend here.

It is worth pointing out that we do not assume as Roth [15] and Blair [7] do that agents have a complete ordering of subsets of agents on the other side of the market. In our approach there is only a *partial* ordering on subsets. In the original college admissions model, for example, if a college with quota 2 ranks students $a > b > c$ then by revealed preference the pair *ab* is preferred to *bc* and *ac*, since given the triple *abc* the pair *ab* is chosen, but the pairs *ac* and *bc* remain incomparable. Indeed, however, for any of the conclusions reached in this paper, it does not matter whether *ac* is preferred to *bc* or vice versa. Thus it is unnecessary to make assumptions about whether, for instance, a firm would rather hire its first, fifth and sixth best worker or its second, third and fourth. The (incomplete) revealed preference ordering turns out to contain all the relevant information.

(3) Recently Baiou and Balinski [4] have generalized the notion of matching to that of a *schedule* matching. In the context of a set of workers W with members w and a set of firms F with members f, the idea is that a firm decides not only which workers it will hire but also how many hours of employment to give each of them. Similarly, the workers must decide how many of their available hours to allocate to each job. A schedule is then a $F \times W$ matrix X whose entries $x(fw)$ give the amount of time worker w works for firm f. The schedule matching is said to be (pairwise) *stable* if there is no pair f and w who could make themselves better off by increasing the hours they work together while not increasing (possibly decreasing) the hours they work with their other partners.[2] This is the natural generalization of (pairwise) stability for ordinary matchings which, in fact, correspond to the special case of schedule matchings where all entries of X are either 0 or 1. In [4] it is assumed that we are in the "classical" case where each agent has a strict ordering of the agents on the other side of the market and preferences on schedules are given by the condition that an agent, say a worker w, is made better off if he can increase the time he works for firm f by reducing the time he works for some less preferred firm f.

The present paper studies schedule matching under more general (revealed) preferences. Our main result shows that under appropriate conditions which include the classical case the set of stable matchings forms a distributive lattice with other interesting structural properties. (For example, a worker may have different schedules under two different stable matchings but he will necessarily work the same number of hours in each.) This extends the results of Alkan [2] for the case of ordinary matchings and some of the arguments below are natural extensions of those in [2].

In the next section we develop the necessary material on the revealed preference ordering of an individual and show that if the choice function is *consistent* and *persistent* (to be defined) then the set of all acceptable schedules has the structure of a (non-distributive) lattice with other important properties. These properties are then used in the following section first to prove that stable schedules always exist (the

[2] We restrict attention to *pairwise* stable matchings because if coalitions other than pairs can form to block a matching, then stable matchings may fail to exist. See [19].

proof uses an extension to schedules of the standard Gale–Shapley algorithm.) We next show how to extend to the case of schedules some basic properties of ordinary stable matchings. The lattice properties of the set of stable matchings are derived in the two subsections that follow. A final section gives examples showing the necessity of the various assumptions on individual preferences and pointing out some properties of matchings which do not generalize to the case of schedules.

It should be mentioned that there is a well-known parallel matching literature of *buyers-and-sellers* where prices or salaries appear explicitly and one looks at the competitive equilibrium allocations. This literature originating from [18] has recently been expanding in remarkable ways (see [3,5,9,12]). Some of our results have their analogues in these works. For comparison it is worth mentioning that the key condition behind those results is the *gross substitutability* condition on *demand correspondences* that was introduced into the matching literature by Kelso and Crawford [13]. The corresponding property for ordinary matchings (under the assumption that preferences are *strict*) has been called *substitutability* by Roth [15] and our key assumption of persistence is simply the generalization of this property to the case of schedules.

2. The individual

In the matching theory of later section we will think of economic agents as firms and workers, or students and colleges, men and women, etc. However, the theory of revealed preference of the individual belongs to the general standard model of consumption or demand theory, and it will be presented in this context here.

An *agent* (consumer) chooses (demands) amounts of *n items* (goods) from given availabilities of each item. This is formalized as follows:

Let R^n_+ be the nonnegative orthant, b an *upper bound* vector and $\mathrm{B} = \{x \in R^n_+ | x \leqslant b\}$. Let $\mathscr{B}$ be a subset of B which is closed under $\vee$ and $\wedge$ (the standard join and meet in R^n). A *choice function* is a map $C : \mathscr{B} \to \mathscr{B}$ such that

$$C(x) \leqslant x$$

for all $x \in \mathscr{B}$. The elements $x = (x(1), \ldots, x(n))$ of the *domain* $\mathscr{B}$ will be called *choice vectors*. The range of C is denoted by $\mathscr{A}$ and its elements are called (acceptable) *schedules*. The most relevant domains for our purposes are the *divisible* domain B itself and the *discrete* domain that consists of all the integer vectors in B. When all bounds are equal to 1, the discrete domain corresponds to the case of ordinary multipartner matching as in college admissions.

An important special case of our model is one in which the items can be measured in some common unit, for example, dollars worth for goods, or man-hours for services. In this case we denote the sum of the entries of a vector x by $|x|$ and call it the *size* of x. In such a model an agent may have a *quota* q which bounds the size of the schedule he can choose. For the college admissions case q is the maximum number of students a college can admit.

A choice function C is called *quota filling* if

$$|C(x)| = q \text{ if } |x| \geqslant q \text{ and } C(x) = x \text{ otherwise.}$$

Two interesting examples of quota filling choice functions are as follows:

Example 1. The items are ranked so that, say, item i is more desirable than $i+1$. Given a choice vector x with $|x| > q$, let j be the item such that $r = \sum^{j} x(i) \leqslant q$ and $r + x(j+1) > q$. Then

$$C(x) = (x(1), \ldots, x(j), q - r, 0, \ldots, 0).$$

Thus, the agent fills as much of his quota as possible with the most desirable items. We will henceforth refer to this C as the *classical* choice function.

Example 2. The domain is B. Given a choice vector x with $|x| > q$, let r be the number such that $\sum_i r \wedge x(i) = q$. Then

$$C(x) = (r \wedge x(1), \ldots, r \wedge x(n)).$$

In words, the agent tries to use all items as equally as possible. (On the discrete domain, there may be more than one such best-schedule hence a tie-breaking criterion is necessary.) We will refer to C as the *diversifying* choice function.

As an illustration, suppose an agent with quota 5 is given the choice vector (2,1,0,4,2). Then, the classical choice function chooses the schedule (2,1,0,2,0) while the diversifying choice function chooses $(4/3, 1, 0, 4/3, 4/3)$.

2.1. The revealed preference lattice

Definition. We say that $x \in \mathscr{A}$ is *revealed preferred* to $y \in \mathscr{A}$, and write $x \succcurlyeq y$, if $C(x \vee y) = x$. We write $x \succ y$ if $x \succcurlyeq y$ and $x \neq y$.

We now impose some standard conditions on the choice function C.

Definition. C is *consistent* if $C(x) \leqslant y \leqslant x$ implies $C(y) = C(x)$.

This is a highly plausible assumption. Applied to college admissions, it says that if some set S of students is chosen for admission from a pool P then the same set will be chosen from any subset of P which contains S.

An immediate consequence of consistency is that $C(x) = x$ if and only if $x \in \mathscr{A}$. Without some further restrictions, revealed preference will not be transitive, hence not a partial ordering, as shown by the following example for the college admissions case.

Example 3. A college can admit two students from two men m, m' and two women w, w'. The pair mw is its first choice, but if either w or m are not available then

(i) $C(mm'w') = mw'$,

(ii) $C(m'ww') = m'w'$.

(In the case of college admissions, we will use the customary notation and represent a choice vector or schedule x by the set of all students s for whom $x(s) = 1$.) Transitivity fails because from (i) we have $mw' \succ m'w'$ and from (ii) $m'w' \succ m'w$ but mw' and $m'w$ are not comparable since $C(mw'm'w) = mw$.

In fact a consistent choice function may exhibit Condorcet type cycles even if it enjoys the quota filling property:

Example 4. A firm with quota 3 may face any subset of 5 workers a, b, c, d, e. Worker b is productive only with a so if a is not available b will not be chosen. Likewise for c and b, respectively, and for a and c. Thus $C(abcde) = C(abcd) = C(abce) = abc$, and $C(bcde) = cde, C(acde) = ade, C(abde) = bde$, so

$$cde \succ ade \succ bde \succ cde.$$

To avoid these situations, we introduce the following condition of *persistence* which, as mentioned earlier, is a generalization of the condition of *substitutability* that has widely been used in ordinary matching models since Roth [15].

Definition. C is *persistent* if $x \geqslant y$ implies $C(y) \geqslant C(x) \wedge y$.

For the college admissions problem, persistence (substitutability) means that if a college offers admission to a student from a given pool of applicants then it will also admit him if the pool of applicants is reduced. This is violated by the choice function in Example 3: the agent likes the couple mw most but prefers $m'w'$ to any other couple if m is not available. In general, persistence rules out the sort of complementarity exhibited here between m' and w'.

It is easy to verify that the classical and diversifying choice functions satisfy consistency and persistence.

An immediate consequence of persistence is that if $x \in \mathscr{A}$ and $x \geqslant y$ then $y \in \mathscr{A}$.

Definition. C is *subadditive* if $C(x \vee y) \leqslant C(x) \vee y$ for all x, y.

Lemma 1. *If C is persistent then it is subadditive.*

Proof. Since $C(x \vee y) \leqslant x \vee y$, we have

$$C(x \vee y) = C(x \vee y) \wedge (x \vee y) = (C(x \vee y) \wedge x) \vee (C(x \vee y) \wedge y) \tag{1}$$

by distributivity. Since $x \leqslant x \vee y$, we have $C(x \vee y) \wedge x \leqslant C(x)$ by persistence. Also $C(x \vee y) \wedge y \leqslant y$. Substituting these two inequalities in (1) gives subadditivity. □

Definition. C is *stationary* if $C(x \vee y) = C(C(x) \vee y)$ for all x, y.

Lemma 2. *If C is subadditive and consistent then it is stationary.*

Proof. By subadditivity $C(x \vee y) \leqslant C(x) \vee y$. Also $C(x) \vee y \leqslant x \vee y$. So $C(C(x) \vee y) = C(x \vee y)$ by consistency. □

In the case of ordinary matching the condition of stationarity has been called *path independence* as in [14] where it was introduced in a somewhat different setup.

It will be assumed from here on that all choice functions are consistent and persistent.

Notation. We write $x \curlyvee y$ for $C(x \vee y)$.

As immediate consequence of stationarity, we have

Corollary 1. *The relation* $\succcurlyeq$ *is transitive and* $x \curlyvee y$ *is the least upper bound of x and y.*

Proof. The operation $\curlyvee$ is associative: $(x \curlyvee y) \curlyvee z = C(C(x \vee y) \vee z) = C((x \vee y) \vee z) = C(x \vee (y \vee z)) = C(x \vee C(y \vee z)) = x \curlyvee (y \curlyvee z)$. Thus, if $x \succcurlyeq y, y \succcurlyeq z$ then $x \curlyvee z = (x \curlyvee y) \curlyvee z = x \curlyvee (y \curlyvee z) = x \curlyvee y = x$ so $x \succcurlyeq z$. Also, if $z \succcurlyeq x, z \succcurlyeq y$ then $z \curlyvee (x \curlyvee y) = (z \curlyvee x) \curlyvee y = z \curlyvee y = z$ so $z \succcurlyeq x \curlyvee y$. □

Thus, the set of schedules $\mathscr{A}$ is an upper-semilattice (with join $\curlyvee$) in the partial order given by $\succcurlyeq$. It is, in fact, a lattice and we will need an expression for its meet $\curlywedge$. First note, it follows at once from stationarity that if $C(x) = z$ and $C(y) = z$ then $C(x \vee y) = z$.

Definition. The *closure* $\bar{x} \in \mathscr{B}$ of $x \in \mathscr{A}$ is $\sup\{y \in \mathscr{B} | C(y) = x\}$.

In the classical college admissions case, $\bar{x}$ consists of x together with all students ranked below the least desired student in x.

We henceforth assume that C is continuous. It then follows that $C(\bar{x}) = x$.

Lemma 3. *The revealed preference meet is given by* $x \curlywedge y = C(\bar{x} \wedge \bar{y})$.

Proof. We must show (i) $C(\bar{x} \wedge \bar{y}) \preccurlyeq x$ (and $C(\bar{x} \wedge \bar{y}) \preccurlyeq y$) and (ii) $z \preccurlyeq x$ and $z \preccurlyeq y$ implies $z \preccurlyeq C(\bar{x} \wedge \bar{y})$.

By definition (i) is true if and only if $C(C(\bar{x} \wedge \bar{y}) \vee x) = x$. By stationarity this is equivalent to $C(x') = x$ where $x' = (\bar{x} \wedge \bar{y}) \vee x$ and, since $x \leqslant x' \leqslant \bar{x}$ and $C(\bar{x}) = x$, the result follows by consistency.

To prove (ii) we must show that $C(C(\bar{x} \wedge \bar{y}) \vee z) = C((\bar{x} \wedge \bar{y}) \vee z)$ (by stationarity) $= C(\bar{x} \wedge \bar{y})$, so note that $z \preccurlyeq x$ means $C(x \vee z) = x$, hence by definition of closure $x \vee z \leqslant \bar{x}$, so $z \leqslant \bar{x}$ and similarly $z \leqslant \bar{y}$ so $z \leqslant \bar{x} \wedge \bar{y}$ so $(\bar{x} \wedge \bar{y}) \vee z = \bar{x} \wedge \bar{y}$ and the result follows. □

Note that in college admissions, $x \curlywedge y$ may include students who are neither in x nor y: Suppose there are four students $1,2,3,4$ ranked in that order, and $x = \{1,3\}, y = \{2,3\}$. Then $x \curlywedge y = \{3,4\}$.

We will need some further properties of the revealed preference lattice.

Lemma 4. *$x \curlywedge y \geqslant x \wedge \bar{y}$.*

Proof. Since $\bar{x} \geqslant \bar{x} \wedge \bar{y}$, we have from persistence $x \curlywedge y = C(\bar{x} \wedge \bar{y}) \geqslant C(\bar{x}) \wedge \bar{x} \wedge \bar{y} = x \wedge \bar{x} \wedge \bar{y} = x \wedge \bar{y}$. □

Lemma 5. *$(x \curlywedge y) \wedge (x \curlyvee y) \leqslant x \wedge y$.*

Proof. Since $\bar{x} \vee \bar{y} \geqslant \bar{x}$, we have from persistence $C(\bar{x}) = x \geqslant C(\bar{x} \vee \bar{y}) \wedge \bar{x} = (x \curlyvee y) \wedge \bar{x}$ from stationarity, and similarly $y \geqslant (x \curlyvee y) \wedge \bar{y}$, so $x \wedge y \geqslant (x \curlyvee y) \wedge (\bar{x} \wedge \bar{y}) \geqslant (x \curlyvee y) \wedge C(\bar{x} \wedge \bar{y}) = (x \curlyvee y) \wedge (x \curlywedge y)$ from Lemma 3. □

2.2. Satiation

In extending the concept of stability from ordinary to schedule matchings in the next section we need to formalize the notion that an agent would not prefer to have more of a given item if it were available. For this purpose the following definition is basic.

Definition. A schedule x is *i-satiated* if $C_i(y) \leqslant x(i)$ for all $y \geqslant x$.

In words, x is i-satiated if the agent would not choose more of item i if it were offered with no reduction in the availability of other items. To illustrate, in the classical case, x is i-satiated if i is the highest ranked item with $x(j) = 0$ for $j > i$. For the diversifying choice function, x is i-satiated if $x(i) = \max_j \{x(j)\}$.

The following properties will be needed in the next section.

Lemma 6. *x is i-satiated if there exists $y \geqslant x$, $y(i) > x(i)$ such that $C(y) = x$.*

Proof. Suppose $z \geqslant x$ and $z(i) > x(i)$ (otherwise there is nothing to prove). Let $y' = z \wedge y$ and note that $y'(i) > x(i)$. Now $y \geqslant y' \geqslant x$ so by consistency $C(y') = C(y) = x$. Also $z \geqslant y'$ so by persistence $x \geqslant C(y') \geqslant C(z) \wedge y'$ in particular $x(i) \geqslant C_i(z) \wedge y'(i)$ but since $y'(i) > x(i)$ we have $C_i(z) \leqslant x(i)$. □

Lemma 7. *x is i-satiated if and only if $\bar{x}(i) = b(i)$.*

Proof. If $x(i) = b(i)$ there is nothing to prove so suppose $x(i) < b(i)$. If $\bar{x}(i) = b(i)$ then x is i-satiated by the previous lemma. If x is i-satiated then let $y = x \vee b^i$ where

b^i is the vector with ith entry $b(i)$ and others 0. Then from satiation $C_i(y) \leqslant x(i)$ and since $C_j(y) \leqslant x(j)$ for $j \neq i$ we have $C(y) \leqslant x \leqslant y$ so by consistency $C(y) = C(x) = x$ so $y \leqslant \bar{x}$ so $\bar{x}(i) = b(i)$. □

Lemma 8. *Suppose $x \succcurlyeq y$.*

(i) *If y is i-satiated then x is i-satiated.*
(ii) *If $x(i) > y(i)$ then y is not i-satiated.*

Proof. (i) Using stationarity and the assumption that $x \succcurlyeq y$, we get $C(\bar{x} \vee \bar{y}) = C(x \vee y) = x$. So by definition of closure $\bar{x} \geqslant \bar{x} \vee \bar{y}$ thus $\bar{x} \geqslant \bar{y}$ in particular $\bar{x}(i) \geqslant \bar{y}(i) = b(i)$ so x is i-satiated by the previous lemma. (ii) Since $x \succcurlyeq y$ we have $x \vee y \geqslant y$ and $C(x \vee y) = x$ so $C_i(x \vee y) = x(i) > y(i)$ so y is not i-satiated. □

Lemma 9. (i) *If x or y is i-satiated then $x \curlyvee y$ is i-satiated.*
(ii) *If x and y are i-satiated then $x \curlywedge y$ is i-satiated.*

Proof. (i) Say x is i-satiated. Then since $x \curlyvee y \succcurlyeq x$ the conclusion follows from Lemma 8(i). (ii) We have $C((x \curlywedge y) \vee b^i) = C(C(\bar{x} \wedge \bar{y}) \vee b^i) = C((\bar{x} \wedge \bar{y}) \vee b^i)$ (by stationarity) $= C((\bar{x} \vee b^i) \wedge (\bar{y} \vee b^i)) = C(\bar{x} \wedge \bar{y})$ (using Lemma 7, since x and y are i-satiated) $= x \curlywedge y$. □

3. Stable matchings

We now consider two finite sets of agents which we interpret as *firms*, F, with members f, and *workers*, W, with members w, having respectively the choice functions C_f, C_w, with ranges $\mathscr{A}_f, \mathscr{A}_w$. We write $\curlyvee_f$, $\curlywedge_f$, $\succcurlyeq_f$ for the join, meet, preference ordering for f, and similarly for w.

A *matching* X is a nonnegative $F \times W$ matrix whose entries, written $x(fw)$, represent the amount of time w works for f. We write $x(f)$ for the f-row and $x(w)$ for the w-column of X. We assume all matchings X are bounded above by some positive matrix B. The choice functions C_F, C_W are defined from C_f, C_w in the natural way.

The revealed preference ordering for agents translates in an obvious way to an ordering on matchings.

Definition (Group preference). The matching X is *preferred* to Y *by* F, written $X \succcurlyeq_F Y$, if $x(f) \succcurlyeq_f y(f)$ for all f in F.

Definition (Acceptability). A matching X is F-*acceptable* if $x(f) \in \mathscr{A}_f$ for all f, and it is W-*acceptable* if $x(w) \in \mathscr{A}_w$ for all w. It is *acceptable* if it is both F and W-acceptable.

The fundamental stability notion is now formalized as follows:

Definition (Stability). An acceptable matching X is *stable* if, for every pair fw, either $x(f)$ is w-satiated or $x(w)$ is f-satiated (or both).

It is straightforward to check that, under persistence, the above definition is precisely the condition that there exists no ‘‘blocking’’ pair.

3.1. Existence

We will show that stable matchings always exist by constructing a sequence of alternately F- and W-acceptable matchings which converge to a stable matching. The method is a natural generalization of the Gale–Shapley algorithm of offers and counteroffers where choice functions are particularly natural. The starting choice vector for each firm f is b_f, namely the vector giving the maximum hours each worker can work with f, and the firms offer the employment vectors $C_f(b_f)$. These employment offers then become the choice vectors for the workers who accept or reject them using their own choice functions and, in turn, the ‘‘counter’’ offers so chosen by the workers determine (in a natural way formalized in the proof below) the new choice vectors for the firms, and so on. Of course the proof must make use of persistence of all firms’ and workers’ choice functions since counterexamples exist if this condition is not satisfied (see Section 4). One difference from the discrete case is the fact that the sequence of acceptable matchings need not terminate after a finite number of iterations and therefore it may be necessary to take the limit of the sequence in order to determine the stable matching.

Theorem 1 (Existence). *There exists a stable matching.*

Proof. Define the sequences $(B^k), (X^k), (Y^k)$ by the following *recursion rule*:

$$B^0 = B,$$

$$X^k = C_F(B^k),$$

$$Y^k = C_W(X^k),$$

and B^{k+1} is obtained from B^k as follows:

$$b^{k+1}(fw) = b^k(fw) \quad \text{if } y^k(fw) = x^k(fw),$$

$$b^{k+1}(fw) = y^k(fw) \quad \text{if } y^k(fw) < x^k(fw).$$

In words: the matrices B^k are the choice matrices for the firms; X^k are the firms’ offers and act as workers’ choice matrices, Y^k are the workers’ counter offers. The recursion follows the rule that (i) if worker w has fully accepted the offer by firm f then f can make any offer to w that it could in the previous round and (ii) if w has not fully accepted the offer by firm f then f cannot offer more hours than those counteroffered by w.

Note that (B^k) is a nonincreasing nonnegative sequence and hence converges, so by continuity of C_F it follows that (X^k) converges, and hence by continuity of C_W it follows that (Y^k) converges. Call the limits $\widehat{B}, \widehat{X}, \widehat{Y}$. We will show,

(i) $\widehat{X} = \widehat{Y}$ and hence it is acceptable,
(ii) $\widehat{X}(= \widehat{Y})$ is stable.

To prove (i), note that $Y^k \leqslant X^k \leqslant B^k$. If, for some fw, $\hat{x}(fw) - \hat{y}(fw) > \varepsilon$, then $x^k(fw) - y^k(fw) > \varepsilon$ for infinitely many k and therefore from the recursion rule $b^k(fw) - b^{k+1}(fw) > \varepsilon$ which is impossible since B^k converges so $\widehat{X} = \widehat{Y}$. (In the special case where X^n is W-acceptable for some n, $Y^n = X^n$ so $B^{n+1} = B^n$ so $X^{n+1} = X^n$ so $\widehat{X} = \widehat{Y} = X^n$.)

To prove (ii), we first show that $Y^{k+1} \succsim_W Y^k$, thus workers are "better off" after each step of the recursion. From the recursion rule $Y^k \leqslant B^{k+1} \leqslant B^k$, so from persistence we have

$$C_F(B^{k+1}) = X^{k+1} \geqslant C_F(B^k) \wedge B^{k+1} \geqslant X^k \wedge Y^k = Y^k,$$

so $Y^{k+1} = C_W(X^{k+1})$ is revealed preferred to Y^k. It follows by continuity that

$$\widehat{Y} \succsim_W Y^k. \tag{2}$$

Now suppose $\hat{y}(f)$ is not w-satiated. Then from Lemma 7 $\hat{y}(fw) < b(fw)$ so from the recursion rule, for some k, $y^k(fw) < x^k(fw)$ so, since $y^k(w) = C_w(x^k(w)) \leqslant x^k(w)$, from Lemma 6 we have $y^k(w)$ is f-satiated and from (2) $\hat{y}(w) \succsim_w y^k(w)$, so from Lemma 8(i) $\hat{y}(w)$ is f-satiated. This proves stability of $\widehat{Y}$. □

3.2. Polarity, optimality, comparative statics

The following are extensions of familiar properties of the ordinary matching market (see [7,8,17]).

Lemma 10. *Let X be a stable matching and let Y be an F-acceptable matching such that $Y \succsim_F X$. Then $C_W(X \vee Y) = X$.*

Proof. If the conclusion is false, then there is some w such that $C_w(x(w) \vee y(w)) = z(w) \neq x(w)$. Hence, $z(w) \succ_w x(w)$, so $z(fw) > x(fw)$ for some f, hence from Lemma 8(ii) $x(w)$ is not f-satiated, but $z(fw) \leqslant y(fw)$ so $x(fw) < y(fw)$ and by hypothesis $y(f) \succsim_f x(f)$ so again by Lemma 8(ii) $x(f)$ is not w-satiated, contradicting stability of X. □

Corollary 2 (Polarity). *If X, Y are stable matchings then $X \succsim_F Y \Leftrightarrow Y \succsim_W X$.*

Theorem 2 (Optimality). *If $\widehat{X}$ is the matching given by the Existence Theorem and X is any other stable matching then $\widehat{X} \succcurlyeq_F X$.*

Proof. Let X be a stable matching. We will show that $X \leqslant \widehat{B}$ where $\widehat{B}$ is the matrix given in the Existence Theorem. Since $\widehat{X} = C_F(\widehat{B})$, the conclusion follows. So suppose not. Then there is an index k such that $B^k \geqslant X^k$ but $b^{k+1}(fw) < x(fw)$ for some fw. From the recursion rules, this means that

$$y^k(fw) < x(fw) \quad \text{and} \quad y^k(fw) < x^k(fw). \tag{3}$$

Now $X^k = C_F(B^k)$ so $X^k \succcurlyeq_F X$ so from Lemma 10 $C_W(X \vee X^k) = X$. But $X \vee X^k \geqslant X^k$ so from persistence $Y^k = C_W(X^k) \geqslant X^k \wedge C(X \vee X^k) = X^k \vee X$ so $y^k(fw) \geqslant x^k(fw) \wedge x(fw)$ for all fw which contradicts (3). □

Suppose a new firm or a new worker enters the market. The following theorem shows that, in the firm-optimal matching, in the first case no firm is better off and no worker worse off, while in the second case no worker is better off and no firm worse off. Formally, let $\widehat{X}$ be the firm-optimal matching in the original $F \times W$ market, and let $\widehat{X^\phi}$ (respectively $\widehat{X^\omega}$) denote the $F \times W$ component of the firm-optimal matching in the market with an additional firm ϕ (worker ω.)

Theorem 3 (Comparative statics). (i) $\widehat{X} \succcurlyeq_F \widehat{X^\phi}$ *and* $\widehat{X^\phi} \succcurlyeq_W \widehat{X}$, (ii) $\widehat{X^\omega} \succcurlyeq_F \widehat{X}$ and $\widehat{X} \succcurlyeq_W \widehat{X^\omega}$.

Proof. To prove (i), we continue the algorithm of the Existence Theorem. The new firm ϕ offers an employment schedule x_ϕ which gives a new offer schedule X' to W where $X' \geqslant X$ and since workers get no worse off with each step of the recursion they are at least well off under $\widehat{X^\phi}$ as under $\widehat{X}$. The firms are no better off since their choice matrix can never exceed $\widehat{B}$.

To prove (ii), we suppose the original market includes ω but $b_\omega = 0$. We denote by $(B^k), (X^k), (Y^k)$ and $(B'^k), (X'^k), (Y'^k)$, respectively, the sequences in the Existence Theorem recursion for the original and new market. Note $B' \geqslant B$. It suffices to show that $B'^k \geqslant B^k$ and $x'^k(w) \leqslant x^k(w)$ for all k and $w \neq w'$. Assume this is true up to k. Since $B'^k \geqslant B^k$, we have by persistence $X^k = C_F(B^k) \geqslant C_F(B'^k) \wedge B^k = X'^k \wedge B^k$ so $x^k(fw) \geqslant x'^k(fw) \wedge b^k(fw)$ but for $w \neq w'$ we have $b^k(fw) = b'^k(fw)$, hence $x^k(w) \geqslant x'^k(w)$. This shows that no W-worker is better off in the new market.

To show that no firm is worse off, we show that $B'^k \geqslant B^k$ for all k. Since $x^k(w) \geqslant x'^k(w)$ for $w \neq w'$, we have by persistence $y'^k(w) \geqslant y^k(w) \wedge x'^k(w)$. There are two cases: If $x'^k(fw) \leqslant y^k(fw)$ then $y'^k(fw) = x'^k(fw)$ so from the recursion rule $b'^{k+1} = b'^k \geqslant b^k \geqslant b^{k+1}$. If on the other hand $x^k(fw) \geqslant x'^k(fw) > y^k(fw)$ then by consistency $y'^k(fw) = y^k(fw)$ so from the recursion rule $b^{k+1}(fw) = y^k(fw) = y'^k(fw) \leqslant b'^{k+1}(fw)$, completing the proof. □

3.3. The stable matching lattice

Let X, X' be an arbitrary pair of matchings fixed throughout this section. We write $X^F = X \curlyvee_F X'$ for the matching whose f-row, $x^F(f)$, is $x(f) \curlyvee_f x'(f)$ and write $X_F = X \curlywedge_F X'$ for the matching whose f-row, $x_F(f)$, is $x(f) \curlywedge_f x'(f)$. We define X^W, X_W via w-columns similarly.

Note that if X, X' are acceptable then X^F is of course F-acceptable but not in general W-acceptable.

The following is a key result.

Lemma 11. *If X and X' are stable matchings then $X^F \leqslant X_W$.*

Proof. We must show that $x^F(fw) \leqslant x_W(fw)$ for all fw:

Case (i) $x^F(fw) \leqslant x(fw) \wedge x'(fw)$. Then, since by Lemma 5 $x(w) \wedge x'(w) \leqslant x(w) \curlywedge_w x'(w) = x_W(w)$, the conclusion follows.

Case (ii) $x(fw) < x^F(fw) \leqslant x'(fw)$. Then, since $x^F(f) \succcurlyeq_f x(f)$, we have by Lemma 9(ii), $x(f)$ is not w-satiated, so by stability $x(w)$ is f-satiated, so from Lemma 8 $\overline{x(w)}(f) = b(fw)$, so $x^F(fw) \leqslant \overline{x(w)}(f) \wedge x'(fw) = (\overline{x(w)} \wedge x'(w))(f) \leqslant (x(w) \curlywedge_w x'(w))(f)$ (again from Lemma 4) $= x_W(fw)$. □

In order to make the above inequality to an equation, it is necessary to make some further assumption. We will assume that the entries of a schedule are measured in some common unit so that it makes sense to add them up. The following condition extends the condition of "cardinal monotonicity" introduced by Alkan [2] (and by Fleiner [10] also independently.)

Definition. The choice function C is *size monotone* if $x \leqslant y$ implies $|C(x)| \leqslant |C(y)|$ for all x, y in $\mathscr{A}$.

Remark. Note that size monotonicity implies that if $x \succcurlyeq y$ then $|x| \geqslant |y|$ since $x \vee y \geqslant y$.

The condition means, for example, that if a worker is forced to cut down on the hours allocated to some firm, then he may choose to work longer for other firms, but he will not increase his total working hours. In the ordinary matching model the condition says that if a firm loses the services of one worker it will replace him by at most one worker. Note that if C is quota filling then it is automatically size monotone. From size monotonicity, we get:

Theorem 4 (Lattice polarity). *If all choice functions are size monotone then $X^F = X_W$.*

Proof. First, since for all $w, x^W(w) \succcurlyeq x_W(w)$, it follows from the remark above that $|x_W(w)| \leqslant |x^W(w)|$ so $|X_W| = \sum_w |x_W(w)| \leqslant \sum_w |x^W(w)| = |X^W|$, and similarly

$|X_F| \leqslant |X^F|$. From the previous Lemma $|X^F| \leqslant |X_W|$, so now $|X_F| \leqslant |X^F| \leqslant |X_W| \leqslant |X^W| \leqslant |X_F|$, so $|X^F| = |X_W|$, so the conclusion follows, and also for any agent, say w,

$$|x^W(w)| = |x_W(w)|. \qquad \square \tag{4}$$

Theorem 5. *The set of stable matchings is a lattice under the orderings $\succcurlyeq_F$ and $\succcurlyeq_W$.*

Proof. It suffices to show that X^F is a stable matching. By definition X^F is F-acceptable and, since by Theorem 4 $X^F = X_W$, it follows that X^F is also W-acceptable. It remains to show stability, so suppose $x^F(f)$ is not w-satiated. Then by Lemma 10(i) $x(f), x'(f)$ are not w-satiated. So by stability $x(w), x'(w)$ are f-satiated, so by Lemma 10(ii) $x_W(w)$ is f-satiated, but by Theorem 4 again $x_W(w)$ is the w-column of X^F, so X^F is stable. $\square$

3.4. Properties of the stable matching lattice

The following property, which says $|x(w)| = |x'(w)|$ for all w, generalizes a result for the classical model.

Theorem 6 (Unisize). *The schedules that an agent may have in any stable matching all have the same size.*

Proof. Note $x^W(w) \curlyvee_w x(w) = x^W(w)$ and $x^W(w) \curlywedge_w x(w) = x(w)$ so from (4) $|x^W(w)| = |x(w)|$ and similarly $|x^W(w)| = |x'(w)|$. $\square$

An immediate consequence is the following result which was first shown by Roth and Sotomayor [16] for the classical college admissions model:

Corollary 3. *If the choice function of an agent is quota filling and he does not fill his quota in a stable matching then he has the same schedule in all stable matchings.*

Proof. Suppose $x(f) \neq x'(f)$ and $|x(f)| = |x'(f)| = c < q$. Then $|x(f) \vee x'(f)| > c$, so by quota filling $|x^F(f)| = |x(f) \vee x'(f)| > c$, contradicting Theorem 6. $\square$

A striking structural property of stable matchings is that, for all pairs fw, $\{x^F(fw), x_F(fw)\} = \{x(fw), x'(fw)\}$, stated equivalently in the following form:

Theorem 7 (Complementarity). *If X and X' are stable matchings $X^F \vee X_F = X \vee X'$ and $X^F \wedge X_F = X \wedge X'$.*

Proof. Let f be any firm. First, from Lemma 5 we have

$$x^F(f) \wedge x_F(f) \leqslant x(f) \wedge x'(f). \tag{5}$$

Secondly, for all w, by lattice polarity (Theorem 4) $x_F(fw) = x^W(fw) = (x(w) \curlyvee_w x'(w))(f) \leqslant (x(w) \vee x'(w))(f) = x(fw) \vee x'(fw)$, thus $x_F(f) \leqslant x(f) \vee x'(f)$ so, since $x^F(f) = x(f) \curlyvee_f x'(f) \leqslant x(f) \vee x'(f)$, we have

$$x^F(f) \vee x_F(f) \leqslant x(f) \vee x'(f), \tag{6}$$

so $|x^F(f)| + |x_F(f)| - |x^F(f) \wedge x_F(f)| = |x^F(f) \vee x_F(f)| \leqslant |x(f) \vee x'(f)| = |x(f)| + |x'(f)| - |x(f) \wedge x'(f)|$, but from the unisize property (Theorem 6) $|x^F(f)| = |x_F(f)| = |x(f)| = |x'(f)|$ so

$$|x^F(f) \wedge x_F(f)| \geqslant |x(f) \wedge x'(f)|, \tag{7}$$

therefore (5) and (7) are equations, hence (6) also is an equation. □

Complementarity implies that the lattice of stable matchings is distributive:

Definition. A lattice $\mathscr{L}$, with join $\curlyvee$ and meet $\curlywedge$, is *distributive* if $z \curlyvee (z' \curlywedge z'') = (z \curlyvee z') \curlywedge (z \curlyvee z'')$ and $z \curlywedge (z' \curlyvee z'') = (z \curlywedge z') \curlyvee (z \curlywedge z'')$ for all z, z', z'' in $\mathscr{L}$.

Remark. A standard fact in lattice theory (Corollary to Theorem II.13 in Birkhoff [6]) is that a lattice $(\mathscr{L}, \curlyvee, \curlywedge)$ is distributive if and only if the following *cancellation law* holds:

If $z \curlyvee z' = z \curlyvee z''$ and $z \curlywedge z' = z \curlywedge z''$ then $z' = z''$ for all z, z', z'' in $\mathscr{L}$.

Theorem 8 (Distributivity). *The* $(\curlyvee_F, \curlywedge_F)$ *and* $(\curlyvee_W, \curlywedge_W)$ *lattices of stable matchings are distributive.*

Proof. Let X, X', X'' be any three stable matchings. If $X \curlyvee_F X' = X \curlyvee_F X''$ and $X \curlywedge_F X' = X \curlywedge_F X''$ then $(X \curlyvee_F X') \vee (X \curlywedge_F X') = (X \curlyvee_F X'') \vee (X \curlywedge_F X'')$ and $(X \curlyvee_F X') \wedge (X \curlywedge_F X') = (X \curlyvee_F X'') \wedge (X \curlywedge_F X'')$, hence by complementarity (Theorem 7) $X \vee X' = X \vee X''$ and $X \wedge X' = X \wedge X''$, so by distributivity of $\vee, \wedge$ using cancellation $X' = X''$. Thus the cancellation law holds for $\curlyvee_F, \curlywedge_F$, similarly for $\curlyvee_W, \curlywedge_W$, and the theorem follows from the remark above. □

An important theorem in the classical case asserts that for stable matchings the schedules $x(f)$ and $x'(f)$ are comparable, that is either they are identical or f prefers one to the other. This was proved for college admissions in [16] and for schedules in [4]. This result does not hold in the general case as we show in the next section. However, we will here show that, for classical agents, it is a direct consequence of complementarity and the unisize property:

Corollary 4. *In the classical case let x and y be schedules where $x \succ y$. Then $x(i) > 0$ implies $x(j) \geqslant y(j)$ for $j < i$.*

Proof. If $y(j) > x(j)$ then for some $\varepsilon > 0$ define the schedule $x_\varepsilon \leqslant x \vee y$ by $x_\varepsilon(i) = x(i) - \varepsilon, x_\varepsilon(j) = x(j) + \varepsilon, x_e(k) = x(k)$ otherwise. Then $x_\varepsilon \succ x$ contradicting $C(x \vee y) = x$. □

Theorem 9. *In the classical case if X and X′ are stable matchings then either* $x(f) \succ_f x'(f), x(f) = x'(f)$, *or* $x(f) \prec_f x'(f)$.

Proof. Let $y(f) = x_F(f) = x(f) \curlywedge_f x'(f)$. By the unisize property we cannot have $y(f) < x(f)$ or $y(f) < x'(f)$. Therefore, if $y(f)$ is distinct from $x(f)$ and $x'(f)$ then by complementarity there is a w such that $y(fw) = x(fw) > x'(fw)$ and there is a w' such that $y(fw') = x'(fw'') > x(fw')$. But if, say, w' is preferred by f to w then since $x(f) \succ_f y(f)$ and $x(fw) > 0$ it follows from Corollary 4 that $x(fw') \geqslant y(fw')$, contradiction. □

4. Examples

In this section we will show by examples the need for our various assumptions. All examples are in the context of the special case of college admissions.

Example 4. If choice functions are consistent and size monotone but not persistent then stable matchings may not exist.

There are two colleges A and B and four students m, w, m', w'. College A has quota 2 and the choice function as in Example 3 so that $mw \succ_A mw' \succ_A m'w' \succ_A m'w$. College B has quota 1 and prefers w to m and will not admit m' or w'. Student m prefers B to A while student w prefers A to B. Students m' and w' prefer being matched with A to being unmatched.

For every assignment of students to A there is a blocking pair as stated below:

(A, mw)	is blocked by B and m,
(A, mw')	is blocked by A and w,
$(A, m'w)$	is blocked by A and w',
$(A, m'w')$ and (B, m)	is blocked by B and w,
$(A, m'w')$ and (B, w)	is blocked by A and m.

Example 5. If preferences are consistent and persistent but not size monotone then stable matchings may not form a lattice. More precisely, the (revealed preference) supremum of stable matchings may not be stable.

There are colleges $A, \ldots, E$ and students $a, \ldots, e$. Preferences are given by the table below: A's first choice is a and second choice ce; similarly for other agents.

Note that the preferences of A and B violate size monotonicity.

A	B	C	D	E	a	b	c	d	e
a^*	$b^\#$	c^*	$d^\#$	e	$C^\#$	D^*	$A^\#$	B^*	$A^\#$
$ce^\#$	de^*	$a^\#$	b^*	—	A^*	$B^\#$	C^*	$D^\#$	B^*
—	—	—	—	—	—	—	—	—	E

One easily verifies that the entries marked $*$ and those marked $\#$ correspond to stable matchings: Namely, in each matching, where a college is matched with its second choice, the preferred student is matched with her first choice. But in the matching which is the college supremum of $*$ and $\#$, both E *and* e are unmatched, hence they block and the college supremum is therefore unstable. Note that the unisize condition is also violated for A and B in the matchings $*$ and $\#$.

The following two examples show that certain results for the classical model do not generalize to the nonclassical model (with consistent, persistent and size monotone choice functions).

Example 6. The college optimal stable matching may not be Pareto optimal for colleges.

There are colleges A, B, Z with quotas $1, 1, 2$, male students m, m' and female students w, w'.

Z chooses mw if all four students are available and otherwise chooses the sexually diverse pair.

The other preferences are given by the table below where the left entry in each pair is the college's ranking of the student and the right entry is the student's ranking of the college.

	m	w	m'	w'
Z	$(-,2)$	$(-,2)^\#$	$(-,1)^{*\#}$	$(-,1)^*$
A	$(1,3)^\#$	$(2,1)^*$	$(4,3)$	$(3,2)$
B	$(2,1)^*$	$(3,3)$	$(4,2)$	$(1,3)^\#$

The only stable matching is the student optimal matching $*$. One sees this by checking from the algorithm that it is also the college optimal matching. But the matching $\#$ makes all colleges (strictly) better off. Of course $\#$ is unstable, being blocked by Z and m.

Example 7. As shown in Corollary 4, in the classical model all stable matchings are comparable for each agent. This need not be so in the nonclassical model.

Colleges A and B have quota 2. Students are m, w, m', w', and A and B are like Z in Example 6 except A most prefers mw and B most prefers $m'w'$. For the students, m and w prefer B to A, m' and w' prefer A to B.

One easily verifies that all four ways of allocating diverse pairs to A and B are stable and also that mw' and $m'w$ are noncomparable in the preferences of both A and B.

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