Qubit state transfer via one-dimensional discrete-time quantum walk

İ. Yalçınkaya* and Z. Gedik
Faculty of Engineering and Natural Sciences, Sabancı University, Tuzla, 34956, İstanbul, Turkey
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We propose a scheme for perfect transfer of an arbitrary qubit state via discrete-time quantum walk on a line or a circle. For this purpose, we take the advantage of one more coin operator which is applied at the end of the walk. This additional coin operator does not depend on the state to be transferred. We show that perfect state transfer over arbitrary distances can be achieved only for identity and flip coin operators. Perfect state transfer can also be realized by using an unbiased coin operator but only for a finite number of distances.

I. INTRODUCTION

Quantum state transfer from one location to another is a significant problem for quantum information processing systems. A quantum computer, which consists of different processing units, requires the quantum states to be transferred between its parts. Therefore, quantum state transfer will be an important part of quantum computer design. In this article, we consider two related fields of research, quantum state transfer and quantum walks on one-dimensional lattices.

Quantum communication through a spin chain was first considered by Bose [1] and since then it has been studied in depth [2–9]. This procedure consists of interacting spins on a chain, whose dynamics is governed by Heisenberg, XX or XY Hamiltonians. Perfect state transfer (PST) through a spin chain, in which adjacent spins are coupled by equal strength, can be achieved only over short distances [10, 11].

Quantum walks (QWs) were introduced as a quantum analogue of classical random walk (CW). Farhi and Gutmann [12] suggested a quantum algorithm, which is now known as the continuous-time QW, that reaches the nth level of a decision tree faster than the CW. *Quantum random walks*, or stated in other words as discrete-time QWs, were introduced by Aharonov [13] as a quantum counterpart of the CW where the walker has larger average path length than that of the CW. This property provides opportunity to define new quantum algorithms [14]. Many physical implementations of QWs have been made and experimental realizations have been reported [15–30].

The time-evolution of qubit state transfer through a spin chain can be interpreted as a continuous-time QW and PST is possible over a spin chain of any length with pre-engineered couplings [10, 11]. Furthermore, this interpretation can be extended to discrete-time QW with a position-dependent coin operator [31]. PST in quantum walks on various graphs has been studied more specifically for the continuous-time model [32]. Transfer of specific quantum states with high fidelities on variants of cycles has been reported for discrete-time model [33] where total space is traced over the coin space at the end of the walk.

In this article, we study the perfect transfer of an arbitrary qubit state between distant sites of one-dimensional lattices in discrete-time QW architecture. We treat the coin as our qubit and assume that the coin is embedded in the walker, i.e., the walker can be thought as a spin-1/2 particle which moves on discrete lattice sites. At the end of the walk, we apply one more coin operator (*recovery operator*) to achieve PST. Recovery operators are fixed for a given coin operator which governs the walk and they can be determined before the walk.

This article is organized as follows. In Sec. II, we present a brief review of discrete-time QWs and introduce spatial and local approaches to the definition of directions for the walker. We define the boundary conditions (N-lines and N-cycles) of the walk. In Sec. III, we introduce the recovery operator and obtain all possible cases where PST occurs for N-lines and N-cycles. In conclusion part, we summarize our results.

II. THE DISCRETE-TIME QW

Formal definition of one-dimensional discrete-time QW consists of two discrete Hilbert spaces. One of them is the position space, denoted by $\mathcal{H}_P$ and spanned by the basis states $\{|x\rangle : x \in \mathbf{Z}\}$, and the other one is the coin space, denoted by $\mathcal{H}_C$ and spanned by basis states $\{|\uparrow\rangle, |\downarrow\rangle\}$. These basis states take the role of position and spin state of a spin-1/2 particle, namely *the walker*. The total quantum state of the walker is determined by both its coin and position degrees of freedom. In other words, the whole space, $\mathcal{H}_C \otimes \mathcal{H}_P$, is spanned by the tensor product of base states which are denoted by $|c, x\rangle$. Time evolution of the walk is governed by a unitary operator which is applied in discrete time intervals (so-called discrete-time) to form the steps of the walk. *One step* is defined by two subsequent unitary operators, *the coin operator* and *the shift operator*, which affect the coin and the position spaces separately. Thus, one step is given by the unitary operator $\mathbf{U} = \mathbf{S}.(\mathbf{C} \otimes \mathbf{I})$, where $\mathbf{S}$, $\mathbf{C}$ and $\mathbf{I}$ are shift, coin and identity operators, respectively. The most general

* iyalcinkaya@sabanciuniv.edu

unitary coin operator can be written as

$$\mathbf{C} = \begin{pmatrix} \sqrt{\rho} & \sqrt{1-\rho}e^{i\theta} \\ \sqrt{1-\rho}e^{i\phi} & -\sqrt{\rho}e^{i(\theta+\phi)} \end{pmatrix} \tag{1}$$

where ρ gives the bias of the coin, i.e., where ρ and $1-\rho$ are probabilities for moving left and right, respectively. Here, θ and ϕ are the parameters defining the most general unitary operator up to a $U(1)$ phase. For $\rho = 1/2$ and $\theta = \phi = 0$, we obtain the well-known Hadamard coin operator (unbiased case). The shift operator is given as

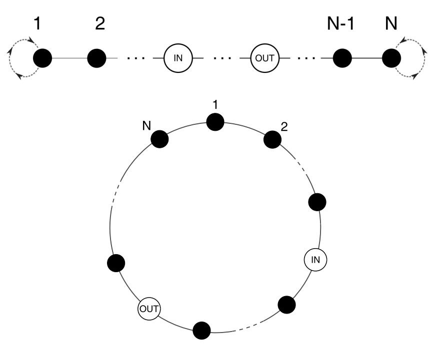
$$\mathbf{S} = |\uparrow\rangle\langle\uparrow| \otimes \sum_x |x+1\rangle\langle x| + |\downarrow\rangle\langle\downarrow| \otimes \sum_x |x-1\rangle\langle x|,$$

where the sum is taken over all discrete positions of the space. The shift operator forces the walker to move in a direction determined by its coin state.

The motion of the walker is restricted to the line where the walker can exist only at N separated sites. Fig. 1 demonstrates two boundary conditions for the walk and these are the ones that we will use throughout the article. Two specific sites are labelled as IN and OUT for reasons which will be clear in Sec. III. In Fig. 1(a), the lattice with N sites and reflecting boundaries (*N-line*) is represented. Self loops at the boundaries indicate that wave function is reflected after the shift operator is applied, similar to the approach used by Romanelli *et al.* for the broken links model [34]. The shift operator is of the form

$$\mathbf{S} = |\uparrow\rangle\langle\downarrow| \otimes |1\rangle\langle 1| + |\downarrow\rangle\langle\uparrow| \otimes |N\rangle\langle N| + \text{b.t.},$$

where *b.t.* stands for the bulk terms which do not include the boundary sites (1 or N). Thus the left (right)-going part at the first (last) site is diverted to the right (left)-going part at the same site to keep the flux conserved. In Fig. 1(b), the lattice with N sites and periodic boundaries (*N-cycle*) is represented. Here, we simply connect the first and the last sites with one more edge.

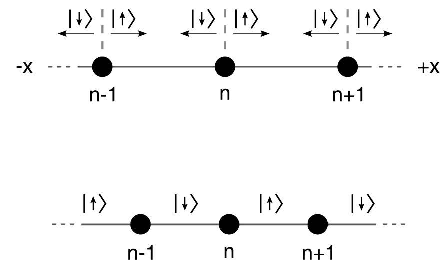


(a)

(b)

FIG. 1. (a) Reflecting boundaries (N-line). (b) Periodic boundaries (N-cycle).

For the walker, directions of motion can be defined in two ways. In the first one, which we shall call as *spatial approach*, the same coin state corresponds to the same spatial direction at every site. Without loss of generality, one can choose the up (down) coin state to correspond the right (left) spatial direction or clockwise (anti-clockwise) rotation. In the second approach, which we shall call as *local approach*, we assign two orthogonal coin states to the two edges of every site in a self-consistent manner. These approaches are summarized in Fig. 2.



(a)

(b)

FIG. 2. Directional approaches for one-dimensional lattice. (a) Spatial approach. (b) Local approach.

The walker starts its motion at an arbitrary position with an arbitrary coin state. In general, the initial position can be a superposition of different basis states, i.e., it can be taken as a de-localized state in the position space. In this work, we will use only localized initial states of the form $|\Psi_0\rangle = |\psi_{0,x}\rangle \otimes |x\rangle$, where $|\psi_{0,x}\rangle = \alpha_{0,x}|\uparrow\rangle + \beta_{0,x}|\downarrow\rangle$ is the initial coin state. The first and the second indices denote the time and position, respectively. After t steps, the initially localized state disperses in the position space and the wave function of the walker becomes

$$|\Psi_0\rangle \xrightarrow{\mathbf{U}^t} |\Psi_t\rangle = \sum_x (\alpha_{t,x}|\uparrow\rangle + \beta_{t,x}|\downarrow\rangle) \otimes |x\rangle. \tag{2}$$

At the end of the walk, the probability of finding the walker at position x is given by summing the probabilities over the coin states

$$P_{t,x} = |\alpha_{t,x}|^2 + |\beta_{t,x}|^2. \tag{3}$$

Time evolution which is given in Eq. (2) can be written as an iterative map which gives the coefficients of $|\uparrow\rangle$ and $|\downarrow\rangle$ at any time for any position. First, consider the effect of the coin operator on the wave function projected on a given position state $|x\rangle$:

$$\begin{aligned} |\Psi_t'\rangle \xrightarrow{\mathbf{C}\otimes\mathbf{I}} & [(\alpha_{t,x}\sqrt{\rho} \ + \beta_{t,x}\sqrt{1-\rho}e^{i\theta})|\uparrow\rangle \ + \\ & (\alpha_{t,x}\sqrt{1-\rho}e^{i\phi} - \beta_{t,x}\sqrt{\rho}e^{i\theta}e^{i\phi})|\downarrow\rangle] \otimes |x\rangle \end{aligned} \tag{4}$$

Coefficients of $|\uparrow\rangle$ and $|\downarrow\rangle$ are the right-going and the left-going probability amplitudes, respectively. Thus the time evolution in Eq. (2) can be written as the map

$$\begin{aligned} \alpha_{t+1,x} &= \alpha_{t,x-1}\sqrt{\rho} + \beta_{t,x-1}\sqrt{1-\rho}e^{i\theta}, \\ \beta_{t+1,x} &= \alpha_{t,x+1}\sqrt{1-\rho}e^{i\phi} - \beta_{t,x+1}\sqrt{\rho}e^{i\theta}e^{i\phi}. \end{aligned} \tag{5}$$

Equation (5) is very useful to keep track of the probability flux on the lattice.

III. RECOVERY OPERATOR AND PST

Motion of the walker starts at an initial site (IN). After t steps, we measure the coin state of the walker at a site (OUT). The similarity between the measured coin state and the initial coin state is given by,

$$f_{t,\text{OUT}} = |\langle \Psi_{\text{OUT}} | \Psi_t \rangle|. \quad (6)$$

Quantity $f_{t,\text{OUT}}$ is called the *fidelity* at time t and site OUT. $|\Psi_{\text{OUT}}\rangle$ is the predefined control state which is given by $|\Psi_{\text{OUT}}\rangle = |\psi_{0,\text{IN}}\rangle \otimes |\text{OUT}\rangle$. $|\psi_{0,\text{IN}}\rangle$ is the coin state at $t = 0$ and site IN. Thus $f_{t,\text{OUT}}$ measures the correlation between $|\psi_{0,\text{IN}}\rangle$ and $|\psi_{t,\text{OUT}}\rangle$. If $f_{t,\text{OUT}} = 0$, there is no correlation between these states and if $f_{t,\text{OUT}} = 1$, these states are exactly the same up to an overall phase. Note that there can be cases where the wave function at Eq. (2) takes the form

$$|\Psi_t\rangle = |\psi_{0,\text{IN}}\rangle \otimes \sum_x \alpha'_{t,x} |x\rangle.$$

While this form satisfies the condition for obtaining the initial coin state, the probability that we find the walker at OUT is less than 1 since the wave function is spread in position space. This does not correspond to PST and the fidelity definition in Eq. (6) reflects this fact. Hence, we are looking for a class of time evolutions of the form

$$|\psi_{0,\text{IN}}\rangle \otimes |\text{IN}\rangle \xrightarrow{?} |\psi_{0,\text{IN}}\rangle \otimes |\text{OUT}\rangle. \quad (7)$$

In general, positions IN and OUT can be chosen as the same site or not. The condition, IN = OUT and $f_{t,\text{OUT}} = 1$, implies that the walk is *periodic* [15, 35, 36]. In this article, we will focus on the latter case and define PST as the time evolution in Eq. (7) with IN $\neq$ OUT. This PST approach has been utilized in [37]. We will show that there are some cases where the walk is periodic but no PST occurs. On the other hand, our numerical calculations have not yielded any non-periodic case where PST occurs.

For a given number of steps, the walker may appear at OUT with probability 1 but with a coin state ($|\psi_{t,\text{OUT}}\rangle$) which is different from the initial one ($|\psi_{0,\text{IN}}\rangle$). Since Eq. (1) includes all possible rotations for a two-dimensional coin, one can transform $|\psi_{t,\text{OUT}}\rangle$ to $|\psi_{0,\text{IN}}\rangle$ with suitable parameters, (ρ',θ',ϕ'). Let us denote this coin operator with primed parameters as $\mathbf{C}_R = \mathbf{C}' \otimes \mathbf{I}$ (*recovery operator*) to distinguish it from the one which governs the walk. We show that, a recovery operator is fixed for a given coin operator and it is independent of the coin state to be transferred for a given lattice. Thus, once we decide on the coin operator which we will use for the walk, we can also determine the recovery operator which will be applied at the end of the walk to achieve PST. This PST scheme can be summarized as

$$|\psi_{0,\text{IN}}\rangle \otimes |\text{IN}\rangle \xrightarrow{\mathbf{U}^t} |\psi_{t,\text{OUT}}\rangle \otimes |\text{OUT}\rangle \xrightarrow{\mathbf{C}_R} |\psi_{0,\text{IN}}\rangle \otimes |\text{OUT}\rangle.$$

In our calculations, IN and OUT are chosen as the outermost sites on the lattice. These are 1st and Nth sites for the N-line, 1st and ($\frac{N}{2}+1$)th sites for the N-cycle with even N, respectively. First we have numerically determined all cases, up to $N = 10$, where the walker is found with probability 1 at IN and OUT for all Bloch states. Without loss of generality, we have restricted the coin operator to $\phi = 0$ [35]. Then, we have analytically studied these cases for their aptness to periodicity and PST, by using Eq. (5).

In Fig. 3, fidelity distributions over initial coin states are given. It can be seen that only limited number of initial coin states are transferred perfectly. For PST, without any knowledge about the initial coin state, one should be able to transfer all coin states with $f_{t,\text{OUT}} = 1$. However, an arbitrary coin operator and an arbitrary lattice do not provide QWs which allow PST in general. Fig. 3 demonstrates only two specific examples. Other coin operators and lattices give similar results except the 4-cycle which will be explained in Sec. III B.

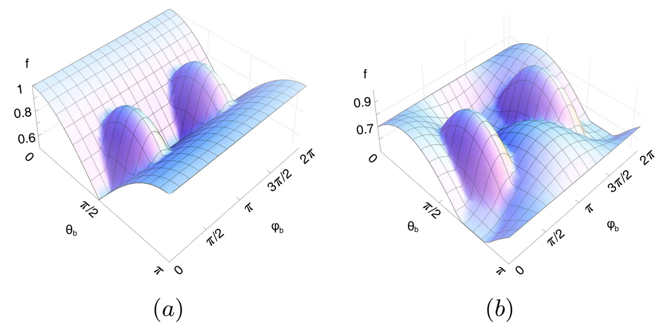


FIG. 3. Fidelities of the initial coin states on 2-line. 1st and 2nd sites are chosen as IN and OUT, respectively. The (θ_b, ϕ_b) plane represents the initial coin states on Bloch sphere and f is the maximum fidelity over time. (a) Identity coin operator is used. Independent of ϕ_b, the coin states $\theta_b = 0$ and $\theta_b = \pi$ are transferred perfectly. (b) Hadamard coin operator is used and no PST is observed within a limited time interval.

A. PST on N-lines

1. *Case:* $\rho \neq 1$

Table I denotes the cases where the walker is found with probability 1 on the 2-line. The wave functions with coin state $|\psi_{0,1}\rangle$ manifest periodicity. To achieve a PST, we consider the other cases where the wave function is

$$|\Psi_t\rangle = [-\beta|\uparrow\rangle + \alpha|\downarrow\rangle] \otimes |2\rangle. \quad (8)$$

After t-steps, we apply appropriate recovery operator, $(\rho', \theta', \phi') = (0, 0, -\pi)$, on Eq. (8). In this way, we obtain the initial coin state and hence PST. Overall process can be written as

$$|\psi_{0,1}\rangle \otimes |1\rangle \xrightarrow{\mathbf{C}_R \mathbf{U}^t} |\psi_{0,1}\rangle \otimes |2\rangle.$$

ρ	Step(t)	Position(x)	Coin state
$\frac{1}{4}$	6	2	$-\beta\vert\uparrow\rangle+\alpha\vert\downarrow\rangle$
	12	1	$\vert\psi_{0,1}\rangle$
$\frac{1}{2}$	4	2	$-\beta\vert\uparrow\rangle+\alpha\vert\downarrow\rangle$
	8	1	$\vert\psi_{0,1}\rangle$
$\frac{3}{4}$	6	1	$\vert\psi_{0,1}\rangle$

TABLE I. For the 2-line, these are the cases where the walker is found with probability 1. The other parameters of the coin operator are chosen as $\theta=\phi=0$.

Thus, any coin state can be transferred on 2-line perfectly with appropriate (ρ,t) values given in Table I. We note that recovery operator is constant for a given coin operator and it provides PST for all initial coin states. In each PST case, the QW is periodic. For example, after 4 steps of the walk with $\rho=1/2$, if the walker proceeds 4 more steps, the initial wave function is recovered. There is also a case with $\rho=3/4$, where 2-line is periodic but it does not lead to PST. PST requires the wave function to localize more than one sites in turn and this process naturally gives rise to periodicity.

Notice that, after applying the recovery operator, we initialize the walker with the initial coin state at a different site. For example, when $\rho=1/2$, if the walker is acted on by sequence of operations, such as $\mathbf{C}_R\mathbf{U}^4\mathbf{C}_R\mathbf{U}^4$, it will be initialized on sites 1 and 2 alternatingly. The sequence of initializations which keeps the initial coin state unchanged, suggest us to define a new classification for discrete-time QWs which we call *n-periodicity.* We can define one step of the walk for the example above as $\mathbf{U}'=\mathbf{C}_R\mathbf{U}^4$. Then, after each step, coin state will be conserved and the only change will occur in the position space. In other words, $\mathbf{U}'$ is same as that of $\mathbf{I}\otimes(|2\rangle\langle1|+|1\rangle\langle2|)$. Since the walker is localized on two sites in an alternating manner, the QW under consideration becomes 2-periodic. In general, the number n gives the total number of sites where initial coin state is localized during the time evolution. If QW is periodic, we will call it 1-periodic, i.e., well-known periodicity concept becomes a member of the general n-periodicity class. Thus, N-line or N-cycle allow maximum N-periodicity. This definition is useful because it generalizes the periodicity definition so that it includes the PST too.

For $\rho\neq1$, reflecting boundaries ensure that there will always be a non-zero probability for finding the walker at IN, independent of t, if there is no destructive interference. However, the dimension of the position space for 2-line allows the wave function to vanish at IN and gives rise to the cases given in Table I.

2. *Case:* $\rho=1$

Independent of the initial coin state, when we restrict the coin operators to $\rho=1$, we find the walker with probability 1 at OUT and IN after specific number of steps for all N-lines. This result can be summarized as

$$t=\begin{cases}N(2l-1), & P_{t,\mathrm{OUT}}=1\\ 2Nl, & P_{t,\mathrm{IN}}\;\;=1\end{cases},$$

where $l\in\mathbb{Z}^+$. To find the coin state of the walker at t, we have derived the wave functions

$$\begin{aligned}|\Psi_{N(2l-1)}\rangle&=e^{i(l-1)\Theta}[-\beta e^{i(\theta+\phi)}|\uparrow\rangle+\alpha|\downarrow\rangle]\otimes|N\rangle,\\ |\Psi_{2Nl}\rangle&=e^{il\Theta}[\alpha|\uparrow\rangle+\beta|\downarrow\rangle]\otimes|1\rangle,\end{aligned}\quad(9)$$

where $\Theta(\theta,\phi,N)=(\theta+\phi)N+\mu\pi$ and θ,ϕ are the parameters of the coin operator. Here, μ is a function which adds the phase π for odd N and it can be defined as $\mu(N)=[1-(-1)^N]/2$. Eq. (9) shows that wave function is periodically localized at opposite sites which agrees with the numerical results. Furthermore, N-line is periodic up to an overall phase. After N steps, we apply recovery operator $(\rho',\theta',\phi')=(0,0,-\theta-\phi-\pi)$ for PST. Recovery operator is a function of θ and ϕ which means that for all coin operators with $\rho=1$, there is always a corresponding recovery operator. Hence, step operator $\mathbf{U}'=\mathbf{C}_R\mathbf{U}^N$ makes N-line 2-periodic for $\rho=1$.

B. PST on N-cycles

Periodicity of 4-cycles has been first discussed by Travaglione and Milburn [15] for the walker with initial coin state $|\uparrow\rangle$. They show that a wave function, initially localized at a site of 4-cycle, is recovered after 8 steps with Hadamard coin. Later, Treggenna *et al.* have extended this result up to 10-cycle [35]. They have shown that, except 7-cycle, all N-cycles with $N<11$, manifest periodicity with appropriate choices of (ρ,θ,ϕ) for every initial coin state. Dukes has analysed the periodicity of N-cycles in detail and presented the general conditions for periodicity [36]. We shall focus on PST rather than periodicity of N-cycles.

1. *Case:* $\rho\neq1$

For 2-cycle, full evolution can simply be written in matrix form as

$$\mathbf{U}^t\leftrightarrow\begin{pmatrix}\sqrt{\rho} & \sqrt{1-\rho}e^{i\theta}\\ \sqrt{1-\rho}e^{i\phi} & -\sqrt{\rho}e^{i(\theta+\phi)}\end{pmatrix}^t\otimes\begin{pmatrix}0&1\\1&0\end{pmatrix}^t.\quad(10)$$

Eq. (10) shows that, shift operator swaps the position of the walker independent of its coin state. At $t=1$, wave function becomes Eq. (4) with $x=2$. Since the coin operator is unitary, $\mathbf{C}_R=\mathbf{C}^\dagger\otimes\mathbf{I}$ leads to PST after first step. If we define one-step as $\mathbf{U}'=(\mathbf{C}^\dagger\otimes\mathbf{I})\mathbf{S}(\mathbf{C}\otimes\mathbf{I})$, QW becomes 2-periodic and it keeps the initial coin state unchanged. In other words, the initial coin state bounces back and forth between two sites. In contrast to 2-line, 2-cycle allows PST for all coin operators with the aid of appropriate recovery operators.

The other PST case on 4-cycle can be achieved by the coin operator with $\rho = 1/4$ or $1/2$ as shown in Table II.

ρ	Step(t)	Position(x)	Coin state
$\frac{1}{4}$	6	3	$\|\psi_{0,1}\rangle$
	12	1	$\|\psi_{0,1}\rangle$
$\frac{1}{2}$	4	3	$(e^{i\pi})\|\psi_{0,1}\rangle$
	8	1	$\|\psi_{0,1}\rangle$

TABLE II. For the 4-line, these are the cases where the walker is found with probability 1. The other parameters of coin operator are chosen as $\theta = \phi = 0$. The overall phase $e^{i\pi}$ for $\rho = 1/2$ appears if $\theta = \pi$.

The 4-cycle is important because it provides PST without any recovery operators as depicted in Fig. 4.

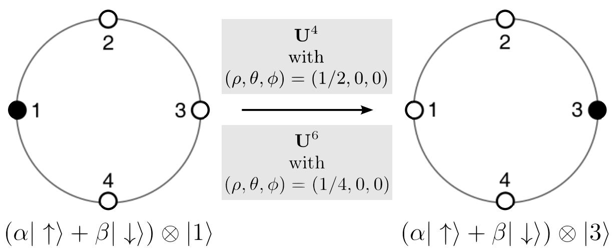


FIG. 4. PST on 4-cycle. This is the only case where discrete-time QW allows PST with Hadamard coin operator or with a biased coin, $\rho = 1/4$, without any recovery operators. Black (hollow) dots indicate $P_{t,x} = 1$ ($P_{t,x} = 0$).

2. *Case:* $\rho = 1$

Now, we consider the N-cycles with even N and $\theta, \phi \neq 0$. Since the coin operator is diagonal, $|\uparrow\rangle$ and $|\downarrow\rangle$ terms do not mix, and generate propagations in opposite directions. After $N/2$ steps, we find the walker at OUT with probability 1. We note that the coin operator adds the phase $e^{i(\theta+\phi+\pi)}$ to the coefficient of $|\downarrow\rangle$ in each step. Thus, after $N/2$ steps, wave function becomes

$$|\mathbf{\Psi}_{N/2}\rangle = (\alpha|\uparrow\rangle + \beta e^{i\frac{N\Theta}{2}}|\downarrow\rangle) \otimes |\tfrac{N}{2}+1\rangle$$

where $\Theta = \theta + \phi + \pi$. Without loss of generality, one can choose $\theta' = 0$ and use the recovery operator $(\rho', \theta', \phi') = (1, 0, -[N\Theta/2] + \pi)$ to achieve PST. The step operator $\mathbf{U}' = \mathbf{C}_R \mathbf{U}^{N/2}$ makes the walk 2-periodic. If N is odd, wave function does not localize at any site except the initial one. The wave function after N steps is

$$|\mathbf{\Psi}_N\rangle = (\alpha|\uparrow\rangle + \beta e^{iN\Theta}|\downarrow\rangle) \otimes |1\rangle.$$

After N steps, appropriate choice for the recovery operator can be given as $(\rho', \theta', \phi') = (1, 0, -N\Theta + \pi)$. The step operator $\mathbf{U}' = \mathbf{C}_R \mathbf{U}^N$ makes the walk 1-periodic.

The coin operators $\rho = 0$ and 1 allow PST if we define the directions for N-cycle with spatial and local approaches, respectively. Although, it has not been indicated in the discussion about N-cycles above, spatial approach has been used intrinsically, i.e., clockwise rotations correspond to $|\uparrow\rangle$. We know that if we use spatial approach with the coin operator $\rho = 1$, the walk gives PST for the N-cycles with even N. When $\rho = 0$, PST is achieved by the local approach. Since, we have to label at least two edges with the same basis state, N-cycles with odd N is ill-defined. Therefore, we consider N-cycles with even N only. If we label all edges as in Fig. 2(b), after $N/2$ steps, the wave function becomes

$$|\mathbf{\Psi}_{N/2}\rangle = \begin{cases} |\psi_{0,in}\rangle \otimes |\frac{N}{2}+1\rangle, & \text{even } N/2, \\ (\alpha e^{i\phi}|\downarrow\rangle + \beta e^{i\theta}|\uparrow\rangle) \otimes |\frac{N}{2}+1\rangle, & \text{odd } N/2. \end{cases}$$

Both case have the overall phase $e^{i\lfloor N/4 \rfloor(\theta+\phi)}$ where $\lfloor\rfloor$ is the floor function. It is clear that we can use $(\rho', \theta', \phi') = (0, -\phi, -\theta)$ to recover the second case and make the walk 2-periodic.

IV. CONCLUSION

We have shown that PST of an unknown qubit state on N-lines and N-cycles is rare in standard discrete-time QW architecture. For $N < 10$, only 4-cycle allows PST between the sites which are furthest apart, with Hadamard operator or a biased coin operator with $\rho = 0.25$. We have proposed a scheme by introducing recovery operators to generate more cases which allow PST. We have demonstrated that one can transfer an unknown qubit state to an arbitrary distance by using identity or flip coin operators with the aid of appropriate recovery operators. With recovery operators, the 2-cycle turns out to be the only lattice which allows PST for all coin operators. Also, the Hadamard coin and biased coin with $\rho = 0.25$ allow PST on 2-line.

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