Intermediate Notions of Rationality for Simple Allocation Problems

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Abstract

In this study, we interpret solution rules on a class of simple allocation problems as data on the choices of a policy-maker. We analyze conditions under which the policy maker's choices are (i) rational on a partition (ii) transitive-rational on a partition. In addition we introduce two new rationaltiy notions: (i) Constant Proportion rationality,(ii) Constant Distance Rationality. Our main results are as follows: (i) if the elements of a partition is closed under coordinate-wise minimum or coordinate-wise maximum operatation, then a well known property in the literature, contraction independence (a.k.a. IIA) is equivalent to rationality on that partition; (ii) if the characteristics vectors falling into the same element of a partition is ordered Weak Axiom of Revealed Preferences (WARP) is equivalent to transitive rationality.

Keywords: partition, rational, contraction independence, weak axiom of revealed preferences, strong axiom of revealed preferences.

BASİT TAHSİS PROBLEMLERİ İÇİN ARA RASYONALİTE KAVRAMLARI

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Özet

Bu çalışmada, bir karar merciinin bir basit tahsis problemi sınıfında sunduğu çözümleri (seçimlerini) veri olarak yorumladık. Karar merciinin seçimlerinin hangi koşullar altında bir bölümleme üzerinde rasyonelite(i), geçişken rasyonalite (ii) kavramlarını sağladığını inceledik. Ek olarak iki yeni rasyonalite kavramını sunduk: (i) Sabit oranlarla Rasyonalite ,(ii) Sabit Farklarla Rasyonalite. Temel sonuçlarımız şunlardır: (i) Bir bölümlemenin elemanları koordinat noktalarının maksimumu ve koordinat noktalarının minimumu operasyonları altında kapalı ise literatürde iyi yer edinmiş bir özellik olan daralmadan bağımsızlık (IIA), bu bölümleme üzerinde rasyonaliteye denktir; (ii) Bir bölümlemenin elemanında yer alan kararteristik vektörler sıralı ise, Açığa Çıkan Tercihlerin Zayıf Aksiyomu (WARP) ile geçişken rasyonalite kavramları denktir.

Anahtar Kelimeler: bölümleme, rasyonel, daralmadan bağımsızlık, açığa çıkan tercihlerin zayıf aksiyomu, açığa çıkan tercihlerin güçlü aksiyomu.

Contents

1 Introduction

Revealed preference theory studies conditions under which by observing the choice behavior of an agent, one can discover the underlying preferences that govern it. Choice rules for which this is possible are called rational. Most of the earlier work on rationality analyzes consumers' demand choices from budget sets (e.g. see Samuelson, 1938, 1948). The underlying premise that choices reveal information about preferences, however, is applicable to a wide range of choice situations. For example, applications of the theory to bargaining games (Nash, 1950) characterize bargaining rules which can be "rationalized" as maximizing the underlying preferences of an impartial arbitrator (Peters and Wakker, 1991; Bossert, 1994; Ok and Zhou, 2000; Sánchez, 2000).

A **simple allocation problem** for a society N is an $|N|+1$ dimensional non-negative real vector $(c_1, ..., c_{|N|}, E) \in \mathbb{R}_+^{N+1}$ satisfying $\sum_N c_i \geqq E$, where E, the **endowment** has to be allocated among agents in N, who are characterized by c, the characteristic vector. An **allocation rule** on a simple allocation problem represents the choices of a decision maker.

There are several applications of simple allocation problems. Some of these applications are;

1. **Permit allocation by the U.S. federal government**: The Environmental Protection Agency allocates each period an amount E of pollution permits among N firms (such as CO_2 emission permits allocated among energy producers). Each firm i, based on its location, is imposed by the local authority an emission constraint c_i on its pollution level (e.g. see Kıbrıs, 2003).

2. **Consumer choice in fixed-price models:** A consumer has to allocate his income E among commodities in N , the price of each fixed and, with appropriate choice of consumption units, normalized to 1. The

consumer faces rationing constraints c_i on how much he can consume of each commodity i (e.g. see Bénassy, 1993, or Kıbrıs and Küçüksenel, 2008).

3. **Demand rationing in supply-chain management:** A supplier is to allocate its production E among demanders in N, each of which demanding c_i units (e.g. see Cachon and Lariviere, 1999).

4. **Single-peaked or saturated preferences:** A social planner is to allocate E units of a perfectly divisible commodity among agents in N, each having preferences with peak (or saturation point) c_i (e.g. see Sprumont, 1991).1

5. **Bargaining with quasilinear preferences and claims**: An arbitrator is to allocate E units of a numeriare good among agents in N , each with quasilinear preferences and each holding a claim.

6. **Taxation:** A public authority is to collect E units of tax among agents in a society N, each agent having an income c_i (e.g. see Edgeworth, 1898, or Young, 1987).

7. **Surplus sharing:** A social planner is to allocate the return E of a project among its investors in N . Each investor i has invested s_i (e.g. see Moulin, 1985, 1987).

8. **Bankruptcy:** A bankruptcy judge is to allocate the remaining assets E of a bankrupt firm among its creditors, N. each agent i has credited c_i to the bankrupt firm and now, claims this amount (e.g. see O'Neill, 1982 or for a review, Thomson, 2003, 2007).

In all of the examples above, a decision maker allocates the resources. In most of the economic models, these decision makers are modelled as maximizers of an objective function, such as social welfare functions. It is an important question that under what conditions such a modelling is possible in terms of understanding the boundaries of the economic models. This paper contributes to the literature which tries to answer that question. Two famous notions, **rationality** and **transitive rationality** are commonly discussed in that literature for the analysis of these economic models.

Kıbrıs analyzed these two notions for simple allocation problems (Kıbrıs, 2008). According to Kıbrıs, an allocation rule is data on the choices of a decision maker. Rationality of a rule is about whether its choices can be modeled as maximization of a binary relation. That is, a rule is said to be **rational** if its choices coincide with maximization of a binary relation on the allocation space.

The maximizing binary relation is independent of the characteristic vector c. Thus, most well-known rules, such as the proportional rule [1], violates rationality because of this requirement. In some applications, such as 1 and 2, the independence of the binary relation from the characteristic vector may be desirable. Yet, in some other applications, it is intuitive to think that the choice of the decision maker may depend on the information that is contained in the characteristic vector. For example, in most countries, bankruptcy laws use the proportional rule in allocation of the remaining asset for shareholders. Thus, the allocation depends on c.

In this paper, we focus on how we can rationalize the choices of such decision makers. To do so, we will weaken the requirement that the maximizing binary relation is completely independent of the information contained in the characteristic vector. An example of such a weakening called **weak rationality** is introduced in Kıbrıs (2008). This property allows a rule to maximize a

[1] The proportional rule, PRO allocates endowment in proportion to the characteristic values of each agent.

different binary relation for each characteristic vector. However, Kıbrıs (2008) shows that every rule satisfies weak rationality.

In this study, our main focus is to introduce *alternative rationality notions* for simple allocation problems, between rationality and weak rationality. That way we may be able to capture the intuition which suggests that an allocation rule may use certain information contained in the characteristics vector.

To introduce these new rationality notions, we make our analysis on a general partition of the space of characteristic vectors. For each partition, we introduce an associated rationality requirement. According to this rationality requirement, for each pair of characteristic vectors that fall into the same element of the partition, the same binary relation must be used for maximization. That is, a rule is **rational on a partition** Π, if and only if for each $\pi \in \Pi$, there exists a binary relation $B(\pi)$, such that the choices of the rule coincide with the maximization of $B(\pi)$ on the allocation space.

In section 2, we introduce our model which follows Kıbrıs (2008). **In section 3**, we analyze the properties of rules that are rational on some partition. We show that a rule that is rational on a partition satisfies a well known property in the literature called **contraction independence** on that partition. However, we show that the reverse relation is not true for all partitions. We, then present conditions under which contraction independence on a partition is equivalent to rationality of an allocation rule on that partition: namely, that partition is closed either under coordinate-wise minimum or coordinate-wise maximum operations. *Theorem 1* shows that contraction independence and another well known property in the literature, **Weak Axiom of Revealed Preferences (WARP)** are equivalent on the partitions that are closed under coordinate-wise minimum or coordinate-wise maximum operations.

An allocation rule is *transitive-rational on a partition* if it can be rationalized by a transitive preference relation on that partition. **In Section 4**, we analyze properties of transitive rationality on a partition for two agents. *Theorem 2* states that for two agents, a rule satisfies *WARP on a partition* (that is, rational on that partition) if and only if it is *transitive rational on that partition.* This result generalizes the result of Kıbrıs (2008) on transitive rationality for two agent case.

In Section 5, we analyze transitive-rational rules for an arbitrary number of agents. We first observe existence of partitions such that there are rational rules on that partition that are not transitive-rational on it. (This is the same as Kıbrıs (2008) and in line with Gale (1960), Kihlstrom, Mas-Colell, and Sonnenschein (1976), and Peters and Wakker (1994) who show that the counterpart of Theorem 2 in consumer choice does not generalize either.) We then observe that if the elements of a partition satisfies some sufficient conditions then WARP and SARP on that partition are equivalent. *Theorem 2* states that if the characteristics vector falling in an element of a partition are ordered then we have the equivalence of WARP and SARP on that partition.

In section 6, we introduce two new rationality notions and characterize the properties of rules which satisfy these alternative notions. These rationality notions are *"constant-proportion rationality"* and *"constant-distance rationality"*. We introduce constant-proportion rationality because most of the people follow the requirement of this rationality notion. Gächter and Riedl (2008) experimentally shows that the proportional rule is the normatively most attractive rule. which constant proportion rationality requirement.

We introduce constant-distance rationality, because this rationality notion is closely related to another well known rule in the related literature, Equal Losses rule, which satisfies constant-distance rationality requirement. [2]

[2] Equal Losses rule equalizes the losses of each agent subject to the constraint that no agent receiving a negative share.

2 Model

Let $N = \{1, 2, ..., n\}$ be set of agents. For $i \in N$, let e_i be the i^{th} unit vector in $\mathbb{R}^N_+$. We use the vector inequalities $\leqq, \leq, <$. For $x, y \in \mathbb{R}^N_+$, let $x \vee y = (\max\{x_i, y_i\})_{i \in N}$ and $x \wedge y = (\min\{x_i, y_i\})_{i \in N}$. Let Δ denote N dimensional simplex, and $int(\Delta)$ its relative interior . Let $d : \mathbb{R}^N \times \mathbb{R}^N \rightarrow \mathbb{R}_+$ denote the Euclidian distance in $\mathbb{R}^N$.

A simple allocation problem for N is a pair $(c, E) \in \mathbb{R}^N_+ \times \mathbb{R}_+$ such that $\sum_N c_i \geqq E$. Let E be the **endowment** and let c be the **characteristic vector**. Let $\mathcal{C}$ be the set of all simple allocation problems, and for all $(c, E) \in \mathcal{C}$, let $X(c, E) = \{x \in \mathbb{R}^N_+ : x \leqq c \text{ and} \sum_N x_i \leqq E\}$ be the **choice set of** (c, E). Let $bd(X(c, E)) = \{x \in \mathbb{R}^N_+ : x \leqq c \text{ and } \sum_N x_i = E\}$.

An allocation **rule** $F : \mathcal{C} \longrightarrow \mathbb{R}^N_+$ assigns each simple allocation problem (c, E) to an allocation $F(c, E) \in X(c, E)$ such that $\sum_N F_i(c,E) = E$.

Here are some well known families of rules. For $\alpha \in int(\Delta)$, the **weighted Proportional rule with weights $\boldsymbol{\alpha}$** allocates the endowment proportional to effective characteristics values of agents, $\alpha_i c_i$, and treats characteristic values of each agent as constraints: for all $i \in N$, $PRO^\alpha_i(c, E) = \min\{\delta\alpha_i c_i, c_i\}$, where $\delta \in \mathbb{R}$ satisfies $\sum_N \min\{\delta\alpha_i c_i, c_i\} = E$. For $\alpha \in int(\Delta)$, the **weighted Gains rule with weights $\boldsymbol{\alpha}$** allocates the endowment proportional to the given weights subject to the constraint that no agent receives more than her characteristic value: for all $i \in N$, $G^\alpha_i(c, E) = \min\{c_i, \delta\alpha_i\}$, where $\delta \in \mathbb{R}$ satisfies $\sum_N \min\{\delta\alpha_i, c_i\} = E$. For $\alpha \in int(\Delta)$, the **weighted Losses rule with weights $\boldsymbol{\alpha}$** equalizes the weighted losses of each agent, $\alpha_i(c_i - x_i)$, subject to the constraint that no agent receiving a negative share: for all $i \in N$, $L^\alpha_i(c, E) = \max\{c_i - \dfrac{\delta}{\alpha_i}, 0\}$ where δ satisfies $\sum_N \max\{c_i - \dfrac{\delta}{\alpha_i}, 0\} = E$. Note that when $\alpha_i = \frac{1}{n}$ for all $i \in N$, $PRO^\alpha = PRO$, **the proportional rule**, $G^\alpha = EG$, **the equal gains rule**, and $L^\alpha = EL$, **the equal losses rule.**

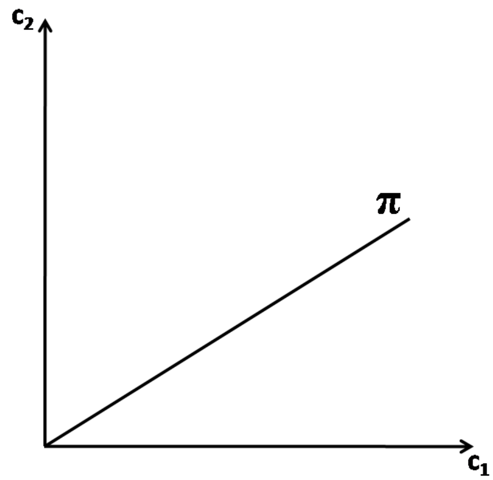

Figure 1: The partition defined in Example 1.

The **Talmud rule**, **applies** equal gains rule**,** until each agents receives half of his characteristics value and then applies equal losses rule :
$TAL_i(c, E) = EG(\frac{c}{2}, \frac{1}{2}\sum_{i \in N} c_i) + EL(\frac{c}{2}, \max\{0, E - \frac{1}{2}\sum_{i \in N} c_i\}.$

Let Π be an arbitrary partition of $\mathbb{R}_+^N$, and let $\pi \in \Pi$ be a member of the partition, that is a set of characteristic vectors. Characteristic vectors, c, c' that fall into the same element π of the partition (i.e. $c, c' \in \pi$) are considered to be similar to each other. That is the partition divides the space of characteristic values into equivalence (or similarity) classes. It is convenient here to introduce some examples of partitions.

Example 1 (Constant Proportion Partition)
Let $\Pi = \{\pi \subseteq \mathbb{R}_+^N \setminus \{0\}$: *for all* c, $c' \in \pi$, $c = \lambda c'$ *for some* $\lambda \in \mathbb{R}_{++}\} \cup \{0\}$. *This partition is constructed according to proportionality.*

Example 2 (Constant Sum Partition) *Let* $n = 2$. *The partition which divides*

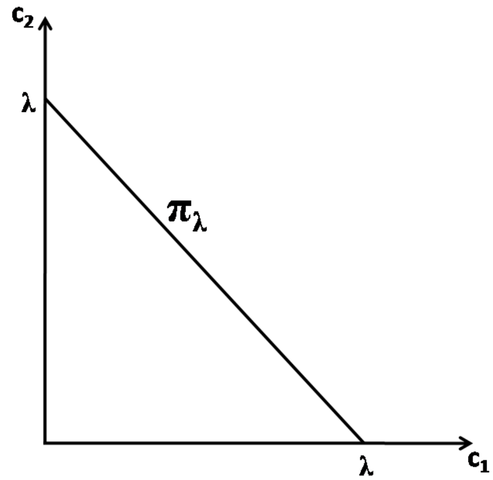

Figure 2: The partition defined in Example 2.

$\mathbb{R}^2_+$ *into hyperplanes of normal vectors:*
$\Pi = \{\pi_\lambda \subseteq \mathbb{R}^2_+ :$ *for each* $c \in \pi_\lambda$, $c_1 + c_2 = \lambda$ *for some* $\lambda \in \mathbb{R}_+\}$*.*

Example 3 (Full Partition)
Let $\Pi = \{\pi \subseteq \mathbb{R}^N_+ : \pi = \{c\}$ *for some* $c \in \mathbb{R}^N_+\}$ *is a partition with singleton sets. That is, each characteristic vector* c *is treated differently on* Π*. This partition is intimately related to weak rationality in Kıbrıs(2008).*

Example 4 (Singleton Partition)
Let $\Pi = \{\pi\}$ *be a partition with a single element,* $\pi = \mathbb{R}^N_+$*. That is, each characteristic vector* c *is treated similarly. This partition is intimately related to weak rationality in Kıbrıs(2008).*

Example 5 (Ordinal Partition)
Let $n = 2$*, and* $\Pi = \{\pi_1, \pi_2, \pi_3\}$ *where* $\pi_1 = \{c \in \mathbb{R}^2_+ : c_1 < c_2\}$,

$\pi_2 = \{c \in \mathbb{R}^2_+ : c_1 > c_2\}$, $\pi_3 = \{c \in \mathbb{R}^2_+ : c_1 = c_2\}$ *is the partition which has three elements. On this partition, the characteristic vectors, in which the ordering of the two characteristic values are the same, are treated similarly.*

Example 6 (Constant Distance Partition)
Let $\Pi = \{\pi \subseteq \mathbb{R}^2_+ : \text{for all } c, c' \in \pi,\ c = c' + \lambda e \text{ for some } \lambda \in \mathbb{R}_+\}$. *Here* c *and* c'*fall into the same* π *if the difference between the two characteristics values is the same. That is,* $c_1 - c_2 = c'_1 - c'_2$.

Let Π and Π' be two partitions. We say, Π is a **refinement of** Π' if each $\pi' \in \Pi'$ can be represented as arbitrary union of the elements in Π. That is, for each $\pi' \in \Pi'$, there exists a collection of sets, $\{\pi_\beta\}_{\beta \in B}$ in Π, such that $\pi' = \cup_{\beta \in B} \pi_\beta$.

Let $\mathcal{C}^\Pi(\pi) = \{(c, E) \in \mathcal{C} \mid c \in \pi\}$ be the set of problems, in which all of the characteristic vectors come from the set $\pi \in \Pi$.

Let $\mathcal{X}(\pi) = \{X(c, E) : (c, E) \in \mathcal{C}^\Pi(\pi)\}$ be the set of feasible allocations defined by the allocation problems in $\mathcal{C}^\Pi(\pi)$.

For a rule F, the **revealed preference relation induced by** F **on** π, $R^F(\pi) \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$ is defined as follows: $xR^F(\pi)\, y$ if and only if there is $(c, E) \in \mathcal{C}^\Pi(\pi)$ such that $x = F(c, E)$ and $y \in Xc, E)$. Similarly, the **strict revealed preference relation induced by** F **on** π is defined as follows; $xP^F(\pi)\, y$ if and only if there is $(c, E) \in \mathcal{C}^\Pi(\pi)$ such that $x = F(c, E)$ and $y \in X(c, E)$ and $x \neq y$.

Remark 1 *Note that all the partitions presented in* Example 1-3, Example 5 and Example 6 *are refinements of the partition presented in* Example 4*. The partition presented in* Example 3 *is a refinement of the partitions presented in* Example1-2, Example 4-5 and Example 6*.*

An allocation rule F is **rational on** Π, if for all $\pi \in \Pi$, there exists a binary relation $B(\pi) \subseteq \mathbb{R}_+^N \times \mathbb{R}_+^N$ such that for all $(c, E) \in \mathcal{C}^{\Pi}(\pi)$,

$$F(c, E) = \{x \in X(c, E) \mid \text{for all } y \in X(c, E),\ xB(\pi)y\}.$$

That is, F is **rational on a partition** Π, if and only if for each $\pi \in \Pi$, there exists a binary relation $B(\pi)$, such that the choices of the rule, coincide with the maximization of $B(\pi)$ on the allocation space.

A rule F is **transitive rational on** Π, if for all $\pi \in \Pi$, there exists a *transitive* binary relation $B(\pi) \subseteq \mathbb{R}_+^N \times \mathbb{R}_+^N$,such that for all $(c, E) \in \mathcal{C}^{\Pi}(\pi)$,

$$F(c, E) = \{x \in X(c, E) \mid \text{for all } y \in X(c, E),\ xB(\pi)y\}.$$

A rule F satisfies **WARP** (**the weak axiom of revealed preferences**) **on** Π if for all $\pi \in \Pi$, $P^F(\pi)$ is *asymmetric* (equivalently if $R^F(\pi)$ is *antisymmetric*).

Remark 2 ***WARP on*** Π *can equivalently be stated as follows: for all* $\pi \in \Pi$ *and for all pairs* $(c, E), (c', E) \in \mathcal{C}^{\Pi}(\pi)$, $F(c, E) \in X(c', E)$ *and* $F(c, E) \neq F(c', E)$ *implies* $F(c', E) \notin X(c, E)$.

We say a rule F satisfies **contraction independence on** Π, if for all $\pi \in \Pi$ and for all pairs (c, E), $(c', E) \in \mathcal{C}^{\Pi}(\pi)$, $F(c, E) \in X(c', E) \subseteq X(c, E)$ implies $F(c', E) = F(c, E)$.

A rule F satisfies **SARP** (**the strong axiom of revealed preferences**) **on** Π if for all $\pi \in \Pi$, $P^F(\pi)$ is *acyclic.*

A rule F satisfies **own-c monotonicity on** Π , if for each $(c, E), (c', E) \in \mathcal{C}^{\Pi}(\pi)$ and $i \in N$, such that $c_i < c'_i$ and $c_{N\setminus\{i\}} = c'_{N\setminus\{i\}}$, we have $F_i(c, E) \leq F_i(c', E)$.

A rule F satisfies **other-c monotonicity on** Π , if for each $(c, E) \in \mathcal{C}^{\Pi}(\pi)$, each $i \in N$ and each $c'_i \in \mathbb{R}_+$ such that $(c'_i, c_{-i}, E) \in \mathcal{C}^{\Pi}(\pi)$, and each $j, k \in N/\{i\}$, $F_j(c, E) > F_j(c'_i, c_{-i}, E)$ implies $F_k(c, E) \geq F_k(c'_i, c_{-i}, E)$.

3 Rationality, WARP, and Contraction Independence

In this section we first ask the following question. Given a partition Π' and a refinement of it, Π, if a rule satisfies a certain property on Π', does it also satisfy this property on Π?. The answer is affirmative for contraction independence, WARP and SARP and rationality.

Proposition 1 *Let F be a rule, Π, Π' be two partitions and let Π be refinement of Π'. Then,*
i) *If F is* contraction independent *on Π', then F is* contraction independent *on Π.*
ii) *If F satisfies* WARP *on Π', then F satisfies* WARP *on Π.*
iii) *If F satisfies* SARP *on Π', then F satisfies* SARP *on Π.*
iv)*If F is* rational *on Π', then F is* rational *on Π.*

Proof. Let Π be a refinement of Π'. Let $\pi \in \Pi$. Then, there exists $\pi' \in \Pi'$ such that $\pi \subseteq \pi'$.

(i) Let F be a contraction independent rule on Π'. We will show that F is contraction independent on Π. Let $\pi \in \Pi$, and $(c, E), (c', E) \in \mathcal{C}^{\Pi}(\pi)$, such that $F(c, E) \in X(c', E) \subseteq X(c, E)$. Since $\pi \subseteq \pi'$, $(c, E), (c', E) \in \mathcal{C}^{\Pi'}(\pi')$. Since F is contraction independent on Π', we have $F(c', E) = F(c, E)$.

(ii) Let F be a rule which satisfies WARP on Π'. We need to show $P^F(\pi)$ is asymmetric. Since $\pi \subseteq \pi'$, $P^F(\pi) \subseteq P^F(\pi')$. Since $P^F(\pi')$ is asymmetric, any subset of it is asymmetric. In particular, $P^F(\pi)$ is asymmetric.

(iii) Let F be a rule which satisfies SARP on Π'. We need to show $P^F(\pi)$ is acyclic. Since $\pi \subseteq \pi'$, $P^F(\pi) \subseteq P^F(\pi')$. Since $P^F(\pi')$ is acyclic, any subset of it is acyclic. In particular, $P^F(\pi)$ is acyclic.

(iv) Let F be a rule which is rational on Π'. Then for each $\pi' \in \Pi'$, there exists a binary relation $B(\pi') \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$, such that for each $(c, E) \in C^{\Pi'}(\pi')$, $F(c, E) = \arg\max_{X(c,E)} B(\pi')$. Since Π is a refinement of Π', there exists a collection of sets, $\{\pi_\beta\}_{\beta \in B}$ in Π, such that $\pi' = \cup_{\beta \in B} \pi_\beta$. Let $\pi \in \{\pi_\beta\}_{\beta \in B}$, and let $B(\pi) = B(\pi')$ for all $\pi_{\beta \in B}$. Then, for each $(c, E) \in C^{\Pi}(\pi)$, $F(c, E) = \arg\max_{X(c,E)} B(\pi)$, since $(c, E) \in C^{\Pi}(\pi)$ implies $(c, E) \in C^{\Pi'}(\pi')$ and $B(\pi) = B(\pi')$. Therefore, F is rational on Π. ■

Kıbrıs(2008) analyzed rationality, contraction independence, WARP and SARP for two extremes. One of this extremes is the singleton partition defined in *Example 4*. The other extreme is the the full partition introduced in *Example 3*. As a partition becomes more refined it becomes easier to satisfy these properties. For instance, since all the partitions are refinements of the singleton partition, if a rule satisfies any of these properties on the singleton partiton then it satisfies that property on any partition. Moreover Kıbrıs (2008) showed that on the full partition these properties are so easy to satisfy that every allocation rule satisfies these properties.

In this study we are analyzing these properties on the partitions which are in between these two extremes, which makes our analysis "intermediate".

Next, we analyze the logical connection between WARP on Π, contraction independence on Π, and rationality on Π. In Kıbrıs (2008), WARP on Π implies rationality on Π. We have a similar result here.

Proposition 2 *If a rule F satisfies* WARP *on Π then it is* rational *on Π.*

Proof. Let F be a rule which satisfies WARP on Π. Thus, $R^F(\pi)$ is anti-symmetric. We want to show that F is rational on Π. Let $\pi \in \Pi$, and let $B(\pi) = R^F(\pi)$. We will show that for any $(c, E) \in C^{\Pi}(\pi)$

$$F(c, E) = \left\{x \in X(c, E) \mid \text{for all } y \in X(c, E),\ xR^F(\pi)\, y\right\}.$$

Let $z \in \left\{x \in X(c,E) \mid \text{for all } y \in X(c,E),\ xR^F(\pi)\, y\right\}$. Hence, $zR^F(\pi)\, F(c,E)$. By definition of $R^F(\pi)$, we have $F(c,E)R^F(\pi)\, z$. Since $R^F(\pi)$ is antisymmetric we have, $F(c,E) = z$. ■

Unlike Kıbrıs(2008), the converse of *Proposition 2* is not true.

Proposition 3 *There are rules which are rational on a partition* Π*, but do not satisfy* WARP *on* Π*.*

Proof. Let $n = 2$. Let Π be Constant Sum Partition defined in *Example 2*. Let $B(\pi) \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$ be defined as follows;
If $\pi \neq \pi_{14}$, then $xB(\pi)y$, if and only if $x_1x_2 \geq y_1y_2$. [3]
If $\pi = \pi_{14}$, then ,

$$xB(\pi)y \iff \begin{cases} [\sum_N x_i \neq 10 \text{ or } \sum_N y_i \neq 10 \text{ and } x_1x_2 \geq y_1y_2] & \text{or} \\ [\sum_N x_i = \sum_N y_i = 10,\ x = (4,6) \text{ and } y \neq (8,2)] & \text{or} \\ [\sum_N x_i = \sum_N y_i = 10,\ x = (6,4),\ y_1 \geq 4] & \text{or} \\ [\sum_N x_i = \sum_N y_i = 10,\ x, y \notin \{(4,6),(6,4)\} \text{ and } x_1x_2 \geq y_1y_2] \end{cases}$$

Now let, $F(c,E) = \arg\max_{x \in X(c,E)} B(\pi)$. By construction, F is a rational rule on the partition Π. However, F violates WARP on Π. To see this, let $E = 10$, $c = (6,8)$, $c' \in (8,6)$. Note that $c, c' \in \pi_{14}$. Then, $F(c,E) = (4,6) \in X(c',E)$, $F(c',E) = (6,4) \in X(c,E)$, but $F(c',E) \neq F(c,E)$. ■

Rationality does not imply WARP on the constant sum partition. This observation suggests that for the equivalence of rationality and WARP on a partition it may be required that the elements of partition needs to possess some extra properties. It turns out that the constant sum partition do not satisfy the sufficient properties, we introduce later in this section, for the equivalence of these notions on a partition.

[3] Note that, $\pi_\lambda = \{c \in \mathbb{R}^2_+ \mid c_1 + c_2 = \lambda\}$

Proposition 4 *Let F be a rational rule on Π, then F satisfies contraction independence on Π.*

Proof. Let F be rational on Π. Since F is rational on Π, then for each $\pi \in \Pi$, there exists $B(\pi) \subseteq \mathbb{R}^N_+ \times \mathbb{R}^N_+$ such that for each $(c, E) \in \mathcal{C}^\Pi(\pi)$, $F(c, E) = \arg\max_{x \in X(c,E)} B(\pi)$. Let $(c, E), (c', E) \in \mathcal{C}^\Pi(\pi)$ such that $F(c, E) \in X(c', E) \subseteq X(c, E)$. Since $F(c, E)$ maximizes $B(\pi)$ on $X(c, E)$ it also maximizes $B(\pi)$ on $X(c', E)$. Hence $F(c, E) = F(c', E)$. ■

Given a partition Π There are rules, which are contraction independent on a partition Π, but violate rationality on Π (and thus, the logically stronger WARP on Π).

Example 7 *Let Π be the partition defined in* Example 2. *Let $F = PRO$. On Π, every rule is contraction independent, but PRO is not rational on that partition. To see this, consider π_{10}. Let $c = (5, 5)$, $c' = (4, 6)$ and $E = 3$. $PRO(c, E) = (1.5, 1.5) \neq (1.2, 1.8) = PRO(c', E)$. If PRO were to be rational on Π, since $PRO(c, E) \in X(c', E)$ and $c', c \in \pi_{10}$, $PRO(c, E)$ would be the choice in $X(c', E)$ as well. Therefore, PRO is not rational on Π. Since PRO is not rational on Π, it violates $WARP$ on Π, by* Proposition 1.

Given a partition Π, contraction independent rules on Π satisfy own-c monotonicity on Π as well. That is, if F is a contraction independent rule on Π, then an increase in the characteristics value of an agent does not decrease the share of that agent.

Lemma 1 *Let Π be a partition and let F be an allocation rule. If F satisfies contraction independent on Π, then F satisfies own-c monotonicity on Π.*

Proof. Assume that F is contraction independent on Π. Suppose for a contradiction that F violates own-c monotonicity on Π. Then there exist

$i \in N$, $\pi \in \Pi$, $(c, E), (c', E) \in \mathcal{C}^{\Pi}(\pi)$ such that $c' = (c'_i, c_{-i})$, $c'_i > c_i$ and $F_i(c', E) < F_i(c, E)$. However, since $c \leq c'$, by contraction independence of F on Π, $F(c, E) = F(c', E)$ which contradicts $F_i(c', E) < F_i(c', E)$. ■

If Π is the singleton partition in *Example 4*, rationality on Π, WARP on Π and contraction independence on Π are equivalent statements (Kıbrıs 2008). However, as we demonstrated above it is not generally true that rationality on a partition, WARP and contraction independence on that partition are equivalent statements. For this equivalence we need the partition we work on to posses certain properties.

Consider the following two properties.

Definition 1 *Let* Π *be a partition, and* $\pi \in \Pi$. *We say* π *is closed under coordinate-wise minimum operation,* $\wedge$ *if and only if for all* $c, \bar{c} \in \pi$, *we have* $c \wedge \bar{c} \in \pi$.

Definition 2 *Let* Π *be a partition, and* $\pi \in \Pi$. *We say* π *is closed under coordinate-wise maximum operation* $\vee$ *if and only if for all* $c, \bar{c} \in \pi$ *we have* $c \vee \bar{c} \in \pi$.

Remark 3 *Each element of the partition introduced in* Example 1*, is closed both under* $\wedge$ *and* $\vee$. *On the other hand, none of the elements of the partition introduced in* Example 2*, are closed under* $\wedge$ *or* $\vee$.

The following is the main result of this section.

Theorem 1 *Let* Π *be a partition such that for each* $\pi \in \Pi$, *either* π *is closed under coordinate-wise minimum or coordinate-wise maximum operations. Then the following are equivalent statements.*
i) *F is contraction independent on* Π.
ii) *F satisfies WARP on* Π.
iii) *F is rational on* Π.

Proof. We will show that **(i)** implies **(ii)**. Then by *Proposition 1* and *Proposition 2*, the result follows. Let $\pi \in \Pi$. Let F be a rule which satisfies contraction independence on Π and suppose for a contradiction that F does not satisfy WARP on Π. Then there exists $(c, E), (c', E) \in \mathcal{C}^{\Pi}(\pi)$ such that $F(c, E) \in X(c', E)$, $F(c, E) \neq F(c', E)$ and $F(c', E) \in X(c, E)$.

First assume π is closed under coordinate-wise minimum operation. Let $c'' = c \wedge c'$. Note that $F(c, E) \leq c$ and $F(c, E) \leq c'$. Thus, $E = \sum_N F_i(c, E) \leq \sum_N \min\{c_i, c'_i\}$. Therefore, $(c'', E) \in \mathcal{C}^{\Pi}(\pi)$. Also, $F(c, E) \leq c'' \leq c$. Then, $F(c, E) \in X(c'', E) \subseteq X(c, E)$. Thus, by contraction independence on Π, we have $F(c, E) = F(c'', E)$. Similarly, $F(c, E) \leq c'' \leq c'$ implies $F(c, E) \in X(c'', E) \subseteq X(c', E)$, which by contraction independence on Π implies $F(c', E) = F(c'', E)$. Therefore, $F(c'', E) = F(c, E) = F(c', E)$, which contradicts with $F(c, E) \neq F(c', E)$.

Now, assume that π is closed under coordinate-wise maximum operation. Let $c'' = c \vee c'$. Then, $(c'', E) \in \mathcal{C}^{\Pi}(\pi)$. We have either $F(c'', E) \leq c \leq c''$ or $F(c'', E) \leq c' \leq c''$. Without loss of generality, assume $F(c'', E) \leq c \leq c''$. Then $F(c'', E) \in X(c, E) \subseteq X(c'', E)$. Hence, by contraction independence on Π, we have $F(c'', E) = F(c, E)$. Since, $F(c'', E) \leq c' \leq c''$, which implies $F(c'', E) \in X(c, E) \subseteq X(c'', E)$. Applying contraction independence on Π, we get $F(c', E) = F(c'', E)$. Hence, we have $F(c'', E) = F(c, E) = F(c', E)$, which contradicts $F(c, E) \neq F(c', E)$. ■

If a domain of simple allocation problems is closed under set union then as a corollary of Hansson (1968), we have the equivalence of WARP on Π, contraction independence on Π, rationality on Π and SARP on Π. However, it is not generally true that, a domain of simple allocation problems is closed under set union. *Example 2* and *Example 4* present domains which are not closed under union. Moreover, closedness under coordinate-wise maximum operation does not imply a simple allocation problem being closed under set union.

Remark 4 *Note that* Example 4 *presents a* π *that is closed under* $\vee$*, but* $\mathcal{X}(\pi)$ *is not closed under union.*

Hansson notes without proof that WARP and contraction independence (*IIA in Hansson*) are equivalent on domains which are closed under intersection. The following proposition establishes the connection between closedness under intersection of a domain of simple allocation problems and closedness under $\wedge$ of π.

Proposition 5 *Let* Π *be a partition and* $\pi \in \Pi$*. Then* π *is closed under* $\wedge$ *if and only if* $\mathcal{X}(\pi)$ *is closed under intersection.*

Proof. "$\Rightarrow$"

Let π be closed under $\wedge$, $c, \bar{c} \in \pi$, and $c' = c \wedge \bar{c}$. Let (c, E), $(\bar{c}, \bar{E}) \in \mathcal{C}^{\Pi}(\pi)$. Then $X(c, E), X(\bar{c}, \bar{E}) \in \mathcal{X}(\pi)$. We want to show, $X(c, E) \cap X(\bar{c}, \bar{E}) \in \mathcal{X}(\pi)$. Let $E' = \min\{E, \bar{E}, \sum_N c_i'\}$. By definition, we have, $\sum c_i' \geq E'$ and $X(c, E) \cap X(\bar{c}, \bar{E}) = X(c', E')$. Since, $c' \in \pi$, we have $(c \wedge \bar{c}, E') \in \mathcal{C}^{\Pi}(\pi)$. Hence, we have $X(c', E') \in \mathcal{X}(\pi)$.

"$\Leftarrow$"

Assume that $\mathcal{X}(\pi)$ is closed under intersection. Let c, $\bar{c} \in \pi$ and $E, \bar{E}$ be such that $(c, E), (\bar{c}, \bar{E}) \in \mathcal{C}^{\Pi}(\pi)$. Then we have $X(c, E) \cap X(\bar{c}, \bar{E}) \in \mathcal{X}(\pi)$. Let $c' = c \wedge \bar{c}$..Thus, $X(c, E) \cap X(\bar{c}, \bar{E}) = X(c', E') \in \mathcal{X}(\pi)$. By definition of $\mathcal{C}^{\Pi}(\pi)$ and $\mathcal{X}(\pi)$ we have $c, \bar{c} \in \pi$ and $c \wedge \bar{c} \in \pi$. That is, π is closed under coordinate-wise minimum operation. ■

4 Transitive Rationality (Two Agents)

In this section, we analyze the properties of transitive rational rules for two agent case. The main result of this section is when there are only two agents, for any partition Π, *WARP on* Π and *SARP on* Π are equivalent statements. Hence, given a rule F which satisfies *WARP on* Π, we can conclude that F is *transitive rational on* Π.

Lemma 2 *Let* Π *be a partition. Let* $n = 2$ *and assume* F *satisfies WARP on* Π*. Then for any* $\pi \in \Pi$*, there does not exist* $x, y, z \in \mathbb{R}^2_+$ *such that* $xP^F(\pi)$*,* $yP^F(\pi)z$ *and* $zP^F(\pi)x$*.*

Proof. Let F be a rule which satisfies WARP on Π, and suppose for a contradiction that there exists a cycle of size three. Then there exists $x, y, z \in \mathbb{R}^2_+$ and $\pi \in \Pi$ such that $xP^F(\pi)y$, $yP^F(\pi)z$ and $zP^F(\pi)x$. Note that, $x \neq y$, $y \neq z$ and $x \neq z$ since $P^F(\pi)$ is asymmetric. By definition of $P^F(\pi)$ there exists $(c^{xy}, E^{xy}), (c^{yz}, E^{yz}), (c^{zx}, E^{zx}) \in \mathcal{C}^{\Pi}(\pi)$ such that $x = F(c^{xy}, E^{xy}), y = F(c^{yz}, E^{yz})$ and $z = F(c^{zx}, E^{zx})$. We have

$$\sum_N x_i \geq \sum_N y_i = E^{yz} \geq \sum_N z_i = E^{zx} \geq \sum_N x_i = E^{xy}$$

Then, $E^{xy} = E^{yz} = E^{xz} = E$. Without loss of generality, assume that $x_1 < y_1$.
Case 1: $z_1 < x_1$. Then, $y = F(c^{yz}, E)$ and $z \leq c^{yz}$. However, since $x_1 < y_1$, $y = F(c^{yz}, E)$ and $x \leq c^{yz}$,which implies $yP^F(\pi)x$, which contradicts with $P^F(\pi)$ is asymmetric.
Case 2: $x_1 < z_1 < y_1$. Then $x = F(c^{xy}, E)$ and $x \leq c^{yz}$. However, since $z_1 < x_1$, $x = F(c^{xy}, E)$ and $z \leq c^{xy}$,which implies $xP^F(\pi)z$, which contradicts with $P^F(\pi)$ is asymmetric.
Case 3: $y_1 < z_1$. Then $z = F(c^{xz}, E)$ and $y \leq c^{yz}$. But then, since $y_1 < z_1$, $z = F(c^{zx}, E)$ and $y \leq c^{zx}$,which implies $zP^F(\pi)y$, which contradicts with $P^F(\pi)$ is asymmetric. ∎

For Π in *Example 4*, Kıbrıs (2008) shows that for two agents and a rule F which satisfies WARP on Π, $P^F(\pi)$ is transitive. However, it may not be true that for any Π, and $\pi \in \Pi$ $P^F(\pi)$ is transitive. We show that for any partitions Π, for a rule which satisfies WARP on Π, $P^F(\pi)$ is acyclic..

Theorem 2 *Let* Π *be a partition,* $n = 2$*. If* F *satisfies WARP on* Π*, then* F *satisfies SARP on* Π*.*

Proof. Let F be a rule which satisfies WARP on Π . Let $\pi \in \Pi$. We want to show $P^F(\pi)$ is acyclic. We will show this by induction on the size of a possible cycle. By *Lemma 2*, there does not exists a cycle of size three. Next, assume that there does not exist a cycle of size $k-1$ and less. Suppose for a contradiction that there exists a cycle of size k. That is, there exists $x^1, ..., x^k$ such that $x^1 P^F(\pi) x^2 P^F(\pi) ... P^F(\pi) x^k P^F(\pi) x^1$. Then there exist $(c^{1,2}, E^{1,2}), (c^{2,3}, E^{2,3}), ..., (c^{k,1}, E^{k,1}) \in \mathcal{C}^{\Pi}(\pi)$, such that $x^1 = F(c^{1,2}, E^{1,2})$, $x^2 = F(c^{2,3}, E^{2,3}), ..., x^k = F(c^{k,1}, E^{k,1})$. Note that

$$E^{1,2} = \sum_N x_i^1 \geq \sum_N x_i^2 = E^{2,3} \geq ..... \geq \sum_N x_i^k = E^{k,1} \geq \sum_N x_i^1 = E^{1,2}.$$

Hence, we have $E^{1,2} = E^{2,3} = ... = E^{k,1} = E$. Moreover, by asymmetry of $P^F(\pi)$, and the assumption that there does not exist a cycle of size $k-1$ and less, for all $i, j \in \{1, ..., k\}$ and $i \neq j$ we have $x^i \neq x^j$. Let x^l be the allocation in which the share of agent 1 is minimum. Without loss of generality, assume that $1 < l < k$. Then, either $x_1^{l+1} < x_1^{l-1}$ or $x_1^{l-1} < x_1^{l+1}$.

Case1: $x_1^{l+1} < x_1^{l-1}$. Then there exists $(c^{l-1,l}, E)$ such that $x^{l-1} = F(c^{l-1,l}, E)$. Note that $x^{l+1} \in X(c^{l-1,l}, E)$. Then $x^{l-1} P^F(\pi) x^{l+1} P^F(\pi) x^{l+2} ... x^{l-2} P^F(\pi) x^{l-1}$, which is a cycle with size $k-1$. This contradicts with the assumption that there does not exist a cycle of size $k-1$ and less.

Case2: $x_1^{l-1} < x_1^{l+1}$.Then there exists $(c^{l,l+1}, E)$ such that $x^l = F(c^{l,l+1}, E)$. Note that $x^{l-1} \in X(c^{l,l+1}, E)$. Then, by definition of $P^F(\pi)$, $x^l P^F(\pi) x^{l-1}$, which contradicts $P^F(\pi)$ being asymmetric. ■

5 Transitive Rationality (n Agents)

For two agent case we showed that WARP on Π and SARP on Π are equivalent statements. However, it is not true when there are more than two agents in a simple allocation problem. The following is an example of a rule F which satisfies WARP on Π, but violates SARP on Π.

Example 8 *Let $n = 3$, and Π be the partition defined in* Example 4. *Let F be a rule defined as follows;*

$$F(c,E) = \begin{cases} (\frac{E}{3}, \frac{E}{3}, \frac{E}{3}) & \textit{if} \quad (\frac{E}{3}, \frac{E}{3}, \frac{E}{3}) \leq c \\ (c_1, c_1, E - 2c_1) & \textit{else if} \quad c_1 < \frac{E}{3} \textit{ and } (c_1, c_1, E - 2c_1) \leq c \\ (E - 2c_2, c_2, c_2) & \textit{else if} \quad c_2 < \frac{E}{3} \textit{ and } (E - 2c_2, c_2, c_2) \leq c \\ (c_3, E - 2c_3, c_3) & \textit{else if} \quad c_3 < \frac{E}{3} \textit{ and } (c_3, E - 2c_3, c_3) \leq c \\ (c_1, c_2, E - c_1 - c_2) & \textit{else if} \quad E - c_1 - c_2 > c_2 \textit{ and } c_1 > c_2 \\ (c_1, E - c_1 - c_3, c_3) & \textit{else if} \quad E - c_1 - c_3 > c_1 \textit{ and } c_3 > c_1 \\ (E - c_2 - c_3, c_2, c_3) & \textit{else if} \quad E - c_2 - c_3 > c_3 \textit{ and } c_2 > c_3 \end{cases}$$

Note that $\Pi = \{\pi\}$, with $\pi = \mathbb{R}^3_+$. Then, F satisfies WARP on Π, but it violates SARP on Π. To see this, let $E = 9$, $c^1 = (1, 9, 9)$, $c^2 = (9, 1, 9)$, and $c^3 = (9, 9, 1)$, $x = (1, 1, 7)$, $y = (7, 1, 1)$, $z = (1, 7, 1)$. Then $F(c^1, E) = x$, $F(c^2, E) = y$ and $F(c^3, E) = z$. Since $x \leq c^2$, $y \leq c^3$, and $z \leq c^1$, we have, $xP^F(\pi)zP^F(\pi)yP^F(\pi)x$.

Proposition 6 *Let Π be a partition for which each $\pi \in \Pi$ the following property.*
property (i) *for each $c, c' \in \pi$ either $c \leq c'$ or $c' \leq c$.*
Then, if F is a rule which satisfies WARP on Π, then F satisfies SARP on Π.

Proof. Let $\pi \in \Pi$ and assume that π satisfies property *(i)*. Assume also that F satisfies WARP on Π. Suppose for a contradiction that F violates SARP on

Π. Then $P^F(\pi)$ is asymmetric but not acyclic. That is, there exist $x^1, x^2, ..., x^l$ such that
$x^1 P^F(\pi) x^2 P^F(\pi) x^3 .... x^l P^F(\pi) x^1$. Then, there exist $(c^1, E), (c^2, E), ...(c^{l-1}, E)$, $(c^l, E) \in C^\Pi(\pi)$ such that $x^1 = F(c^1, E)$, $x^2 = F(c^2, E), ..., x^{l-1} = F(c^{l-1}, E)$ and $x^l = F(c^l, E)$. Note that for each $1 \leq i \leq l$ we have $c^i \in \pi$. Note also that for each $i, j \in \{1, ..., l\}$ we have either $c^i \leq c^j$ or $c^j \leq c^i$, which implies $X(c^i, E) \subseteq X(c^j, E)$ or $X(c^j, E) \subseteq X(c^i, E)$. Let $c^m = c^1 \vee c^2 \vee .... \vee c^l$. Then for each i such that $1 \leq i \leq l$, we have $X(c^i, E) \subseteq X(c^m, E)$. Without loss of generality, assume $1 < m < l$. Then $x^{m-1} \in X(c^m, E)$ and $x^m = F(c^m, E)$. Therefore, $x^m P^F(\pi) x^{m-1}$ which contradicts with $P^F(\pi)$ being asymmetric. ■

6 Alternative Rationality Notions

In this section we will define two new rationality notions. The first one is constant-proportion rationality, and the second one is constant-distance rationality.

6.1 Constant-Proportion Rationality

Let $\Pi^{CPRO} = \{\pi \subseteq \mathbb{R}^N_+ \setminus \{0\} : \text{for all } c, c' \in \pi, c = \lambda c' \text{ for some } \lambda \in \mathbb{R}_{++}\} \cup \{0\}$.

Definition 3 *A rule F satisfies* constant-proportion rationality, *if it is* rational *on Π^{CPRO}.*

Remark 5 *Note that since Π^{CPRO} is closed under $\wedge$ and $\vee$. Therefore, the results we obtained in* Section 3 *hold for constant-proportion rationality. Moreover, the characteristics vectors falling into the same element of Π^{CPRO} are ordered. Therefore, WARP on Π^{CPRO} and SARP on Π^{CPRO} are equivalent.*

Definition 4 *We say F satisfies constant-proportion contraction independence if F satisfies contraction independence on Π^{CPRO}.*

Let $int(\Delta^N)$ denote the interior of N dimensional simplex.

Proposition 7 *Let $\alpha \in int(\Delta)$. Then Proportional rule with weights α,* PRO^{α} *satisfies* constant-proportion rationality.

Proof. Let $\pi \in \Pi^{CPRO}$ and $(c, E), (c', E) \in \mathcal{C}^{\Pi}(\pi)$ such that $c = \lambda c'$ for some $\lambda \in \mathbb{R}_+$. We will show that, PRO^{α} satisfies constant-proportion contraction independence. We have $PRO_i^{\alpha}\ (c, E) = \min\{\delta\alpha_i c_i, c_i\}$ for all $i \in N$, and $\delta \in \mathbb{R}_+$ is such that $\sum_N \min\{\delta\alpha_i c_i, c_i\} = E$. Assume that $PRO^{\alpha}(c, E) \leq c' < c$ We will show that, $PRO^{\alpha}\ (c', E) = PRO^{\alpha}\ (c, E)$. First note that, for all $i \in N$, $PRO_i^{\alpha}\ (c, E) \leq c_i' < c_i$. Hence, we have for all $i \in N$, $PRO_i^{\alpha}\ (c, E) = \delta\alpha_i c_i$. Now, we will show that, for all $i \in N$, $PRO_i^{\alpha}\ (c', E) = \delta'\alpha_i c_i'$, with $\sum_N \delta'\alpha_i c_i' = E = \sum_N \min\{\delta\alpha_i c_i, c_i\} = \sum_N \delta\alpha_i c_i$, which implies $\delta' = \dfrac{\delta}{\lambda}$.Suppose for a contradiction that there exists $j \in N$, such that $c_j' = PRO_j^{\alpha}\ (c', E) > \delta'\alpha_j c_j'$. That is, $c_j' = \lambda c_j < \lambda\delta'\alpha_j c_j$. Then we have, $\delta'\alpha_j c_j > c_j > \delta\alpha_j c_j \geq c_j'$. Without loss of generality assume also that there is only one such j. Then we have, $\sum_{i \neq j} PRO_i^{\alpha}(c', E) + c_j' = E = \sum_{i \neq j} PRO_i(c, E) + c_j > \sum_{i \in N} PRO_i(c, E) = E$, a clear contradiction. ■

Remark 6 *The proportional rule,* PRO *is a member of* PRO^{α}*, with symmetric weights. (i.e. for all $i \in N$, $\alpha_i = \dfrac{1}{N}$). This rule also satisfies constant-proportion rationality.*

Remark 7 *Equal Gains rule, EG, satisfies constant proportion contraction independence, since it satisfies contraction independence.*

Example 9 *Equal Losses rule,(EL), and Talmud Rule (TAL) do not satisfy constant-proportion contraction independence.*

Let $E = 50$, $N = \{1, 2\}$, and $c' = (20, 60)$, $c = (30, 90)$, $EL(c, E) = (0, 50) \in X(c', E) \subseteq X(c, E)$, and $EL(c', E) = (5, 45) \neq EL(c', E)$. That is, EL does not satisfy constant proportion contraction independence. For Talmud Rule we get, $TAL(c, E) = (15, 35) \in X(c', E) \subseteq X(c, E)$.However, $TAL(c', E) = (10, 40) \neq TAL(c, E)$. That is, TAL does not satisfy constant proportion contraction independence. Therefore it does not satisfy constant proportion rationality.

The rules which satisfy constant-proportion rationality given in the examples so far are continuous rules. There are discontinuous rules, which satisfies constant-proportion rationality as well.

Example 10 *Let F be a rule defined as follows;*

$$F_i(c, E) = \begin{cases} \frac{E}{N} & \text{if } \frac{E}{N} \in X(c, E) \\ PRO(c, E) & \text{otherwise} \end{cases} \quad \text{for all } i \in N$$

Let $\pi \in \Pi^{CPRO}$, and $c, c' \in \pi$, and suppose that $c' \leqq c$. Hence $X(c', E) \subseteq X(c, E)$. The rule, F satisfies constant proportion contraction independence, since as long as we have equal division is feasible for every one, F allocates endowment equally. If equal division is not available in $X(c', E)$, then, we have $F(c, E) \in X(c', E) \subseteq X(c, E) \implies F(c', E) = F(c, E)$,since $F(c, E) = PRO(c, E)$, and PRO satisfies constant-proportion contraction independence.

The following proposition, characterizes the rules which are continuous and constant proportionally rational.

Proposition 8 *Let $n = 2$. A rule F is continuous and constant-proportion contraction independent if and only if for each $\pi \in \Pi^{CPRO}$, there exists a continuous function $r(.; \pi) : \mathbb{R}_+ \mapsto \mathbb{R}^N_+$ such that $F(c, E) = \arg\min_{x \in bd(X(c,E))} d(x, r(E; \pi))$ and $r(E; \pi)$ is continuously changing with the angle θ that π makes with the x_1 axis.*

Proof. " $\Rightarrow$ "

Let F be a continuous and constant proportion contraction independent rule, and $\pi \in \Pi^{CPRO}$. For each $E \in \mathbb{R}_+$ let $c_E \in \pi$ be such that
$c_E = \inf\{c \in \pi : c_i \geq E$, for all $i \in N\}$, and let $r(E;\pi) = F(c_E, E)$. Since F is a continuous in E $r(E;\pi)$ is continuous in E and continuously changes with the angle θ, since F is continuous in c. Now we want to show that, for each $(c, E) \in C^{\Pi}(\pi)$, $F(c, E) = \arg\min_{x \in bd(X(c,E))} d(x, r(E;\pi))$.
Let $(c, E) \in C^{\Pi}(\pi)$ be given. If $c \geq c_E$, then by constant-proportion contraction independence $F(c, E) = F(c_E, E) = r(E;\pi)$. If $c < c_E$ and $F(c_E, E) \in X(c, E)$, we have $F(c, E) = r(E;\pi)$, by constant proportion contraction independence of F. Without loss of generality assume that $F_1(c_E, E) \geq c_1$. If $F(c_E, E) \notin X(c, E)$ then by continuity and constant proportion contraction independence of F, we have $F_1(c, E) = c_1$. Hence, $F(c, E) = \min\{c_1, E - c_1)$, which is the smallest distance to $F(c_E, E) = r(E;\pi)$. Therefore, for all cases we can write, $F(c, E) = \arg\min_{x \in bd(X(c,E))} d(x, r(E;\pi))$.

" $\Leftarrow$ "

Assume that for each $\pi \in \Pi^{CPRO}$ there exist sa continuous function
$r(E;\pi) : \mathbb{R}_+ \to \mathbb{R}^N_+$ such that $F(c, E) = \arg\min_{x \in bd(X(c,E))} d(x, r(E;\pi))$ and $r(E;\pi)$ is continuously changing with the angle θ. By construction F is continuous. We want to show, F is constant proportion contraction independent, as well. Let E be given and $c, c' \in \pi$ be such that $F(c', E) \in X(c, E) \subseteq X(c', E)$. By definition of F, we have $F(c, E) = F(c', E)$. To see this, let $x^* \in bd(X(c, E)) \subseteq bd(X(c', E))$ be the minimizer of $d(x, r(E;\pi))$ on $bd(X(c', E))$. Then x^* is also the minimizer of it on $bd(X(c, E))$. ■

6.2 Constant-Distance Rationality

Definition 5 *A rule F satisfies* constant-distance rationality,*if it is* rational *on $\Pi^{CD} = \{\pi \subseteq \mathbb{R}^N_+ :$ for all $c, c' \in \pi$, $c = c' + \lambda e$ for some $\lambda \in \mathbb{R}^N_+\}$.*

Remark 8 *Note that since* Π^{CD} *is closed under* $\wedge$ *and* $\vee$. *Therefore, the results we obtained in* Section 3 *hold for constant-proportion rationality. Moreover, the characteristics vectors falling into the same element of* Π^{CPRO} *are ordered. Therefore, WARP on* Π^{CD}*and SARP on* Π^{CD} *are equivalent.*

Proposition 9 *Let* $\alpha \in int(\Delta^N)$*.Then weighted losses rule with weights,*α*,* L^{α} *satisfies* constant-distance rationality*.*

Proof. The proposition can be proved with a similar argument of the proof of Proposition 8. ■

7 Concluding Remarks and Open Questions

In this study we focused on the analysis of intermediate notions of rationality for simple allocation problems. We presented two alternative rationality notions, constant proportion and constant distance rationality. Our results generalizes the previous results introduced in Kıbrıs (2008). We have four possible extensions for this research.

First extension is to weaken the sufficient conditions we imposed for the equivalence of rationality and contraction independence on a partition. Since it is easy to come up with partitions which are not closed under coordinate wise minimum or maximum operations, such an extension would be very useful for the analysis of a partition in terms of rationality. However, we want to note that the possibility set of such a weakening is so huge. Because, we are looking for a property to impose on a partition among any properties.

The second possible extension is to weaken the assumption (if possible, coming up with a necessary condition) imposed on a partition which makes WARP and SARP on that partition equivalent statements. As it is the case for the first extension, dealing with artbitary properties would possibly make it hard to come up with necessary conditions. However, a weaker sufficient condition may not be that hard to find.

Third extension is to analyze the representability of a rule by a function on a partition. That way, one can easily represent the choices of decision maker by a function and simplify her analysis. Such an extension would also be useful because it contributes to the literature by generalizing the result of Kıbrıs (2008) on representability.

Fourth and last extension we think of is characterizing the allocation rules which satisfies the new rationality notions we introduced for population size $|N| \geq 3$.

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