Optimality of Linearity with Collusion and Renegotiation*

Mehmet Barlo† Ayça Özdoğan‡

Sabancı University TOBB–ETU

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Abstract

This study analyzes a continuous–time N–agent Brownian moral hazard model with constant absolute risk aversion (CARA) utilities, in which agents' actions jointly determine the mean and the variance of the outcome process. In order to give a theoretical justification for the use of linear contracts, as in Holmstrom and Milgrom (1987), we consider a variant of its generalization given by Sung (1995), into which collusion and renegotiation possibilities among agents are incorporated. In this model, we prove that there exists a linear and stationary optimal compensation scheme which is also immune to collusion and renegotiation.

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†FASS, Sabancı University, Tuzla, Istanbul, 34956, Turkey; +90 216 483 9284; barlo@sabanciuniv.edu

‡**Corresponding Author**: TOBB University of Economics and Technology, Department of Economics, Söğütözü Cad. No:43 Söğütözü, Ankara, 06560, Turkey; +90 312 292 4543; aozdogan@etu.edu.tr

1 Introduction

We analyze contracting between a principal and a team of agents, where the outcome process is governed by a Brownian motion. Agents have CARA utilities and jointly determine the drift and diffusion rates. Each of them can observe others' behavior and exploit any collusion and renegotiation opportunities at every instant via enforceable side–contracts contingent on effort levels and realized outcomes. We establish a theoretical justification for the use of linear contracts by proving that there are optimal stationary and linear sharing rules that are immune to collusion and renegotiation.[1] Thus, it is as if agents were to choose the mean and variance only once and the principal were restricted to employ stationary and linear sharing rules.

Agents' ability to observe and verify others' actions and their knowledge of how each one of them affects the mean and variance as well as how these contribute to their costs bring about collusion and renegotiation concerns.[2] These, in turn, imply that agents' agreements have to be efficient. Alternatively, they have to solve a utilitarian bargaining in every date and state. The principal who cannot observe or verify agents' behavior only knows that agents' bargaining (induced by her own offer) must result in an efficient outcome. Hence, the optimal contract she offers (i.e. individually rational sharing rules and control laws, drift and diffusion rates) must solve in every date and state agents' bargaining problem for some bargaining weights. Efficiency with CARA preferences delivers a useful aggregation result which we employ to establish that the principal can contract with the team as if she is contracting with a representative agent having CARA preferences, because we prove the following: Given optimal control laws for the drift and diffusion rates and an optimal compensation for the team, agents' compensations obtained from the efficient distribution of team's compensations employing the stationary bargaining weights that are stated in our main result and

[1] Contacts generally have simpler forms (such as linear) compared to the ones predicted by the theory. As far as empirical evidence is concerned, Lafontaine (1992) reports that "franchise contracts generally involve the payment, from the franchisee to the franchisor, of a lump–sum franchise fee as well as a proportion of sales in royalties, with the latter usually constant over all sales levels." And, Slade (1996) notes that only linear contracts are used by the oil companies engaged in franchising in retail–gasoline markets in Vancouver.

[2] This formulation suits cases in which agents are better informed than the principal about the managerial details and interim outcomes of the project. This can occur when the principal does not have the necessary technical training (e.g., lacking the expertise to operate a nuclear power plant) to deal with the associated details which agents (well trained in nuclear physics and details about how to operate that power plant) are supposed to be fluent with in the first place. Or, when she is far away (e.g., in another country) from the agents (working in an overseas factory producing a technical product) and information technologies are not sufficient (possibly due to language barriers) so that the principal has to base her contract only on the final output, while agents working together (and speaking the same language) can observe and verify others' choices.

the very same control laws, also solve agents' bargaining problem with their "real" date and state specific bargaining weights starting every date and state. Therefore, the fact that these are not necessarily agents' real bargaining weights (that the principal is not necessarily aware of) turns out not to be important. Because that, now, the principal is contracting with a representative agent having CARA preferences, her problem can be analyzed using techniques in Sung (1995) which enable us to obtain that there is an optimal and stationary linear contract for the team. As linearity is preserved during the corresponding efficient redistribution of team's compensation to agents, our main result is established.[3]

Holmstrom and Milgrom (1987), the pioneer work displaying the optimality of linear contracts in a repeated agency setting with exponential utilities, considers a principal–agent pair where the agent determines the drift rate of a Brownian motion.[4] Schättler and Sung (1993) generalizes the continuous–time principal–agent problem with exponential utility to a larger class of stochastic processes.[5] The key restriction in these two models is that the agent is not allowed to control the variance of the outcome process. Sung (1995) extends Holmstrom and Milgrom (1987)'s Brownian model to the case where the agent can also control the diffusion rate of the Brownian motion. The resulting problem becomes similar to that in Holmstrom and Milgrom (1987) with an additional time–state independent constraint and he proves that the linearity in outcome result holds. Koo, Shim, and Sung (2008), on the other hand, present a continuous–time principal–agent model under moral hazard with many agents. Their model is a continuous–time counterpart of Holmstrom (1982) and an extension of Holmstrom and Milgrom (1987) with a team of N agents. The principal has N production tasks one for each agent who cannot observe each other. They show that optimal contracts are also linear in all outcomes (produced separately by each agent). For their linearity

[3]We thank an anonymous referee for pointing out that our analysis can be associated with bonus pools in investment banks. Our results indicate that optimality calls for the shareholders of an investment bank to use a contract in which employees' compensations are composed of (1) an employee specific part that is constant regarding the bank's profits, and (2) the access to one internal bonus pool, which is equal to an optimal fraction of the profits. An agent's share in this pool is given by a fraction that only depends on employees' CARA coefficients, and the bonus system treats two equally risk averse employees equally (even though they were to have different outside opportunities and costs, and these differences would be reflected by the constant part of their compensations). So, when each employee has the same CARA coefficient, each agent gets an equal share from the bonus pool.

[4]Lack of income effects with exponential utilities and time–state independent cost functions, imply that the optimal control the agent chooses is time–state independent. Stationarity of the environment implies that among all possible compensation schemes, an optimal one is stationary and linear in the final output.

[5]Schättler and Sung (1993) use martingale methods to derive necessary conditions for optimality of the agent's problem and provide sufficiency conditions for the validity of this first–order approach.

result, the formulation involving the simultaneous–move game played by agents is important to preserve stationary decision making environment for the principal. It needs to be emphasized that our model does not feature separate production processes, and our agents can perfectly observe each other and can engage in renegotiable side–contracting.

The paper is organized as follows. Section 2 presents the model and the principal's problem. Section 3 presents the main result and its proof.

2 Model and Preliminaries

The principal and N agents interact over time interval $t \in [0,1]$. At an instant t, agent $i \in N \equiv \{1,\ldots,N\}$ chooses an effort level $e_t^i \in E_i$, E_i a compact interval, and these choices are observable and verifiable by all the other agents, but not the principal. The probability space is given by $(\Omega, \mathcal{F}, P)$ where Ω is the space $C = C([0,1])$ of all continuous functions on the interval $[0,1]$ with values in $\Re$. So, a particular event $w \in \Omega$ is of the form $w : [0,1] \to \Re$. The effort choices $e : [0,1] \to \times_{i\in N} E_i$, where $e_t = (e_t^i)_{i\in N}$, imply control laws μ and σ which are assumed to be $\mathcal{F}_t$–predictable mappings, $\mu : [0,1] \times \Omega \to U$ and $\sigma : [0,1] \times \Omega \to \mathbb{S}$, where U is a bounded open subset of $\Re$ and $\mathbb{S}$ is a compact subset of $\Re_{++}$. Controls μ and σ determine the instantaneous drift, μ_t, and diffusion rates, σ_t, of a stochastic process, $\{X_t\}_t$, governed by a Brownian motion defined by $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t$. Indeed, $\mu_t \equiv \mu(t,X)$ and $\sigma_t \equiv \sigma(t,X)$.[6]

The intermediate outcome X_t should be thought of as the total returns up to period $t \in [0,1]$, and B_t is the standard Wiener process. The drift and diffusion rates and the intermediate accumulated returns are neither observable nor verifiable by the principal. However, X_1, the level of accumulated returns at the end of the project, is both observable and verifiable by the principal. At the beginning of the project, the principal and agents agree upon a contract, i.e. salary rules $(S_i)_{i\in N}$ with $S_i : \Omega \to \Re$ for all $i \in N$ and control laws (μ, σ) with the restriction that salaries are payable at the end of the project according to the rules agreed upon at time 0 which depend only on X_1.[7]

[6]We assume σ satisfies a uniform Lipschitz condition: There exists a constant K such that for $Z, \bar{Z} \in C[0,1]$, $|\sigma(t,Z) - \sigma(t,\bar{Z})| \leq K \sup_{0\leq s\leq t} |Z(s) - \bar{Z}(s)|$. Even though this condition may be weakened (as was suggested by an anonymous referee) by noticing that our process is one dimensional and by employing Revuz and Yor (1999, Theorem 3.5, p.390; Exercises 3.13-14, p.397) (while it would still hold for the optimal contract), we use this Lipschitz condition (so, Revuz and Yor (1999, Theorem 2.1, p.375)) in order to have a parallel presentation with Sung (1995).

[7]This formulation is consistent with our hypothesis of the mean and variance being unobservable and nonverifiable by the principal. If $(S_i)_{i\in N}$ were to depend on the entire process $\{X_t\}_t$, implying the requirement that $\{X_t\}_t$ is both

Agents' instantaneous time–state independent cost functions are given by $c_i(\mu_t, \sigma_t)$, where $c_i : U \times \mathbb{S} \to \Re$, $i \in N$, and it is assumed to be twice continuously differentiable. We assume c_i and $c_{i\mu}$ (derivative with respect to mean) are bounded, and both $c_{i\mu}$ and $c_{i\mu\mu}$ (second derivative with respect to mean) are strictly positive. The total costs incurred by agent $i \in N$ is given by $\int_0^1 c_i(\mu_t, \sigma_t)\mathrm{d}t$.

This formulation handles situations where agents' costs depend on the mean and variance of returns. Thus, there exists an interaction effect on the two moments of the outcome process, and on the costs of agents. Yet, it also handles the standard environment with two agents in which one agent determines only the mean and the other agent only the variance, and the interaction effect on the costs is assumed to be minimal.

All have CARA utilities where coefficients of the principal and agent $i \in N$ are given by R and r_i, respectively. The reservation certainty equivalent figures for the agents are given by W_{i0}, $i \in N$. We assume that at each $t \in [0, 1]$, agents observe $\{X_s, \mu_s, \sigma_s, (e^i_s)_{i \in N}\}_{s \leq t}$. Agent i's expected continuation utility at time t given S_i and (μ, σ) (computed with the information at time t) is $V_{it} \equiv E\left[-\exp\left\{-r_i W_i^S(X; \mu, \sigma)\right\}\middle| \mathcal{F}_t\right]$ where $W_i^S(X; \mu, \sigma) = \left(S_i(X) - \int_0^1 c_i(\mu_s, \sigma_s)\mathrm{d}s\right)$ is his net payoff at the end of the project.

We assume that at any $t \in [0, 1]$ the whole history $\{X_s, \mu_s, \sigma_s, (e^i_s)_{i \in N}\}_{s \leq t}$ is observable and verifiable by all agents, and salaries $(S_i)_{i \in N}$ are determined by the principal at the beginning of the project. Thus, at any instant a *utilitarian bargaining problem* among agents emerges due to collusion opportunities. The outcome of this bargaining then can be implemented via state–contingent binding contracts drafted and agreed upon in period zero, specifying the arrangement among agents for each possible date and state.[8] As for any given history agents' arrangement ensures optimality from that state onwards, our formulation involves renegotiation concerns.

Collusion implies that the outcome of agents' bargaining is ex–ante efficient, necessitating that there should not be any history, state, and any other feasible contract that every agent (strictly) prefers to the one that was agreed upon. This brings about optimal risk sharing. Given $\mathcal{F}_t$–predictable salary control laws $\mathbf{S}_i : [0, 1] \times \Omega \to \Re$ for $i \in N$, we let agent i's induced salary at time t under $\mathbf{S} \equiv (\mathbf{S}_i)_{i \in N}$ be $\mathbf{S}_i(t) : \Omega \to \Re$, denoting the salary arrangement (on compensations

observable and verifiable by the principal, then she could infer $\{\mu_t\}_t$ and/or $\{\sigma_t\}_t$. For more on this issue, we refer the reader to Sung (1995) footnotes 7 and 8.

[8] Because that this state–contingent contract is drafted and agreed upon at date zero, we will have the participation constraint in the agents' problem formulated with only the date zero information.

to be made at the end of the project) to i under $\mathbf{S}$ at t. Given principal's offer, i.e. salaries $(S_i)_i$ and control laws (μ, σ), and bargaining weights $\theta : [0,1] \times \Omega \to \text{int}(\Delta)$ that are assumed to be $\mathcal{F}_t$–predictable mappings where $\text{int}(\Delta)$ denotes the interior of the N dimensional simplex, side–contracting via salary control laws $\hat{\mathbf{S}}_i : [0,1] \times \Omega \to \Re$ for $i \in N$, $\hat{\mu} : [0,1] \times \Omega \to U$, and $\hat{\sigma} : [0,1] \times \Omega \to \mathbb{S}$ solves *agents' problem* if for any given $t \in [0,1]$,

$$\sum_{i \in N} \theta_{it} E\left[-\exp\left\{ -r_i W_i^{\hat{\mathbf{S}}(t)}(X; \hat{\mu}, \hat{\sigma}) \right\} \Big| \mathcal{F}_t \right] \tag{1}$$

is maximized where $W_i^{\hat{\mathbf{S}}(t)}(X; \hat{\mu}, \hat{\sigma}) \equiv \left(\hat{\mathbf{S}}_i(t)(X) - \int_0^1 c_i(\hat{\mu}_s, \hat{\sigma}_s) \mathrm{d}s \right)$ for any $X \in \Omega$, and

$$\begin{aligned}
&\mathrm{d}X_\tau = \tilde{\mu}_\tau \mathrm{d}\tau + \tilde{\sigma}_\tau \mathrm{d}B_\tau, \tau \geq t, \\
&\sum_{i=1}^{N} \hat{\mathbf{S}}_i(\tau)(X) \leq \sum_{i=1}^{N} S_i(X), \tau \geq t, X \in \Omega, \\
&E\left[-\exp\left\{ -r_i W_i^{\hat{\mathbf{S}}(0)}(X; \hat{\mu}, \hat{\sigma}) \right\} \Big| \mathcal{F}_0 \right] \geq E\left[-\exp\left\{ -r_i W_i^{S}(X; \mu, \sigma) \right\} \big| \mathcal{F}_0 \right], i \in N.
\end{aligned}$$

While the first of these requirements is a natural feasibility constraint, the second is a balanced budget condition, and the third i's date–0 participation constraint.[9]

Agents' problem is to be solved in every date and state insisting on the optimality from that date and state onwards. So, it involves renegotiation concerns. The resulting arrangements, control laws, are implemented via date and state dependent binding side–contracts. And, agents' bargaining weights are not necessarily stationary, but date and state dependent.

The principal is aware of the collusion capabilities and bargaining among agents, so knows that she is restricted to offer contracts that are ex–ante efficient in every date and state. However, she is not aware of the specific bargaining weights that are to be employed. Consequently, we have:

Definition 1 (Principal's Problem) *Principal chooses salary functions* $(\hat{S}_i)_{i \in N}$ *and control laws* $(\hat{\mu}, \hat{\sigma})$ *such that*

$$\left((\hat{S}_i)_{i \in N}, \hat{\mu}, \hat{\sigma} \right) \in \text{argmax}_{((S_i)_{i \in N}, \mu, \sigma)} E\left[-\exp\left\{ -R\left(X_1 - \sum_{i=1}^{N} S_i(X) \right) \right\} \Big| \mathcal{F}_0 \right]$$

[9] The date–0 participation constraint considers the grand coalition/team, and not sub–coalitions. This is because each agent has the ability to report his observable and verifiable information regarding the past to the principal, whenever he obtains lower payoffs due to some sub–coalition's arrangements.

subject to

i. *Feasibility:* $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t$, $t \in [0,1]$;

ii. *Individual Rationality:* $E\left[-\exp\left\{-r_i W_i^S(X;\mu,\sigma)\right\}\middle|\mathcal{F}_0\right] \geq -\exp\{-r_i W_{i0}\}$, $i \in N$;

iii. *Agents' Problem:* $(S_i)_{i\in N}$*, and* (μ,σ) *must be such that there exists (1) a profile of control laws* $\mathbf{S}_i : [0,1]\times\Omega \to \Re$ *for* $i \in N$ *satisfying* $\mathbf{S}_i(1)(X) = S_i(X)$ *for all* $i \in N$ *and for all* $X \in \Omega$*, and (2) a control law* $\theta : [0,1]\times\Omega \to \mathrm{int}(\Delta)$*, so that* $((\mathbf{S}_i)_i, \mu, \sigma)$ *solves agents' problem.*

In Definition 1, feasibility and individual rationality are standard, and collusion is handled by requiring the principal's offer to solve agents' problem.

The interaction among agents is similar to that in Bone (1998) which identifies a useful aggregation result that we employ to obtain a new version of the principal's problem with a "representative agent" (the team of all agents).[10] In that study a group of agents with CARA utilities jointly choose between uncertain prospects. Although a static environment is modeled, the following two key aspects are common with our setting: (1) the choice of any prospect must be unanimously agreed, and (2) the uncertain outcomes from the chosen prospect are distributed among agents according to some unanimously made prior agreements. So, in that model agents are involved in a static utilitarian bargaining with given bargaining weights, and the outcome must be ex–ante efficient.

Given control laws $((\mathbf{S}_i)_{i\in N}, \mu, \sigma)$, we know that (due to condition 4 of Bone (1998) which is an immediate consequence of first–order analysis) the induced salary profile at time t, $\mathbf{S}_i(t) : \Omega \to \Re$ for $i \in N$, is efficient in the agents' (static) problem starting from instant t with given control laws μ and σ is equivalent to the existence of $(\theta_{it})_i$ such that for a.e. $X \in \Omega$ we have that for any $i, j \in N$, $\theta_{it} r_i \exp\left\{-r_i W_i^{\mathbf{S}(t)}(X;\mu,\sigma)\right\} = \theta_{jt} r_j \exp\left\{-r_j W_j^{\mathbf{S}(t)}(X;\mu,\sigma)\right\}$. Following the same arithmetic manipulations of Bone (1998) (calling for a careful summation of this equation across

[10] An earlier study, Brennan and Kraus (1978), shows that an aggregation leading to a representative agent representation is possible when agents have either CARA utilities or HARA (hyperbolic absolute risk aversion) preferences with equal exponents. And, it is shown in section 4 in Bone (1998) that this conclusion does not hold with non-identical exponents. Moreover, with identical exponents the representative agent's utility function is not necessarily negative exponential. However, CARA utilities' property of not involving any income effects and the use of stochastic processes with the martingale property are essential for obtaining a stationary decision making environment for the agents: The history in our setting determines the accumulated returns which do not influence agents' decisions due to lack of income effects; and, incremental future returns is expected not to be different from today's due to the martingale property. This stationarity is a key feature in the search for optimality of linearity.

agents and obtaining the aggregated terms, his conditions 9–13), it can easily be shown that efficient risk sharing occurs if and only if there exists $(\theta_{it})_i$ such that for all $i \in N$ and a.e. $X \in \Omega$,

$$W_i^{\mathbf{S}(t)}(X;\mu,\sigma) = k_{it} + \frac{r_c}{r_i}\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma), \tag{2}$$

where $r_c = \left(\sum_j 1/r_j\right)^{-1}$, $k_{it} = (r_c/r_i)\left(\sum_j \left(\ln(\theta_{it}r_i) - \ln(\theta_{jt}r_j)\right)/r_j\right)$ for $i \in N$, and $\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma) = \sum_i W_i^{\mathbf{S}(t)}(X;\mu,\sigma)$. So, given the history and control laws, efficiency implies that agent i's payment in instant t from the total payments (the team's state–contingent compensation) involves a (state–independent) constant payment, and a fraction which depends on agents' CARA coefficients and not the bargaining weights. Moreover, summing across agents these fractions add up to unity while the fixed payments sum to zero.

Therefore, whenever $((\mathbf{S}_i)_i,\mu,\sigma)$ is efficient, then for any t there is θ_t with associated k_{it} with the requirement that $\sum_i k_{it} = 0$. This efficient distribution induces a total compensation of $\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma)$. For any other efficient arrangement $((\mathbf{S}'_i)_i,\mu',\sigma')$ (associated with weights θ'_t and fixed payments k'_{it}) that induces the total compensation of $\bar{W}^{\mathbf{S}'(t)}(X;\mu,\sigma)$, it can be seen that there exists an efficient redistribution employing θ_t (and corresponding $(k_{it})_i$ that satisfy $\sum_i k_{it} = 0$) with the same total compensation $\bar{W}^{\mathbf{S}'(t)}(X;\mu,\sigma)$ using the results from the previous paragraph and the construction given in equation 2. That is, an efficient prospect $((\mathbf{S}'_i)_i,\mu',\sigma')$ with some θ'_t is also efficient with θ_t inducing the same total compensation $\bar{W}^{\mathbf{S}'(t)}(X;\mu,\sigma)$ but different fixed payments k_{it}. Then, as $V_{it} = \exp\{-r_i k_{it}\}E\left[-\exp\left\{-r_c\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma)\right\}\middle|\mathcal{F}_t\right]$, given any two efficient prospects $((\mathbf{S}_i)_i,\mu,\sigma)$ and $((\mathbf{S}'_i)_i,\mu',\sigma')$, every agent i prefers prospect $((\mathbf{S}_i)_i,\mu,\sigma)$ to $((\mathbf{S}'_i)_i,\mu',\sigma')$ if and only if $E\left[-\exp\left\{-r_c\bar{W}^{\mathbf{S}(t)}(X;\mu,\sigma)\right\}\middle|\mathcal{F}_t\right]$ strictly exceeds $E\left[-\exp\left\{-r_c\bar{W}^{\mathbf{S}'(t)}(X;\mu',\sigma')\right\}\middle|\mathcal{F}_t\right]$ for every t. This means that the interests of agents when dealing with efficient prospects are perfectly aligned, and collective behavior can be represented by a representative agent with CARA utilities with a coefficient r_c whenever efficiency is ensured.

Next we define *the team's problem*: Given salaries $(S_i)_i$, and control laws μ and σ, $\mathcal{F}_t$-predictable control laws $\hat{\mathbf{S}}_c : [0,1]\times\Omega \to \Re$, $\hat{\mu} : [0,1]\times\Omega \to U$, and $\hat{\sigma} : [0,1]\times\Omega \to \mathbb{S}$ solves the *team's problem* if for any given $t \in [0,1]$ the following is maximized

$$E\left[-\exp\left\{-r_c W^{\hat{\mathbf{S}}_c(t)}(X;\hat{\mu},\hat{\sigma})\right\}\middle|\mathcal{F}_t\right],$$

where $W^{\hat{\mathbf{S}}(t)}(X;\hat{\mu},\hat{\sigma}) \equiv \left(\hat{\mathbf{S}}_c(t)(X) - \sum_i \left(\int_0^1 c_i(\hat{\mu}_s,\hat{\sigma}_s)\mathrm{d}s\right)\right)$ for any $X \in \Omega$, subject to

$$\mathrm{d}X_\tau = \hat{\mu}_\tau \mathrm{d}\tau + \hat{\sigma}_\tau \mathrm{d}B_\tau, \tau \geq t,$$
$$\hat{\mathbf{S}}_c(\tau)(X) \leq \sum_i S_i(X), \tau \geq t, X \in \Omega.$$

Thus, for any given control laws $(\mathbf{S}_c^T, \mu^T, \sigma^T)$ that solves the team's problem, by only using CARA coefficients $(r_i)_i$ (regardless of the date and state and the level of what θ_t supposed to be), we can construct $((\mathbf{S}_i^T)_i, \mu^T, \sigma^T, \theta^*)$, where $(\theta_{it}^*)_i \in \mathrm{int}(\Delta)$ is defined by $\theta_{it}^* = r_c/r_i$, such that it is on the efficiency frontier. This follows from equation 2 and the definition of θ^* ensuring that the associated $(k_{it}^*)_i$ satisfies $\sum_i k_{it}^* = 0$, in turn, implying efficiency as discussed above.

Having dealt with efficient redistributions of salaries, we let the principal contract with the team:

Definition 2 *Principal chooses a salary for the team $\hat{S}_c$, which depend only on X_1, and control laws $(\hat{\mu},\hat{\sigma})$, such that*

$$\left(\hat{S}_c,\hat{\mu},\hat{\sigma}\right) \in \mathrm{argmax}_{(S_c,\mu,\sigma)} E\left[-\exp\left\{-R\left(X_1 - S_c(X)\right)\right\} \middle| \mathcal{F}_0\right]$$

subject to

i. *Feasibility:* $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t$, $t \in [0,1]$*;*

ii. *The team's participation:* $E\left[-\exp\left\{-r_c W^{\hat{S}_c}(X;\hat{\mu},\hat{\sigma})\right\}\middle| \mathcal{F}_0\right] \geq -\exp\{-r_c \sum_{i\in N} W_{i0}\}$.

iii. *The team's problem: S_c and (μ,σ) must be such that there exists a control law $\mathbf{S}_c : [0,1]\times\Omega \to \Re$ satisfying $\mathbf{S}_c(1)(X) = S_c(X)$ for all $X \in \Omega$ so that $(\mathbf{S}_c,\mu,\sigma)$ solves the team's problem.*

3 Optimality of Linearity

The following theorem proves that the linearity results of Holmstrom and Milgrom (1987), Schättler and Sung (1993), and Sung (1995) are robust with respect to collusion and renegotiation. The main reason of this observation is the lack of income effects and the nature of the bargaining among agents (both of which are due to CARA utilities), and the use of a Markovian stochastic process satisfying the martingale property.

Theorem 1 *There exists a stationary and linear optimal collusion proof and renegotiation proof contract.*

The optimal contract of Theorem 1 is given by stationary salaries $(S_i^*)_{i\in N}$ and stationary control laws (μ^*,σ^*) with $\mu_t^* = m^*$ and $\sigma_t^* = s^*$ for all $t \in [0,1]$ where $S_i^*(X) = c_i(m^*,s^*) + W_{i0} + \theta_i^* \left[S_c^*(X) - \sum_{j\in N} (c_j(m^*,s^*) + W_{j0}) \right]$, and

$$
\begin{aligned}
S_c^*(X) &= \sum_{i\in N} (W_{i0} + c_i(m^*,s^*)) + \left[\sum_{i\in N} c_{i\mu}(m^*,s^*) \right] ((X_1 - X_0) + m^*) + \frac{r_c}{2} \left[\sum_{i\in N} c_{i\mu}(m^*,s^*) \right]^2 s^{*2}, \\
r_c &= \left(\sum_{i\in N} \frac{1}{r_i} \right)^{-1} = \frac{\prod_{i\in N} r_i}{\sum_{i\in N} \prod_{j\neq i} r_j}, \\
\theta_i^* &= \frac{r_c}{r_i} = \frac{\prod_{j\neq i} r_j}{\sum_{i\in N} \prod_{j\neq i} r_j},
\end{aligned}
$$

and (m^*,s^*) solves the following maximization problem:

$$
\Phi^p(\hat{m},\hat{s}) \equiv \hat{m} + R \left[\sum_{i\in N} c_{i\mu}(\hat{m},\hat{s}) \right] \hat{s}^2 - \sum_{i\in N} c_i(\hat{m},\hat{s}) - \frac{1}{2}(R + r_c) \left(\sum_{i\in N} c_{i\mu}(\hat{m},\hat{s}) \right)^2 \hat{s}^2 - \frac{R}{2}\hat{s}^2
$$

subject to $(\hat{m},\hat{s})$ in $\operatorname{argmax}_{m,s} \Phi^a(m,s \mid \hat{m},\hat{s})$ where it is defined by

$$
\Phi^a(m,s \mid \hat{m},\hat{s}) \equiv \left[\sum_{i\in N} c_{i\mu}(\hat{m},\hat{s}) \right] m - \sum_{i\in N} c_i(m,s) - \frac{1}{2} r_c \left(\sum_{i\in N} c_{i\mu}(\hat{m},\hat{s}) \right)^2 s^2.
$$

Thus, the principal is acting as if she is contracting with the representative agent while making sure that the total salary is distributed efficiently among the agents employing θ^*, as was discussed in the final parts of the previous section. It is imperative to point out that, while doing this the principal is not required to be aware of agents' "real" bargaining powers.

The rest of the section concerns the proof of Theorem 1.

The problem in Definition 2 belongs to the class studied in Sung (1995), and due to its Proposition 2 there exists stationary salaries S_c^* and control laws (μ^*,σ^*) as defined above.[11]

[11]Sung (1995) uses the first–order approach, introduced by Schättler and Sung (1993), by allowing agents to control the variance as well as the mean of the process. The first–order necessary conditions lead to a semi–martingale representation of agent's salary function. The principal's relaxed problem is formulated by replacing the

The second step entails the principal's use of bargaining weights θ^* in the formulation of efficient redistribution of S_c^* among agents' and propose contracts accordingly. Hence,

$$\begin{aligned} S_i^*(X) &= c_i(m^*, s^*) + W_{i0} + \theta_i^* \left[S_c^*(X) - \sum_{j \in N} (c_j(m^*, s^*) + W_{j0}) \right] \\ &= c_i(m^*, s^*) + W_{i0} + \frac{r_c}{r_i} \left(A_1(X_1 - X_0) + A_2 \right), \end{aligned}$$

where $A_1 = \sum_{j \in N} c_{j\mu}(m^*, s^*)$ and $A_2 = \left(\sum_{j \in N} c_{j\mu}(m^*, s^*)\right) m^* + \frac{r_c}{2} \left(\sum_{j \in N} c_{j\mu}(m^*, s^*)\right)^2 s^{*2}$. As $c_{i\mu}, c_{i\mu\mu}$ are strictly positive, both A_1 and A_2 are strictly positive.

Next, we show that i's individual rationality, condition (ii) in Definition 1, follows because $E_0 \left[-\exp \left\{ -r_i \left(S_i^* - \int_0^1 c_i(\mu^*, \sigma^*) \mathrm{d}t \right) \right\} \right]$ equals

$$\begin{aligned} & E_0 \left[-\exp \left\{ -r_i \left(W_{i0} + \frac{r_c}{r_i} \left(A_1(X_1 - X_0) + A_2 \right) \right) \right\} \right] \\ & \quad = -\exp\{-r_i W_{i0}\} \, E_0 \left[-\exp \left\{ -r_c \left(A_1(X_1 - X_0) + A_2 \right) \right\} \right] \\ & \quad = -\exp\{-r_i W_{i0}\} \, E_0 \left[-\exp \left\{ -r_c \left(S_c^* - \sum_{j \in N} W_{j0} - \sum_{j \in N} c_j(m^*, s^*) \right) \right\} \right] \end{aligned}$$

and the individual rationality constraint of the team is satisfied.

Finally, $(S_i^*)_{i \in N}$ are linear in X_1 since the total salary is linear in the final outcome.

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salary function with this semi–martingale representation. The sufficiency conditions for the validity of the first–order approach given in Schättler and Sung (1993) are met in our Brownian setting.

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