Bargaining with Nonanonymous Disagreement: Nondecomposable Rules*

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Abstract

We analyze bargaining situations where the agents' payoffs from disagreement depend on who among them breaks down the negotiations. We model such problems as a superset of the standard domain of Nash (1950). We first show that this domain extension creates a very large number of new rules that could not have been analyzed under the standard domain. Particularly, the rules associated with the Nash domain constitute a nowhere dense subset of all possible rules. Next, we analyze monotone path rules on our domain and show that they are characterized by a set of properties that are not jointly compatible on the Nash domain. We also analyze and characterize a Proportional rule on our domain.

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1 Introduction

A typical bargaining problem, as modeled by Nash (1950) and the vast literature that follows, is made up of two elements. The first is a set of alternative agreements on which the agents negotiate. The second element is an alternative realized in case of disagreement. This "disagreement outcome" does not however contain detailed information about the nature of disagreement. Particularly, it is assumed in the existing literature that the realized disagreement alternative is independent of who among the agents disagree(s).

In real life examples of bargaining, however, the identity of the agent who terminates the negotiations turns out to have a significant effect on the agents' "disagreement payoffs". The reunion negotiations between the northern and the southern parts of Cyprus that took place at the beginning of 2004 constitute a good example. Due to a very strong support from the international community towards the island's reunion, neither party preferred to be the one to disagree. Also, each preferred the other's disagreement to some agreements which they in turn preferred to leaving the negotiation table themselves.[1] Wage negotiations between firms and labor unions constitute another example to the dependence of the disagreement payoffs on the identity of the disagreer. There, the disagreement action of the union, a strike, and that of the firm, a lockout, can be significantly different in terms of their payoff implications.[2] Finally, note that the bargaining framework is frequently used in economic models of family (see *e.g.* Becker (1981), Manser and Brown (1980), Sen (1983) and the following literature): the married couple bargains on alternative joint-decisions and divorce is their disagreement alternative. In the current models, payoffs from divorce do not depend on who in the couple leaves the marriage. However, it seems to us that this is hardly the case in reality.

[1] There is a vast number of articles that discuss the issue. For example, see the *Economist* articles dated April 17, 2004 (volume 371, issue 8371), *Cyprus: A Greek Wrecker* (page 11) and *Cyprus: A Derailment Coming* (page 25); also see *Greece's Election: Sprinting Start?* dated March 13, 2004 (volume 370, issue 8366, page 31). Also see the special issue on Cyprus of *International Debates* (2005, 3:3). Finally, an interview (in Turkish) with a former Turkish minister of external affairs, which appeared in the daily newspaper Radikal on February 16, 2004, presents a detailed discussion of the implications of disagreement.

[2] A similar case may arise between two countries negotiating at the brink of a war. Among the two possible disagreement outcomes, each country might prefer the one where it leaves the negotiation table first and makes an (unexpected) "preemptive strike" against the other.

Note that, neither of these examples can be fully represented in the confines of Nash's (1950) standard model. We therefore extend this model to a **nonanonymous-disagreement** model of bargaining by allowing the agents' payoffs from disagreement to depend on who among them disagrees. For this, we replace the disagreement *payoff vector* in the Nash (1950) model with a disagreement *payoff matrix*. The i^{th} row of this matrix is the payoff vector that results from agent i terminating the negotiations. The standard (**anonymous-disagreement**) domain of Nash (1950) is a "measure-zero" subset of ours where all rows of the disagreement matrix are identical.[3]

Our contribution to the literature is three-fold. First, we introduce a model that enables us to focus on an important aspect of most real-life negotiations. Since what we offer is a domain extension, this comes at no cost. Furthermore, our domain extension significantly increases the amount of admissible bargaining rules. Every bargaining rule on the Nash domain has counterparts on our domain (we call such rules **decomposable** since they are a composition of a bargaining rule from the Nash domain and a function that transforms disagreement matrices to disagreement vectors). But our domain also offers an abundance of rules that are **nondecomposable** (that is, they are not counterparts of rules from the Nash domain). In **Subsection 3.1**, we show that the class of decomposable rules is a *nowhere dense* subset of all bargaining rules. That is, the interior of its complement is sufficient to approximate any bargaining rule (*i.e.* interior of the class of nondecomposable rules is *dense*). This result implies that restricting oneself to the Nash domain involves a significant loss of generality.

Our second contribution to the literature is an analysis of a class of **monotone path rules** on our extended domain (**Subsection 3.2**). Rules in this class assign each disagreement matrix to an agenda in which the agents jointly improve their payoffs until doing so is no more feasible (this agenda is represented as a monotone increasing path in the payoff

[3] One restriction of our model is that it does not specify the outcome of a coalition of agents jointly terminating the negotiations. Modeling coordinated disagreement by a coalition would bring in questions about the bargaining process in that coalition and move us further towards a non-transferable utility game analysis. In this paper, we choose to remain in the bargaining framework and only consider individual deviations.

space). A version of this class is introduced by Thomson and Myerson (1980) for a class of problems with no disagreement vector and a normalized version is discussed by Peters and Tijs (1984) for Nash type problems (also see Thomson [23] for an extensive discussion). On the Nash domain, monotone path rules satisfy a **strong monotonicity** property (introduced by Kalai (1977)) which states that an expansion of the set of possible agreements should not make any agent worse-off. They however violate another central property called **scale invariance** (introduced by Nash (1950)); this property ensures the invariance of the physical bargaining outcome with respect to utility-representation changes that leave the underlying von Neumann-Morgenstern (1944) preferences intact. On our domain, however, a subset of monotone path rules satisfy both of these properties. In *Subsection 3.2*, we characterize the class of monotone path rules that satisfy *weak Pareto optimality*, *strong monotonicity*, *scale invariance*, and a *continuity* property. We also show that dropping *scale invariance* from this list characterizes the whole class of monotone path rules.

In this subsection, we also focus on two-agent bargaining problems, since they are central to bargaining theory. On the basis of how the agents rank the two disagreement outcomes, we partition this class into three. On each subclass, we present a simple definition of monotone path rules. Using them, a monotone path rule for two-agent problems can be fully defined by the specification of at most eight monotone paths. We also analyze the implications of a weak anonymity property called **symmetry**. We describe a monotone path rule (the **Cardinal Egalitarian rule**) which uniquely satisfies *weak Pareto optimality*, *strong monotonicity*, *scale invariance*, and *symmetry.*

Finally, in **Subsection 3.3** we analyze a version of the Kalai-Smorodinsky (1975) bargaining rule on our domain (that we call the **Proportional rule**). We show that, the Proportional rule satisfies a **restricted monotonicity** property (introduced by Roth (1979) as a weakening of the individual monotonicity property of Kalai-Smorodinsky (1975))[4] as well as a property (**limited sensitivity,** introduced by Gupta and Livne (1988)) that re-

[4]Either property, together with *Pareto optimality, symmetry*, and *scale invariance*, characterize the Kalai-Smorodinsky bargaining rule. The relationship between the two monotonicity properties as well as a characterization of rules that satisfy all the above properties but *symmetry* can be found in Peters and Tijs (1984).

quires certain changes in the disagreement matrix not to affect the outcome. We then show that, on two-agent bargaining problems the Proportional rule uniquely satisfies *weak Pareto optimality, restricted monotonicity, scale invariance, symmetry,* and *limited sensitivity*.

In a companion paper (Kıbrıs and Gürsel (2006)), we enquire if the counterparts of some standard results on the Nash (1950) domain continue to hold for decomposable rules on our extended domain. We first show that a unique extension of the Kalai-Smorodinsky bargaining rule is the only rule that satisfies the Kalai-Smorodinsky (1975) properties. This uniqueness result, however, turns out to be an exception. An infinite number of decomposable rules survive the Nash (1950), Kalai (1977), Perles-Maschler (1981), and Thomson (1981) properties even though, on anonymous-disagreement problems each of these results characterizes a single rule. In that paper, we also observe that extensions to our domain of a standard independence property (by Peters, 1986) imply decomposability.

We end this section with a review of some other extensions of the Nash (1950) model. Gupta and Livne (1988) analyze bargaining problems with an additional reference point (in the feasible set). They interpret it as a past agreement that the agents can refer to when negotiating. In this model, they analyze the implications of some basic properties as well as *limited sensitivity* and show that a version of what we define as the "Proportional rule" in Subsection 3.3 uniquely satisfies them.[5] Chun and Thomson (1992) analyze an alternative model where the reference point is not feasible (and is interpreted as a vector of "incompatible" claims). They analyze the implications of several properties regarding changes in the feasible set, the disagreement point, and the reference point. Their results also characterize a rule that allocates gains proportionally to the reference point. Neither of these two papers however focus on disagreement.

Chun and Thomson (1990a and 1990b) extend the Nash (1950) model to allow probabilistic disagreement points. Their model represents cases where the agents are not certain about the implications of disagreement. The authors, in (1990a) show that a basic set of properties characterize the weighted Egalitarian rules. In (1990b), they show that the Nash

[5] More precisely, Gupta and Livne show that a proportional rule on their domain uniquely satisfies *weak Pareto optimality, scale invariance, symmetry, restricted monotonicity, limited sensitivity,* and *independence of non-individually rational alternatives*. Except the last property, this is consistent with our result.

bargaining rule uniquely satisfies an alternative set of properties. While the Chun-Thomson framework focuses on the implications of disagreement, in their model (as in that of Nash), these implications are independent of who disagrees.

Finally, there are many papers that discuss disagreement-related properties but remain on the Nash (1950) model. For example, see Dagan, Volij, and Winter (2002), Livne (1986), Thomson (1987), and Peters and van Damme (1991).

2 Model

2.1 Preliminaries

Let $N = \{1, ..., n\}$ be the set of agents. For each $i \in N$, let $e_i \in \mathbb{R}^N$ be the vector whose i^{th} coordinate is 1 and every other coordinate is 0. Let $\mathbf{1} \in \mathbb{R}^N$ (respectively, $\mathbf{0}$) be the vector whose every coordinate is 1 (respectively, 0). For vectors in $\mathbb{R}^N$, inequalities are defined as: $x \leqq y$ if and only if $x_i \leqq y_i$ for each $i \in N$; $x \leq y$ if and only if $x \leqq y$ and $x \neq y$; $x < y$ if and only if $x_i < y_i$ for each $i \in N$. For each $S \subseteq \mathbb{R}^N$, $Int(S)$ denotes the interior of S and $Cl(S)$ denotes the closure of S. For each $S \subseteq \mathbb{R}^N$ and $s \in S$, $conv\{S\}$ denotes the convex hull of S and $s\text{-}comp\{S\} = \{x \in \mathbb{R}^N \mid s \leqq x \leqq y \text{ for some } y \in S\}$ denotes the s-comprehensive hull of S. The set S is *s-comprehensive* if $s\text{-}comp\{S\} \subseteq S$. The set S is *strictly s-comprehensive* if it is *s-comprehensive* and for each $x, y \in S$ such that $x \geq y \geq s$, there is $z \in S$ such that $z > y$.

Let the **Euclidean metric** be defined as $\|x - y\| = \sqrt{\sum (x_i - y_i)^2}$ for $x, y \in \mathbb{R}^N$ and let the **Hausdorff metric** be defined as

$$\mu^H(S^1, S^2) = \max_{i \in \{1,2\}} \max_{x \in S^i} \min_{y \in S^j} \|x - y\|$$

for compact sets $S^1, S^2 \subseteq \mathbb{R}^N$.

Let

$$D = \begin{bmatrix} D_{11} & \cdots & D_{1n} \\ \vdots & \ddots & \vdots \\ D_{n1} & \cdots & D_{nn} \end{bmatrix} = \begin{bmatrix} D_1 \\ \vdots \\ D_n \end{bmatrix} \in \mathbb{R}^{N \times N}$$

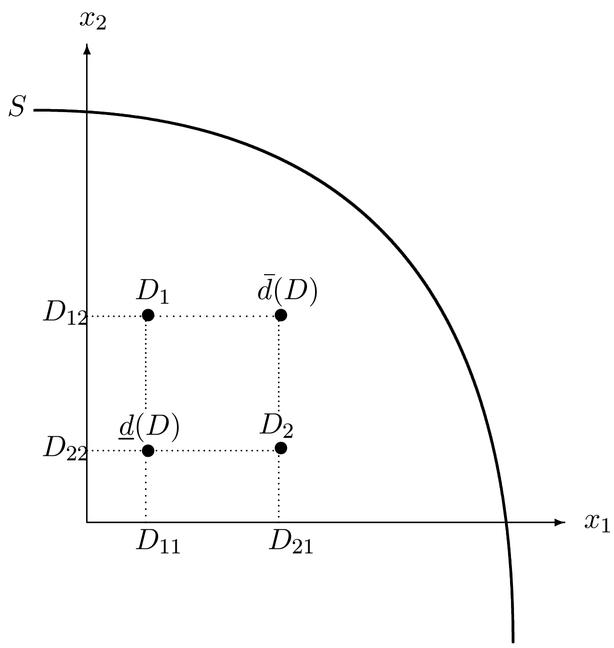

Figure 1: A typical bargaining problem with nonanonymous disagreement.

be a matrix in $\mathbb{R}^{N\times N}$. The i^{th} row vector $D_i = (D_{i1}, ..., D_{in}) \in \mathbb{R}^N$ represents the disagreement payoff profile that arises from agent i terminating the negotiations. For each $i \in N$, let $\overline{d}_i(D) = \max\{D_{ji} \mid j \in N\}$ be the maximum payoff agent i can get from disagreement and let $\underline{d}_i(D) = \min\{D_{ji} \mid j \in N\}$ be the minimum payoff. Let $\overline{d}(D) = (\overline{d}_i(D))_{i\in N}$ and $\underline{d}(D) = (\underline{d}_i(D))_{i\in N}$. Let the **metric $\boldsymbol{\mu}^{\mathbf{M}}$ on $\mathbb{R}^{\mathbf{N\times N}}$** be defined as $\mu^M(D, D') = \max_{i\in N} \|D_i - D'_i\|$ for $D, D' \in \mathbb{R}^{N\times N}$.

Let Π be the set of all permutations π on N. A function $f : \mathbb{R} \to \mathbb{R}$ is *positive affine* if there is $a \in \mathbb{R}_{++}$ and $b \in \mathbb{R}$ such that $f(x) = ax + b$ for each $x \in \mathbb{R}$. Let Λ be the set of all $\lambda = (\lambda_1, ..., \lambda_n)$ where each $\lambda_i : \mathbb{R} \to \mathbb{R}$ is a positive affine function.

For $\pi \in \Pi$, $S \subseteq \mathbb{R}^N$, and $D \in \mathbb{R}^{N\times N}$, let $\pi(S) = \{y \in \mathbb{R}^N \mid y = (x_{\pi(i)})_{i\in N}$ for some $x \in S\}$ and $\pi(D) = (D_{\pi(i)\pi(j)})_{i,j\in N}$. The set S (respectively, the matrix D) is **symmetric** if for every permutation $\pi \in \Pi$, $\pi(S) = S$ (respectively, $\pi(D) = D$). For $\lambda \in \Lambda$, let $\lambda(S) = \{(\lambda_1(x_1), ..., \lambda_n(x_n)) \mid x \in S\}$ and

$$\lambda(D) = \begin{bmatrix} \lambda_1(D_{11}) & \cdots & \lambda_n(D_{1n}) \\ \vdots & \ddots & \vdots \\ \lambda_1(D_{n1}) & \cdots & \lambda_n(D_{nn}) \end{bmatrix} = \begin{bmatrix} \lambda(D_1) \\ \vdots \\ \lambda(D_n) \end{bmatrix} \in \mathbb{R}^{N\times N}.$$

A **(nonanonymous-disagreement) bargaining problem for** N is a pair (S, D) where

$S \subseteq \mathbb{R}^N$ and $D \in \mathbb{R}^{N \times N}$ satisfy (see Figure 1):

1. for each $i \in N$, $D_i \in S$;
2. S is compact, convex, and $\underline{d}(D)$-comprehensive;
3. there is $x \in S$ such that $x > \overline{d}(D)$.

Assumptions 1, 2 and a counterpart of 3 are standard in the literature. They essentially come out of problems where the agents have expected utility functions on a bounded set of lotteries.

Let $\mathcal{B}$ be the class of all bargaining problems for agents in N. Let $\mathcal{B}_= = \{(S, D) \in \mathcal{B} \mid D_1 = D_2 = ... = D_n\}$ be the subclass of **problems with anonymous disagreement**. Let $\mathcal{B}_{\neq} = \mathcal{B} \setminus \mathcal{B}_=$ be the subclass of **problems with nonanonymous disagreement**.

Remark 1 *We will abuse notation and use* (S, d) *to represent an anonymous-disagreement problem* $(S, D) \in \mathcal{B}_=$ *where* $D_1 = ... = D_n = d$.

Let the **metric $\mu^{\mathcal{B}}$ on** $\mathcal{B}$ be defined as $\mu^{\mathcal{B}}((S, D), (S', D')) = \max\{\mu^H(S, S'), \mu^M(D, D')\}$ for $(S, D), (S', D') \in \mathcal{B}$. Given $(S, D) \in \mathcal{B}$, the set of **Pareto optimal alternatives** is $PO(S, D) = \{x \in S \mid y \geq x \text{ implies } y \notin S\}$, the set of **weakly Pareto optimal alternatives** is $WPO(S, D) = \{x \in S \mid y > x \text{ implies } y \notin S\}$ and the set of **individually rational alternatives** is $I(S, D) = \{x \in S \mid x_i \geqq D_{ii} \text{ for each } i \in N\}$.[6] The **maximal individually rational payoff of** $i \in N$ is $\overline{m}_i(S, D) = \max\{x_i \mid x \in I(S, D)\}$.

2.2 Bargaining Rules

A **(nonanonymous-disagreement) bargaining rule** $F : \mathcal{B} \to \mathbb{R}^N$ is a function that satisfies $F(S, D) \in S$ for each $(S, D) \in \mathcal{B}$. Let $\mathcal{F}$ be the class of all bargaining rules. Let the **metric $\mu^{\mathcal{F}}$ on** $\mathcal{F}$ be defined as $\mu^{\mathcal{F}}(F, F') = \sup\{\frac{\|F(S,D)-F'(S,D)\|}{1+\|F(S,D)-F'(S,D)\|} \mid (S, D) \in \mathcal{B}\}$ for $F, F' \in \mathcal{F}$. Given a bargaining rule $F \in \mathcal{F}$, a feasible set $S \subseteq \mathbb{R}^N$, and an agreement $x \in S$,

[6] For a more detailed discussion of individual rationality and properties related to it, please see Kıbrıs and Gürsel (2006).

the **anonymous inverse set of F at (S, x)**, which contains all disagreement vectors d that produce x, is defined as

$$F_{=}^{-1}(S,x) = \{d \in \mathbb{R}^N \mid (S,d) \in \mathcal{B}_{=} \text{ and } F(S,d) = x\}.$$

We first introduce some standard properties. A bargaining rule F is **Pareto optimal** if for each $(S,D) \in \mathcal{B}$, $F(S,D) \in PO(S,D)$. It is **weakly Pareto optimal** if for each $(S,D) \in \mathcal{B}$, $F(S,D) \in WPO(S,D)$. It is **symmetric** if for each $(S,D) \in \mathcal{B}$ with *symmetric* S and D, $F(S,D)$ is also *symmetric*, that is, $F_1(S,D) = ... = F_n(S,D)$.

The following property requires small changes in the data of a problem not to have a big effect on the agreement. A bargaining rule F is **continuous** if for every sequence $\{(S^m, D^m)\}_{m\in\mathbb{N}} \subseteq \mathcal{B}$ that converges with respect to $\mu^{\mathcal{B}}$ to some $(S,D) \in \mathcal{B}$, we have $\lim_{m\to\infty} F(S^m, D^m) = F(S,D)$. It is **set-continuous** if $F(.,D)$ is continuous for every $D \in \mathbb{R}^{N\times N}$.

The following is a central property in bargaining theory. It requires the physical bargaining outcome to be invariant under utility-representation changes as long as the underlying von Neumann-Morgenstern (1944) preference information is unchanged. A bargaining rule F is **scale invariant** if for each $(S,D) \in \mathcal{B}$ and each $\lambda \in \Lambda$, $F(\lambda(S), \lambda(D)) = \lambda(F(S,D))$.

The following "monotonicity" properties are also standard in the literature. They require that an expansion of the set of possible agreements make no agent worse-off. A bargaining rule F is **strongly monotonic** (Kalai, 1977) if for each $(S,D),(T,D) \in \mathcal{B}$, $T \subseteq S$ implies $F(T,D) \leqq F(S,D)$. The following is a weaker version introduced by Roth (1979). It restricts the monotonicity requirement to expansions that leave the agents' maximal individually rational payoffs unchanged. A bargaining rule F is **restricted monotonic** if for each $(S,D),(T,D) \in \mathcal{B}$, $T \subseteq S$ and $\overline{m}(S,D) = \overline{m}(T,D)$ imply $F(T,D) \leqq F(S,D)$.

The following is an adaptation of a property proposed by Gupta and Livne (1988) for their domain. It requires that the solution be independent of a change in an agent's payoff from his own disagreement as long as this change does not affect the agents' maximal individually rational payoffs. A bargaining rule F is **limitedly sensitive (to changes in the disagreement matrix)** if for all $(S,D),(S,D') \in \mathcal{B}$, $\overline{m}(S,D) = \overline{m}(S,D')$ and $D_{ij} = D'_{ij}$ for all $i \neq j$ imply $F(S,D) = F(S,D')$. Note that this is a very weak property since only the D_{ii} are allowed to change and their changes are restricted to those that do not affect $\overline{m}(S, D)$ (this is, for example, not possible on strictly $\underline{d}(D)$-comprehensive problems).

3 Results

3.1 Disagreement Simplicity

Note that the literature on the Nash (1950) model analyzes rules that are defined on *anonymous-disagreement problems*, $\phi : \mathcal{B}_= \to \mathbb{R}^N$ (hereafter, **anonymous-disagreement rules**). Two well-known examples are the bargaining rules of Nash (1950) and Kalai-Smorodinsky (1975).

Anonymous-disagreement rules can be extended to our domain via a function that aggregates the multiple disagreement points $D = (D_i)_{i \in N} \in \mathbb{R}^{N \times N}$ to a single one $d \in \mathbb{R}^N$. A function $\alpha : \mathbb{R}^{N \times N} \to \mathbb{R}^N$ is an **aggregator function** if for each $D \in \mathbb{R}^{N \times N}$, $\underline{d}(D) \leqq \alpha(D) \leqq \overline{d}(D)$. This property guarantees $(S, \alpha(D)) \in \mathcal{B}$ (please see Remark 1) and for $d \in \mathbb{R}^N$, $\alpha(d, ..., d) = d$.

A bargaining rule $F : \mathcal{B} \to \mathbb{R}^N$ is **decomposable** if there is an *anonymous-disagreement rule* $\phi : \mathcal{B}_= \to \mathbb{R}^N$ and an *aggregator function* $\alpha : \mathbb{R}^{N \times N} \to \mathbb{R}^N$ such that $F = \phi \circ \alpha$ (more precisely, for each $(S, D) \in \mathcal{B}$, $F(S, D) = \phi(S, \alpha(D))$ holds).[7] Otherwise, F is called **nondecomposable**. Let $\mathcal{F}_=$ be the class of *decomposable* bargaining rules. Let $\mathcal{F}_{\neq} = \mathcal{F} \backslash \mathcal{F}_=$ be the class of *nondecomposable* bargaining rules.

In this section, we analyze the extent to which the imposition of *decomposability* restricts the class of admissible bargaining rules. Since decomposable rules are intimately linked to the Nash (1950) domain, this will give us an idea on how big a class of rules was ruled out by the anonymous-disagreement assumption of Nash.

Consider the following rather technical property. Fix any disagreement matrix, $D \in \mathbb{R}^{N \times N}$. Then for each feasible set $S \subseteq \mathbb{R}^N$, consider the set of anonymous disagreement vectors $d \in \mathbb{R}^N$ for which $F(S, d) = F(S, D)$ (note that the set of such d is the anonymous

[7] For example, by taking a composition of the Nash bargaining rule ν and the aggregator function $\alpha(.) = \overline{d}(.)$, one can define a decomposable rule F as $F(S, D) = \nu(S, \overline{d}(D))$.

inverse set $F_{=}^{-1}(S, F(S, D)))$. Let the correspondence $\delta^F : \mathbb{R}^{N\times N} \Longrightarrow \mathbb{R}^N$ be defined as follows: for each $D \in \mathbb{R}^{N\times N}$,

$$\delta^F(D) = \bigcap_{\substack{S\subseteq\mathbb{R}^N \ s.t. \\ (S,D)\in\mathcal{B}}} F_{=}^{-1}(S, F(S, D)).$$

That is, $\delta^F(D)$ is the set of all d that is contained in every $F_{=}^{-1}(S, F(S, D))$, independent of S. Note that $\underline{d}(D) \leqq d \leqq \overline{d}(D)$ holds for all $d \in \delta^F(D)$. A bargaining rule F is **disagreement-simple** if the correspondence δ^F is nonempty-valued. In other words, F is *disagreement-simple* if for each $D \in \mathbb{R}^{N\times N}$ there is $d \in \mathbb{R}^N$ such that $F(., d) = F(., D)$. We thus have the following lemma (for its proof, please see the Appendix).

Lemma 2 *A bargaining rule is* decomposable *if and only if it is* disagreement-simple.

Note that *disagreement simplicity* is a very demanding property. It is thus satisfied by only a very small subset of all bargaining rules. That is, bargaining rules on our extended domain are mostly *nondecomposable*. We next present a result that supports this intuition. We show that the class of decomposable rules $\mathcal{F}_{=}$ is *nowhere dense* in $\mathcal{F}$, that is $Int(Cl(\mathcal{F}_{=})) = \emptyset$. The class $\mathcal{F}_{=}$ is *nowhere dense* in $\mathcal{F}$ if and only if the interior of its complement, $Int(\mathcal{F}_{\neq})$, is *dense in* $\mathcal{F}$ (see Sutherland (2002), pg. 63-64).[8] This means that the interior of the class of nondecomposable rules, $\mathcal{F}_{\neq}$ (which by definition is an open set) is so big that it can be used to approximate any bargaining rule.

Theorem 3 *The class of decomposable rules $\mathcal{F}_{=}$ is* nowhere dense *in $\mathcal{F}$.*[9]

Proof. We first show that any decomposable bargaining rule can be approximated by a nondecomposable bargaining rule.

Step 1. For each $F \in \mathcal{F}_{=}$ and $\varepsilon > 0$, there is $F^{\varepsilon} \in \mathcal{F}_{\neq}$ such that $\mu^{\mathcal{F}}(F, F^{\varepsilon}) < \varepsilon$.

Fix some $(S^*, D^*) \in \mathcal{B}_{\neq}$ and let $x^* = F(S^*, D^*)$. Without loss of generality assume $\varepsilon < 1$ and let $0 < \delta < \frac{\varepsilon}{2(1-\varepsilon)}$. Let $x^{\varepsilon} \in S^* \setminus \{x^*\}$ be such that $\|x^* - x^{\varepsilon}\| < \delta$.

[8] A subset A of $\mathcal{F}$ is dense if $Cl(A) = \mathcal{F}$.

[9] This statement is stronger than the class of nondecomposable rules $\mathcal{F}_{\neq}$ being *dense* in $\mathcal{F}$. For example, rational numbers $\mathbb{Q}$ is *dense* in $\mathbb{R}$ but its complement is not *nowhere dense*.

Let

$$F^{\varepsilon}(S,D) = \begin{cases} x^{\varepsilon} & \begin{array}{c}\text{if } S = S^* \text{ and } D \in F_{=}^{-1}(S^*, x) \\ \text{for some } x \in S^* \text{ such that } \|x^* - x\| < \delta,\end{array} \\ \\ F(S,D) & \text{otherwise.} \end{cases}$$

Since $F \in \mathcal{F}_{=}$, $\delta^{F}(D^*) \neq \emptyset$ and thus, $F_{=}^{-1}(S^*, x^*) \neq \emptyset$. This implies $F^{\varepsilon} \neq F$. Note that for $x \in S^*$ such that $\|x^* - x\| < \delta$, $\|x^* - x^{\varepsilon}\| < \delta$ implies $\|x - x^{\varepsilon}\| < 2\delta$. Thus for each $(S,D) \in \mathcal{B}$, $\|F(S,D) - F^{\varepsilon}(S,D)\| < 2\delta$. Therefore

$$\mu^{\mathcal{F}}(F, F^{\varepsilon}) \leq \frac{2\delta}{1+2\delta} < \varepsilon.$$

Finally, to see that F^{ε} is nondecomposable, note that for $x \in S^* \backslash \{x^{\varepsilon}\}$ such that $\|x^* - x\| < \delta$, we have $(F^{\varepsilon})_{=}^{-1}(S^*, x) = \emptyset$. Particularly,

$$(F^{\varepsilon})_{=}^{-1}(S^*, x^*) = (F^{\varepsilon})_{=}^{-1}(S^*, F^{\varepsilon}(S^*, D^*)) = \emptyset.$$

Thus, $\delta^{F^{\varepsilon}}(D^*) = \bigcap_{\substack{S \subseteq \mathbb{R}^N \ s.t. \\ (S,D^*) \in \mathcal{B}}} (F^{\varepsilon})_{=}^{-1}(S, F^{\varepsilon}(S, D^*)) \subseteq (F^{\varepsilon})_{=}^{-1}(S^*, F^{\varepsilon}(S^*, D^*)) = \emptyset$. Thus F^{ε} violates *disagreement simplicity* and by Lemma 2, is *nondecomposable*.

We will next show that $F^{\varepsilon} \in Int(\mathcal{F}_{\neq})$.

Step 2. There is $\gamma > 0$ such that for all $F^{\gamma} \in \mathcal{F}$ satisfying $\mu^{\mathcal{F}}(F^{\varepsilon}, F^{\gamma}) < \gamma$, we have $F^{\gamma} \in \mathcal{F}_{\neq}$.

Let $\gamma = \frac{\|x^* - x^{\varepsilon}\|}{2 + \|x^* - x^{\varepsilon}\|}$ and $\eta = \frac{\gamma}{1-\gamma}$. Since $\mu^{\mathcal{F}}(F^{\varepsilon}, F^{\gamma}) < \gamma$, $\|F^{\varepsilon}(S,D) - F^{\gamma}(S,D)\| < \eta$ for each $(S,D) \in \mathcal{B}$. Particularly,

$$\|F^{\varepsilon}(S^*, D^*) - F^{\gamma}(S^*, D^*)\| = \|x^* - F^{\gamma}(S^*, D^*)\| < \eta.$$

Now let $d \in \mathbb{R}^N$ be such that $(S^*, d) \in \mathcal{B}_{=}$. Since either $F^{\varepsilon}(S^*, d) = x^{\varepsilon}$ and thus, $\|x^* - F^{\varepsilon}(S^*, d)\| = 2\eta$ or $\|x^* - F^{\varepsilon}(S^*, d)\| \geq \delta > 2\eta$, we have $\|x^* - F^{\varepsilon}(S^*, d)\| \geq 2\eta$. By triangle inequality,

$$2\eta \leq \|x^* - F^{\varepsilon}(S^*, d)\| \leq \|x^* - F^{\gamma}(S^*, d)\| + \|F^{\gamma}(S^*, d) - F^{\varepsilon}(S^*, d)\|.$$

But $\mu^{\mathcal{F}}(F^\gamma, F^\varepsilon) < \gamma$ implies $\|F^\gamma(S^*, d) - F^\varepsilon(S^*, d)\| < \eta$. Therefore,

$$\|x^* - F^\gamma(S^*, d)\| > \eta.$$

Combining the two displayed inequalities, we obtain $F^\gamma(S^*, D^*) \neq F^\gamma(S^*, d)$ for any $d \in \mathbb{R}^N$ such that $(S^*, d) \in \mathcal{B}_=$. Therefore, $\delta^{F^\gamma}(D^*) \subseteq (F^\gamma)^{-1}_=(S^*, F^\gamma(S^*, D^*)) = \emptyset$. Thus F^γ violates *disagreement simplicity* and by Lemma 2, is *nondecomposable.* This establishes that $F^\varepsilon \in Int(\mathcal{F}_{\neq})$.

Finally, we will show that any rule at the closure of $\mathcal{F}_=$ can be approximated by rules from $Int(\mathcal{F}_{\neq})$.

Step 3. For each $F \in \mathcal{F}_{\neq} \backslash Int(\mathcal{F}_{\neq})$ and $\varepsilon > 0$ there is $F' \in Int(\mathcal{F}_{\neq})$ such that $\mu^{\mathcal{F}}(F, F') < \varepsilon$.

Since $F \notin Int(\mathcal{F}_{\neq})$, there is $F'' \in \mathcal{F}_=$ such that $\mu^{\mathcal{F}}(F, F'') < \frac{\varepsilon}{2}$. Now since $F'' \in \mathcal{F}_=$, by steps 1 and 2, there is $F' \in Int(\mathcal{F}_{\neq})$ such that $\mu^{\mathcal{F}}(F', F'') < \frac{\varepsilon}{2}$. But then, by triangle inequality, $\mu^{\mathcal{F}}(F, F') < \varepsilon$ proves the claim.

■

While the class of nondecomposable rules contains an open and dense subset of $\mathcal{F}$, it is not open itself. Example 13 in the Appendix constructs a nondecomposable bargaining rule whose every neighborhood contains a decomposable rule.

Note that the relationship between anonymous-disagreement problems $\mathcal{B}_=$ and nonanonymous-disagreement problems $\mathcal{B}_{\neq}$ is quite similar to that between $\mathcal{F}_=$ and $\mathcal{F}_{\neq}$.

Remark 4 *The class $\mathcal{B}_=$ is* nowhere dense *in $\mathcal{B}$.*

3.2 Monotone Path Rules

Our first observation is that a very large class of rules simultaneously satisfy three properties which, on the class of anonymous-disagreement problems, $\mathcal{B}_=$, are incompatible: *weak Pareto optimality, strong monotonicity,* and *scale invariance* (see Thomson [23] for a discussion). We will introduce them next.

A **monotone path** on $\mathbb{R}^N$ is the image $G \subseteq \mathbb{R}^N$ of a function $g : \mathbb{R}_+ \to \mathbb{R}^N$ which is such that for all $i \in N$, $g_i : \mathbb{R}_+ \to \mathbb{R}$ is continuous and nondecreasing and for some $j \in N$, $g_j(\mathbb{R}_+) = [g_j(0), \infty)$. Let $\mathbb{G}$ be the set of all monotone paths.

Let $p : \mathbb{R}^{N\times N} \to \mathbb{G}$ be a **generator function** that maps each disagreement matrix D to a monotone path $p(D)$ such that (i) $x = \min p(D)$ satisfies $\underline{d}(D) \leqq x \leqq \overline{d}(D)$ and $x_i = \overline{d}_i(D)$ for some $i \in N$, and (ii) there are no $x, y \in p(D)$ such that $x \neq y$ and $x_i = y_i > \overline{d}_i(D)$ for some $i \in N$. Condition (i) requires the path $p(D)$ to start from a point x that is at the weak Pareto boundary of the rectangular set in-between $\underline{d}(D)$ and $\overline{d}(D)$. Condition (ii) strenghtens the monotonicity requirement on the path $p(D)$. For example, in $\mathbb{R}^2$ it requires the path not to be vertical (respectively, horizontal) on the half-space of vectors whose first (respectively, second) coordinate is greater than that of $\overline{d}(D)$. The generator function p is **scale invariant** if for each $D \in \mathbb{R}^{N\times N}$ and $\lambda \in \Lambda$, $p(\lambda(D)) = \lambda(p(D))$.

The **monotone path rule $\mathbf{F^p} : \mathcal{B} \to \mathbb{R}^N$ with respect to the generator function p** maps each $(S, D) \in \mathcal{B}$ to the maximal point of S along $p(D)$, that is, $F^p(S, D) = WPO(S, D) \cap p(D)$.[10]

The class of monotone path rules contains both decomposable and nondecomposable rules. The following remark is useful in identifying them.

Remark 5 *A monotone path rule F^p is* decomposable *if for each $D \in \mathbb{R}^{N\times N}$ there is $d \in \mathbb{R}^N$ such that $\underline{d}(D) \leqq d \leqq \overline{d}(D)$ and $p(D) \subseteq p(d)$.*

Next, we present a characterization of monotone path rules. In the result, the domain reduces from $\mathcal{B}$ to $\mathcal{B}_{\neq}$ as we introduce *scale invariance*. This is because the given properties are not compatible on $\mathcal{B}_{=}$.

Theorem 6 *A bargaining rule $F : \mathcal{B} \to \mathbb{R}^N$ is* weakly Pareto optimal, strongly monotonic, *and* set-continuous *if and only if it is a monotone path rule F^p. A bargaining rule on $\mathcal{B}_{\neq}$, $F : \mathcal{B}_{\neq} \to \mathbb{R}^N$ is* weakly Pareto optimal, strongly monotonic, set-continuous, *and* scale invariant *if and only if it is a monotone path rule F^p where p is* scale invariant.

Proof. Monotone path rules by definition are *weakly Pareto optimal. Strong monotonicity* follows from the monotonicity of the paths $p(D)$ and *set-continuity* follows from Condition (ii) on the generator function p.

[10] Continuity and monotonicity of the path $p(D)$ guarantee that this intersection is nonempty while Condition (ii) on the generator function p guarantees that it is a singleton.

For the uniqueness part of the first statement, let $F : \mathcal{B} \to \mathbb{R}^N$ be a bargaining rule that is *weakly Pareto optimal, strongly monotonic,* and *set-continuous.*

Step 1 *(define p):* Fix arbitrary $D \in \mathbb{R}^{N \times N}$ and for each $r \in \mathbb{R}_+$, let $x^r(D) = \overline{d}(D) + r\mathbf{1}$. Let $f : \mathbb{R}_+ \to [0,1]$ be an increasing continuous function such that $f(0) = 0$ and $\lim_{r\to\infty} f(r) = 1$. For each $i \in N$, let $Y_i^r \in \mathbb{R}^N$ be such that $Y_{ii}^r = -1$ and for $j \neq i$, $Y_{ij}^r = f(r)$; let $L_i^r(D) = \{x^r(D) + lY_i^r \mid 0 \leqq l \leqq x_i^r(D) - \underline{d}_i(D)\}$. Now let $S_D^r = \underline{d}(D)$-*comp*$\{$*conv*$\{L_1^r(D), ..., L_n^r(D)\}\}$ and note that for all $r \in \mathbb{R}_{++}$, (i) S_D^r is *strictly* $\underline{d}(D)$-*comprehensive* and (ii) $F(S_D^r, D) \in PO(S_D^r, D)$. Finally, let $\underline{x}(D) = \lim_{r\to 0} F(S_D^r, D)$. Now define $p(D) = \{\underline{x}(D)\} \cup \{F(S_D^r, D) \mid r \in \mathbb{R}_{++}\}$.

Step 2 *($F = F^p$):* Now let $(S, D) \in \mathcal{B}$ and let $x^* = F^p(S, D)$. Then by definition of p, there is $r \in \mathbb{R}_{++}$ such that $x^* = F^p(S_D^r, D) = F(S_D^r, D)$. Let $T = S \cap S_D^r$ and note that $x^* \in T$. Since $x^* \in PO(S_D^r, D)$, we also have $x^* \in PO(T, D)$. Now, by *strong monotonicity* of F, $F(T, D) \leqq x^*$. First assume that $y \leq x^*$ implies $y \notin WPO(T, D)$. Then *weak Pareto optimality* of F implies $F(T, D) = x^*$. Alternatively if there is $y \leq x^*$ such that $y \in WPO(T, D)$, let $\{T^m\}_{m\in\mathbb{N}} \to T$ be such that for each $m \in \mathbb{N}$, $T^m \subseteq S_D^r$ is *strictly* $\underline{d}(D)$-*comprehensive* (this is possible since S_D^r is strictly $\underline{d}(D)$-comprehensive) and $x^* \in PO(T^m, D)$ (this is possible since $x^* \in PO(S_D^r, D)$). Then by the previous case, $F(T^m, D) = x^*$ for each $m \in \mathbb{N}$ and by *set-continuity* of F, $F(T, D) = x^*$. Finally, $T \subseteq S$, by *strong monotonicity* of F implies $x^* \leqq F(S, D)$. If $x^* \in PO(S, D)$, this implies $x^* = F(S, D)$. Alternatively if $x^* \in WPO(S, D) \setminus PO(S, D)$, let $\{S^m\}_{m\in\mathbb{N}} \to S$ be such that for each $m \in \mathbb{N}$, $T \subseteq S^m$ and $x^* \in PO(S^m, D)$ (this is possible since $x^* \in PO(T, D)$). Then by the previous case, $F(S^m, D) = x^*$ and by *set-continuity* of F, $F(S, D) = x^*$.

For the second statement, let $F : \mathcal{B}_{\neq} \to \mathbb{R}^N$ be a bargaining rule that is *weakly Pareto optimal, strongly monotonic, set-continuous,* and *scale invariant.* We already established that $F = F^p$ for some generator function p. We next show that $F = F^p$ is *scale invariant* if and only if p is *scale invariant.* For this, let $(S, D) \in \mathcal{B}_{\neq}$ and $\lambda \in \Lambda$.

First note that

$$F^p(\lambda(S), \lambda(D)) = WPO(\lambda(S), \lambda(D)) \cap p(\lambda(D)).$$

Next, note that $WPO(\lambda(S), \lambda(D)) = \lambda(WPO(S, D))$. Then,

$$\begin{aligned}\lambda(F^p(S, D)) &= \lambda(WPO(S, D) \cap p(D)) \\ &= \lambda(WPO(S, D)) \cap \lambda(p(D)) = WPO(\lambda(S), \lambda(D)) \cap \lambda(p(D)).\end{aligned}$$

Therefore, $p(\lambda(D)) = \lambda(p(D))$ implies $F^p(\lambda(S), \lambda(D)) = \lambda(F^p(S, D))$ (therefore, *scale invariance* of p implies *scale invariance* of F^p). Alternatively assume $p(\lambda(D)) \neq \lambda(p(D))$. For each $r > \sum \overline{d}_i(\lambda(D))$, let $T^r = \{x \in \mathbb{R}^N \mid \sum x_i \leqq r \text{ and } x \geqq \underline{d}(\lambda(D))\}$ and note that $(T^r, \lambda(D)) \in \mathcal{B}_{\neq}$. Now there is $r^* > \sum \overline{d}_i(\lambda(D))$ such that

$$WPO(T^{r^*}, \lambda(D)) \cap p(\lambda(D)) \neq WPO(T^{r^*}, \lambda(D)) \cap \lambda(p(D))$$

since otherwise, the two paths coincide. This implies, $F^p(\lambda(T^{r^*}), \lambda(D)) \neq \lambda(F^p(T^{r^*}, D))$ (therefore, *scale invariance* of F^p implies *scale invariance* of p).

■

On two-agent problems, *scale invariant* monotone path rules take a very simple form. Each can be completely characterized by at most eight distinct paths. With more agents, this upper bound jumps to infinity.[11] Since two-agent problems are central in bargaining theory, in what follows we will make use of this property to focus on them further.

Two-agent bargaining problems (which we will represent with $\mathcal{B}^2$) can be grouped into three distinct classes. In the first class of problems, the disagreement of one agent is strictly preferred by both agents to the disagreement of the other.[12] We represent this class with

$\mathcal{B}^2_{>} = \{(S, D) \in \mathcal{B}^2_{\neq} \mid \text{there is } i, j \in \{1, 2\} \text{ such that } i \neq j \text{ and for all } k \in \{1, 2\}, D_{ik} > D_{jk}\}$.

[11] This has got to do with the fact that with two agents, all disagreement matrices are divided into eight equivalence classes: two matrices in the same class are related by a positive affine transformation. In an equivalence class, it is sufficient to specify a monotone path for one matrix; scale invariance then defines the paths of the other matrices. With three or more agents however, the number of equivalence classes becomes infinite.

[12] In a private communication, Steve Brams suggested the attack on Pearl Harbor as an example to this case. At the time, both parties preferred Japan's war declaration to that of U.S.. For the U.S. government this would strenghten the domestic support for the war effort. As we now know, the Japanese government preferred the advantage of a preemptive strike. Of course, whether the U.S. government expected such an attack is an interesting issue and raises questions about asymmetric information on the implications of disagreement.

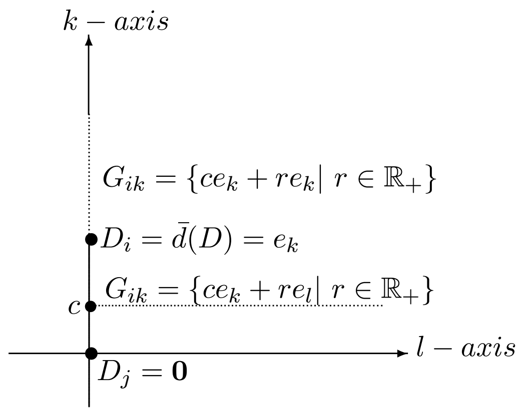

Figure 2: The configuration of the monotone paths in Corollary 8.

Two monotone paths suffice to describe all scale invariant monotone path rules on $\mathcal{B}^2_>$.

Corollary 7 *On $\mathcal{B}^2_>$, a bargaining rule is* weakly Pareto optimal, strongly monotonic, set-continuous, *and* scale invariant *if and only if it is a monotone path rule F^p of the following kind : there are two monotone paths $G_1 = p\left(\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}\right)$ and $G_2 = p\left(\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}\right)$ and for each $(S, D) \in \mathcal{B}^2_>$ satisfying $D_i > D_j$, $F^p(S, D)$ is the inverse image under the positive affine transformation $\lambda \in \Lambda$ defined by $\lambda(D_j) = \mathbf{0}$ and $\lambda(D_i) = \mathbf{1}$ of the maximal point of $\lambda(S)$ along G_i. That is, $F^p(S, D) = \lambda^{-1}(WPO(\lambda(S), \lambda(D)) \cap G_i)$.*

Please see the Appendix for the proofs of Corollary 7 and the following Corollary 8.

The second class of problems represents cases where one agent is indifferent between the two disagreement alternatives and the other has strict preferences. To represent this class, let

$$\mathcal{B}^2_{\geq} = \{(S, D) \in \mathcal{B}^2_{\neq} \mid \text{there is } i, j, k, l \in \{1, 2\} \text{ such that } i \neq j,\ k \neq l,\ D_{ik} > D_{jk} \text{ and } D_{il} = D_{jl}\}.$$

On this class, a *scale invariant* monotone path rule is specified by four monotone paths that have very particular shapes (see Figure 2).

Corollary 8 *On $\mathcal{B}^2_{\geq}$, a bargaining rule is* weakly Pareto optimal, strongly monotonic, set-continuous, *and* scale invariant *if and only if it is a monotone path rule F^p of the following kind: there are four monotone paths $G_{11} = p\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\right)$, $G_{12} = p\left(\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right)$, $G_{21} =$*

$p\left(\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\right)$, $G_{22} = p\left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\right)$ *and for each* $(S, D) \in \mathcal{B}^2_{\geq}$ *satisfying* $D_{ik} > D_{jk}$ *and* $D_{il} = D_{jl}$, $F^p(S, D)$ *is the inverse image under the positive affine transformation* $\lambda \in \Lambda$ *defined by* $\lambda(D_j) = \mathbf{0}$ *and* $\lambda(D_i) = e_k$ *of the maximal point of* $\lambda(S)$ *along* G_{ik}. *That is,* $F^p(S, D) = \lambda^{-1}(WPO(\lambda(S), \lambda(D)) \cap G_{ik})$. *Furthermore, for each* $i, k \in \{1, 2\}$, *there is* $c \in [0, 1]$ *and* $m \in \{1, 2\}$ *such that* G_{ik} *is defined as* $G_{ik} = \{ce_k + re_m \mid r \in \mathbb{R}_+\}$.

In the third class of problems, the agents disagree on their (strict) ranking of the two disagreement alternatives.[13] We represent this class with

$$\mathcal{B}^2_{\ngtr} = \{(S, D) \in \mathcal{B}^2_{\neq} \mid \text{there is } i, j, k, l \in \{1, 2\} \text{ such that } i \neq j,\ k \neq l,\ D_{ik} > D_{jk}, \text{ and } D_{il} < D_{jl}\}.$$

For this subclass, a result very similar to Corollary 7 holds: *scale invariant* monotone path rules can be fully described by the specification of two paths, $G_1 = p\left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$ and $G_2 = p\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\right)$. Among them, the only *symmetric* rule is the following.

The **Cardinal Egalitarian rule**, F^{CE}, assigns each $D \in \mathbb{R}^{N \times N}$ to the linear monotone path that passes through $\underline{d}(D)$ and $\overline{d}(D)$: $p^{CE}(D) = \{\overline{d}(D) + r(\overline{d}(D) - \underline{d}(D)) \mid r \in \mathbb{R}_+\}$. This rule is well-defined for all nonanonymous-disagreement problems, $\mathcal{B}_{\neq}$, independent of the number of agents. Additional to the properties stated in the next proposition, it is *set-continuous.*

Proposition 9 *The Cardinal Egalitarian rule,* F^{CE}, *is* weakly Pareto optimal, strongly monotonic, scale invariant, *and* symmetric *on* $\mathcal{B}_{\neq}$. *Furthermore on* $\mathcal{B}^2_{\ngtr}$, *it is the unique bargaining rule that satisfies these properties.*

Proof. It is straightforward to show that F^{CE} satisfies these properties. Conversely let F be any bargaining rule on $\mathcal{B}^2_{\ngtr}$ that satisfies them. Take any $(S, D) \in \mathcal{B}^2_{\ngtr}$. We want to show that $F(S, D) = F^{CE}(S, D)$.

[13] As discussed in the introduction, the Cyprus negotiations at 2004 fall into this class since both parties seem to have preferred the other leaving the table to themselves doing so.

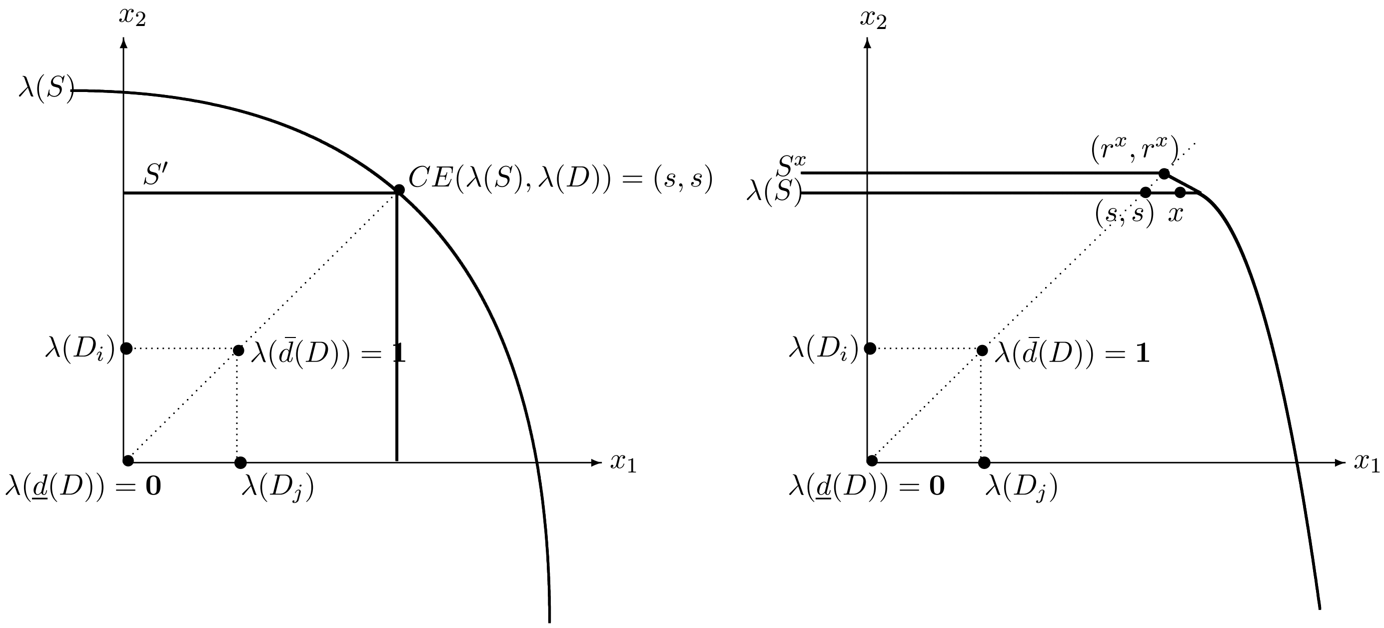

Figure 3: Constructing S' (on the left) and S^x (on the right) in Case 1, proof of Proposition 9.

Consider the positive affine transformation $\lambda \in \Lambda$ such that $\lambda_i(x) = \frac{x_i - \underline{d}_i(D)}{\bar{d}_i(D) - \underline{d}_i(D)}$ for $i \in N$. Note that $\lambda(\bar{d}(D)) = \mathbf{1}$ and $\lambda(\underline{d}(D)) = \mathbf{0}$. Then, by definition $F^{CE}(\lambda(S),\lambda(D)) = (s,s)$, for some $s > 1$. Consider $S' = 0\text{-}comp\{(s,s)\}$. Note that $\lambda(D) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\lambda(D) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. Since S' is symmetric, $(S',\lambda(D))$ is a symmetric problem. Then, by *symmetry* and *weak Pareto optimality* of F, $F(S',\lambda(D)) = (s,s)$. Since $\lambda(S) \supseteq S'$, *strong monotonicity* of F implies $F(\lambda(S),\lambda(D)) \geqq F(S',\lambda(D))$.

Now if $(s,s) \in PO(\lambda(S),\lambda(D))$ (as in the left problem in Figure 3), then $F(\lambda(S),\lambda(D)) = (s,s) = F^{CE}(\lambda(S),\lambda(D))$. Alternatively, assume that $(s,s) \in WPO(\lambda(S),\lambda(D))$ (as in the right problem in Figure 3). Suppose $F(\lambda(S),\lambda(D)) = x \geq (s,s)$. Let $r^x \in \mathbb{R}$ be such that $s < r^x < \max\{x_1, x_2\}$. Let $S^x \subseteq \mathbb{R}^N$ be such that $S^x = conv\{\underline{d}(D)\text{-}comp\{r^x, r^x\}, \lambda(S)\}$. Note that $(S^x,\lambda(D)) \in \mathcal{B}_{\neq}$ and $F^{CE}(S^x,\lambda(D)) = (r^x, r^x) \in PO(S^x,\lambda(D))$. So by the previous argument, $F(S^x,\lambda(D)) = (r^x,r^x)$. Also since $s < r^x$, $S^x \supseteq \lambda(S)$. Thus by *strong monotonicity* of F, $F(S^x,\lambda(D)) = (r^x,r^x) \geqq x = F(\lambda(S),\lambda(D))$, contradicting $r^x < \max\{x_1,x_2\}$. Therefore, $F(\lambda(S),\lambda(D)) = (s,s)$.

Finally, by *scale invariance* of F and F^{CE}, $F(S,D) = \lambda^{-1}(F(\lambda(S),\lambda(D))) = \lambda^{-1}(F^{CE}(\lambda(S), \lambda(D))) = F^{CE}(S,D)$.

■

Since there are no symmetric problems in $\mathcal{B}^2_> \cup \mathcal{B}^2_{\geq}$, any bargaining rule is *symmetric* on those classes of problems. Therefore, the properties of Proposition 9 do not pinpoint a single rule on $\mathcal{B}^2_> \cup \mathcal{B}^2_{\geq}$.[14]

It follows from the above results that the construction of a *scale invariant* monotone path rule on $\mathcal{B}^2_{\neq}$ involves the specification of at most eight monotone paths (two for $\mathcal{B}^2_>$, two for $\mathcal{B}^2_{\not>}$, and four for $\mathcal{B}^2_{\geq}$).

Remark 10 *The properties* weak Pareto optimality, strong monotonicity, set-continuity, scale invariance, *and* symmetry *are logically independent. For this, note that the rule* F^1 *defined as* $F^1(S,D) = \overline{d}(D)$ *satisfies all properties except* weak Pareto optimality*. The rule* F^2 *defined as* $F^2(S,D) = (max\{x_1 \mid\ x \in S \text{ and } x_2 = \underline{d}_2(D)\}, \underline{d}_2(D))$ *satisfies all properties except* symmetry*. The rule* F^3 *defined as*

$$F^3(S,D) = \arg\max_{x \in S} \min_{i \in N} \; x_i - \underline{d}_i(D)$$

satisfies all properties except scale invariance*. Finally, let* $m_i^* = max\{x_i \mid x \in S \text{ and } x \geqq \underline{d}(D)\}$ *and define* F^4 *as*

$$F^4(S,D) = \arg\max_{x \in S} \min_{i \in N} \frac{x_i - \overline{d}_i(D)}{m_i^* - \overline{d}_i(D)}.$$

This rule satisfies all properties except strong monotonicity*. Finally, let* F^5 *coincide with* F^{CE} *everywhere except* $\mathcal{B}^2_>$*. There, let* F^5 *be as explained in Corollary 7 except let* $G_i = [\mathbf{1}, \mathbf{1} + 1e_i] \cup \{\mathbf{1} + 1e_i + re_j \mid r \in \mathbb{R}_+\}$ *for* $i \neq j$*. Since each* G_i *violates Condition (ii) in the definition of the generator functions,* F^5 *violates* set-continuity *but it satisfies all the other properties above.*

[14] Also, we do not state uniqueness for more than two agents. The following is an example of a rule that satisfies all the properties in the proposition and that is different from the Cardinal Egalitarian rule for problems with more than two agents. Let $\xi : \mathbb{R}^{N\times N} \to \mathbb{R}^{N \times N}$ be defined as

$$[\xi(D)]_{ji} = \begin{cases} D_{ji} & \text{if } D_{ji} = \underline{d}_i(D), \\ \\ \min\{D_{ji} \mid D_{ji} \neq \underline{d}_i(D)\} & \text{otherwise.} \end{cases}$$

Then, let $F^\xi \in \mathcal{F}_{\neq}$ be defined as $F^\xi(S,D) = F^{CE}(S, \xi(D))$ for each $(S,D) \in \mathcal{B}_{\neq}$.

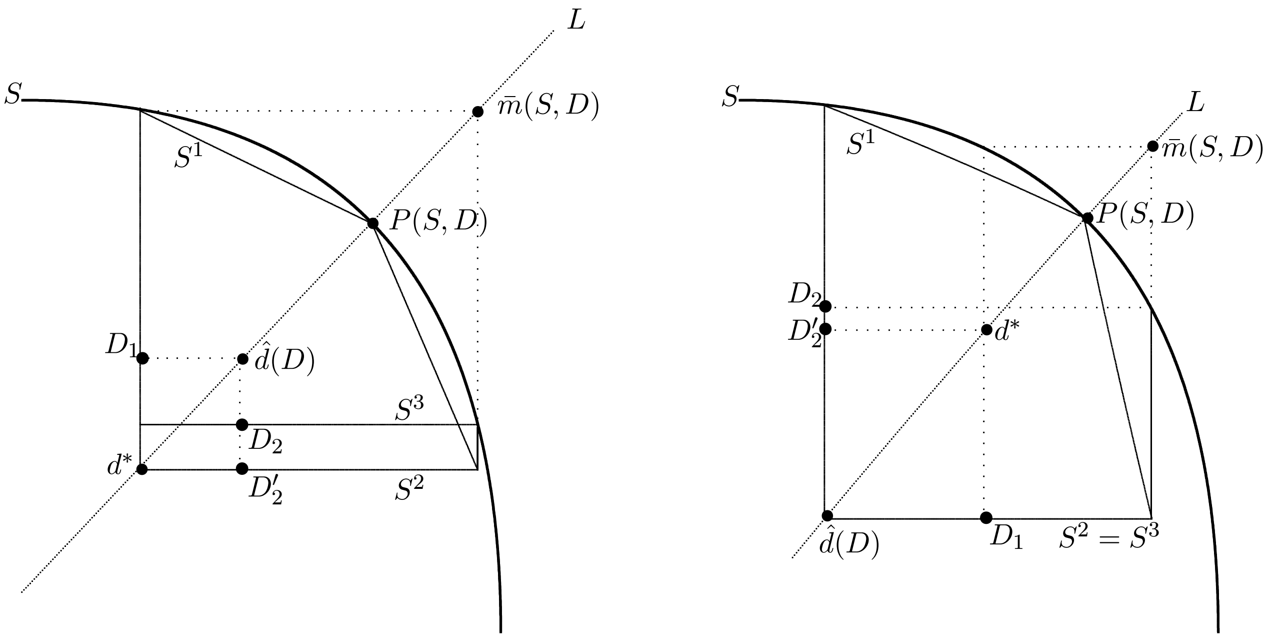

Figure 4: Two alternative configurations in Case 1.1, proof of Theorem 11 ($i = 1$ on the left and $i = 2$ on the right).

3.3 Proportional Rule

Weakening the *strong monotonicity* requirement of Kalai (1977) to the *restricted monotonicity* property of Roth (1979) significantly enlarges the class of admissible rules (as well as their domain from $\mathcal{B}_{\neq}$ to $\mathcal{B}$); even when one continues to demand all the other properties of the previous subsection. The following *nondecomposable* rule, the **Proportional Rule**, P, is a member of this class and is defined as follows. For each $(S, D) \in \mathcal{B}$ and $i \in N$, let $\hat{d}_i(D) = max\{D_{ji} | j \in N \setminus \{i\}\}$. Then

$$P(S, D) = \arg\max_{x \in S} \min_{i \in N} \frac{x_i - \hat{d}_i(D)}{\overline{m}_i(S, D) - \hat{d}_i(D)}.$$

The *Proportional rule*, unlike the *Cardinal Egalitarian rule*, assigns different roles to an agent's payoff from his own disagreement (D_{ii}) and his payoffs from others' disagreements (D_{ji}). The former determines $\overline{m}_i(S, D)$ while the latter determines $\hat{d}_i(D)$.

Theorem 11 *The Proportional rule, P, is* weakly Pareto optimal, restricted monotonic, scale invariant*,* symmetric*, and* limitedly sensitive *on $\mathcal{B}$. Furthermore on $\mathcal{B}^2$, it is the unique bargaining rule that satisfies these properties.*

Proof. It is straightforward to show that P satisfies these properties. Conversely let F be any bargaining rule on $\mathcal{B}^2$ that satisfies them. Take any $(S, D) \in \mathcal{B}^2$. We want to show that $F(S, D) = P(S, D)$. First note that on $\mathcal{B}^2$, P is also *(strongly) Pareto optimal*. Thus, $P(S, D) \in PO(S, D)$.

Case 1: Suppose that $\bar{m}(S, D) \notin S$.

<u>*Claim:*</u> For any $x \in S$ such that $x \leq P(S, D)$, $x \notin WPO(S, D)$.
<u>*Proof of Claim:*</u> Let $x \in S$. If $x < P(S, D)$, trivially $x \notin WPO(S, D)$. Alternatively, assume $x_1 = P_1(S, D)$ and $x_2 < P_2(S, D)$. Since S is convex and $(\bar{m}_1(S, D), D_{22}) \in S$, $K = conv\{(\bar{m}_1(S, D), D_{22}), P(S, D)\} \subseteq S$. Since $\bar{m}_1(S, D) > P_1(S, D) = x_1$, for each $y \in K \setminus \{P(S, D)\}$, we have $y_1 > x_1$. Also, since $P_2(S, D) > x_2$, there is $y^* \in K \setminus \{P(S, D)\}$ such that $y_2^* > x_2$. But then $y^* > x$ and $y^* \in S$ imply $x \notin WPO(S, D)$. A similar argument for the alternative case where $x_1 < P_1(S, D)$ and $x_2 = P_2(S, D)$ concludes the proof of the claim.

Let $i \in N$ and $j \neq i$ satisfy $D_{i1} \leqq D_{j1}$ and if $D_{i1} = D_{j1}$ then $D_{i2} \leqq D_{j2}$. Let L be the line passing through $\bar{m}(S, D)$ and $\hat{d}(D)$. Let $d^* \in L$ satisfy $d_1^* = D_{11}$.

Case 1.1: Suppose that $[d_1^* = \underline{d}_1(D)$ and $d_2^* \leq \underline{d}_2(D)]$ or $\left[d_1^* = \overline{d}_1(D)\right.$ and $\left.d_2^* < \overline{d}_2(D)\right]$ (see Figure 4). Then, let S^1, S^2, and S^3 be as follows:

$$S^1 = \begin{cases} conv\{d^*, (\bar{m}_1(S, D), d_2^*), P(S, D), (d_1^*, \bar{m}_2(S, D))\} & \text{if } D_{i1} = D_{j1}, \\ conv\{d^*, (\bar{m}_1(S, D), d_2^*), P(S, D), (d_1^*, \bar{m}_2(S, D))\} & \text{if } D_{i1} \neq D_{j1} \text{ and } d_1^* = D_{i1}, \\ conv\{(D_{i1}, D_{j2}), (\bar{m}_1(S, D), D_{j2}), P(S, D), (D_{i1}, max\left\{x_2 \mid \begin{array}{c} x \in S, \\ x_1 = D_{i1} \end{array}\right\})\} & \text{if } D_{i1} \neq D_{j1} \text{ and } d_1^* = D_{j1}. \end{cases}$$

$$S^2 = \begin{cases} d^*\text{-}comp\{x \in S \mid x \geq (D_{i1}, D_{j2})\} & \text{if } D_{i1} = D_{j1}, \\ d^*\text{-}comp\{x \in S \mid x \geq (D_{i1}, D_{j2})\} & \text{if } D_{i1} \neq D_{j1} \\ & \text{and } d_1^* = D_{i1}, \\ \hat{d}(D)\text{-}comp\{x \in S \mid x \geq (D_{i1}, D_{i2})\} & \text{if } D_{i1} \neq D_{j1} \\ & \text{and } d_1^* = D_{j1}. \end{cases}$$

$$S^3 = \underline{d}(D)\text{-}comp\{S^2\}.$$

Note that $S^1 \subseteq S^2$, $S^3 \subseteq S^2$, and $S^3 \subseteq S$, but not always $S^2 \subseteq S$. Let $D_2' = (D_{21}, d_2^*)$. First, we consider $(S^1, D_1, D_2') \in \mathcal{B}^2$. Let $\lambda(x) = (\frac{x_1 - d_1^*}{\bar{m}_1(S,D) - d_1^*}, \frac{x_2 - d_2^*}{\bar{m}_2(S,D) - d_2^*})$. Note that, $\lambda(d^*) = \mathbf{0}$, $\lambda(\bar{m}(S,D)) = \mathbf{1}$ and $\lambda(S^1)$ is symmetric. Also note that $\lambda_1(D_{11}) = \lambda_2(D_{22}')$ and $\lambda_1(D_{21}') = \lambda_2(D_{12})$. Therefore, $(\lambda(S^1), \lambda(D_1), \lambda(D_2'))$ is a symmetric problem and by *symmetry* of F, $F_1(\lambda(S^1), \lambda(D_1), \lambda(D_2')) = F_2(\lambda(S^1), \lambda(D_1), \lambda(D_2'))$. By *weak Pareto optimality* of F, $F(\lambda(S^1), \lambda(D_1), \lambda(D_2')) = P(\lambda(S^1), \lambda(D_1), \lambda(D_2'))$. Since F and P are *scale invariant*, $F(S^1, D_1, D_2') = P(S^1, D_1, D_2')$. Note that, $P(S^1, D_1, D_2') = P(S^2, D_1, D_2') = P(S^2, D) = P(S^3, D) = P(S, D)$. Since $S^1 \subseteq S^2$, by *restricted monotonicity* of F, $F(S^1, D_1, D_2') \leqq F(S^2, D_1, D_2')$. Since $F(S^1, D_1, D_2') = P(S, D) \in PO(S^2, D_1, D_2')$, $F(S^1, D_1, D_2') = F(S^2, D_1, D_2')$. By *limited sensitivity* of F, $F(S^2, D_1, D_2') = F(S^2, D_1, D_2)$. Since $S^3 \subseteq S^2$, by *restricted monotonicity* of F, $F(S^3, D) \leqq F(S^2, D)$. But since $F(S^2, D) = P(S^3, D)$, $x \leq F(S^2, D)$ implies by the Claim that $x \notin WPO(S^3, D)$. Therefore, $F(S^3, D) = F(S^2, D)$. Finally since $S^3 \subseteq S$, by *restricted monotonicity* of F, $F(S^3, D) \leqq F(S, D)$ and since $F(S^3, D) = P(S, D) \in PO(S, D)$, $F(S, D) = P(S, D)$.

Case 1.2: Suppose that $[d_1^* = \underline{d}_1(D)$ and $d_2^* > \underline{d}_2(D)]$ or $[d_1^* = \overline{d}_1(D)$ and $d_2^* \geq \overline{d}_2(D)]$ (see Figure 5). Let $\tilde{d}^* \in L$ satisfy $\tilde{d}_2^* = D_{22}$. Define $\tilde{S}^1$, $\tilde{S}^2$, and $\tilde{S}^3$ as follows:

$$\tilde{S}^1 = \begin{cases} conv\{(D_{i1}, D_{j2}), max\{x \in S \mid x_2 = D_{j2}\}, P(S,D), (D_{i1}, \bar{m}_2(S,D))\} & \text{if } \tilde{d}_2^* > \underline{d}_2(D), \\ conv\{\tilde{d}^*, (\bar{m}_1(S,D), \tilde{d}_2^*), P(S,D), (\tilde{d}_1^*, \bar{m}_2(S,D))\} & \text{if } \tilde{d}_2^* = \underline{d}_2(D). \end{cases}$$

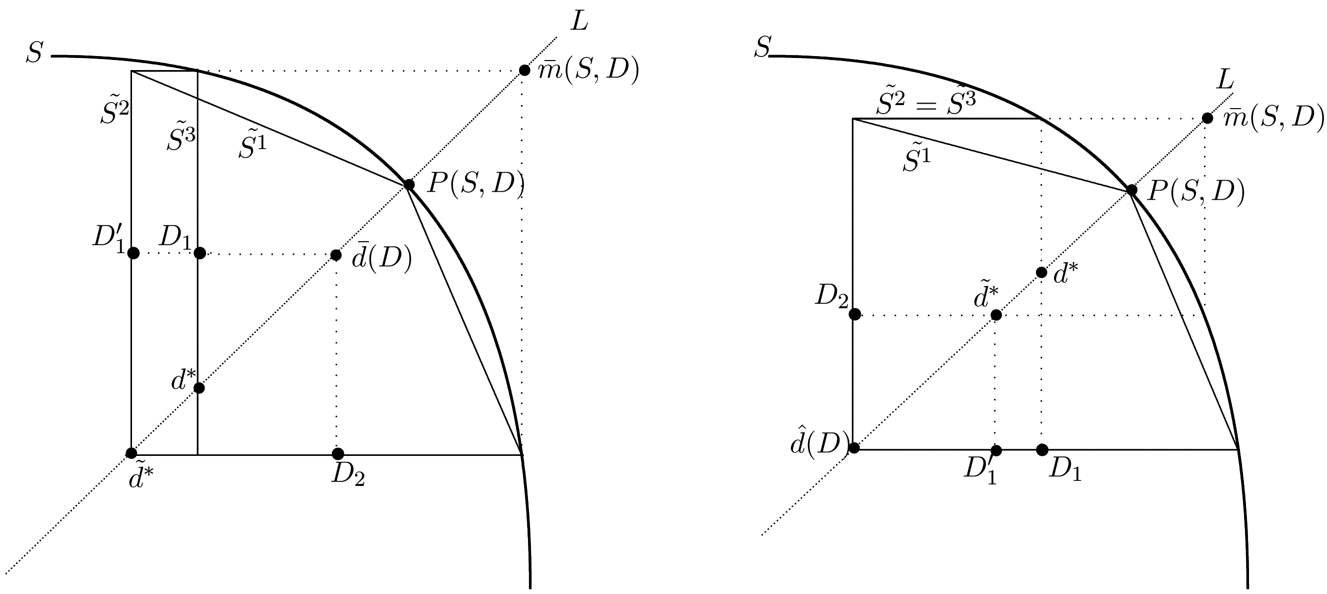

Figure 5: Two alternative configurations in Case 1.2, proof of Theorem 11 ($i = 1$ on the left and $i = 2$ on the right).

$$\tilde{S}^2 = \begin{cases} \hat{d}(D)\text{-}comp\{x \in S \mid x_2 \leq \bar{m}_2(S,D)\} & \text{if } \tilde{d}_2^* > \underline{d}_2(D), \\ \\ \tilde{d}^*\text{-}comp\{x \in S \mid x_2 \leq \bar{m}_2(S,D)\} & \text{if } \tilde{d}_2^* = \underline{d}_2(D) \end{cases}$$

$$\tilde{S}^3 = \underline{d}(D)\text{-}comp\{\tilde{S}^2\}.$$

The rest of the proof is the same as in Case 1.1 except that $\tilde{d}^*$, $\tilde{S}^1$, $\tilde{S}^2$, and $\tilde{S}^3$ replace d^*, S^1, S^2, and S^3, respectively (and we now use $D_1' = (\tilde{d}_1^*, D_{12})$).

Case 2: Suppose that $\bar{m}(S,D) \in S$.

Let $\hat{S} = \hat{d}(D)$-*comp*$\{S\}$. First, consider the problem $(\hat{S}, \hat{d}(D))$. Let $\lambda_i(x) = (\frac{x_i - \hat{d}_i(D)}{\bar{m}_i(S,D) - \hat{d}_i(D)})$ for $i \in \{1,2\}$. Note that $\lambda(\hat{d}(D)) = \mathbf{0}$, $\lambda(\bar{m}(S,D)) = \mathbf{1}$, and $\lambda(\hat{S})$ is symmetric. Therefore, $(\lambda(\hat{S}), \lambda(\hat{d}(D)))$ is a symmetric problem and by *symmetry* of F, $F_1(\lambda(\hat{S}), \lambda(\hat{d}(D))) = F_2(\lambda(\hat{S}), \lambda(\hat{d}(D)))$. By *weak Pareto optimality* of F, $F(\lambda(\hat{S}), \lambda(\hat{d}(D))) = P(\lambda(\hat{S}), \lambda(\hat{d}(D)))$. Since F and P are *scale invariant*, $F(\hat{S}, \hat{d}(D)) = P(\hat{S}, \hat{d}(D))$. Note that, $P(\hat{S}, \hat{d}(D)) = P(S, \hat{d}(D)) = P(S,D) = \bar{m}(S,D)$. Since $\hat{S} \subseteq S$, by *restricted monotonicity* of F, $F(\hat{S}, \hat{d}(D)) \leqq$

$F(S, \hat{d}(D))$. Since $F(\hat{S}, \hat{d}(D)) = P(S, D)$ is *Pareto optimal however*, $F(\hat{S}, \hat{d}(D)) = F(S, \hat{d}(D))$. Then by *limited sensitivity* of F, $F(S, \hat{d}(D)) = F(S, D)$. Thus, $F(S, D) = P(S, D)$.

■

For problems with more than two agents, there are other rules that satisfy the properties in Theorem 11. For example, consider the alternative rule which is obtained by replacing the $\widehat{d}_i(D)$ with $\widetilde{d}_i(D) = \min\{D_{ji} \mid j \in N \setminus \{i\}\}$ in the definition of the Proportional rule.

Remark 12 *The properties in Theorem 11 are logically independent. For this, note that the rule F^1 defined as $F^1(S, D) = \hat{d}(D)$ satisfies all properties except* weak Pareto optimality. *The rule F^2 defined as $F^2(S, D) = (\max\{x_1 \mid x \in S \text{ and } x_2 = \hat{d}_2(D)\}, \hat{d}_2(D))$ satisfies all properties except* symmetry. *The rule F^3 defined as*

$$F^3(S, D) = \arg\max_{x \in S} \min_{i \in N} x_i - \hat{d}_i(D)$$

satisfies all properties except scale invariance. *The rule F^4 defined as $F^4(S, D) = \nu(S, \hat{d}(D))$ (where ν is the Nash (1950) bargaining rule) satisfies all properties except* restricted monotonicity. *Finally, the rule F^5 defined as*

$$F^5(S, D) = \arg\max_{x \in S} \min_{i \in N} \frac{x_i - \overline{d}_i(D)}{\bar{m}_i(S, D) - \overline{d}_i(D)}$$

satisfies all properties except limited sensitivity.

4 Appendix

Proof. (Lemma 2) Take any bargaining rule F. First note that, if F is decomposable with $F = \phi \circ \alpha$, then $\phi = F \mid_{\mathcal{B}_=}$. To see this, let $(S, D) \in \mathcal{B}_=$. Then $D_1 = ... = D_n = d$ and thus, $\alpha(D) = d$. We then have, $F(S, D) = \phi(S, \alpha(D))$. Noting Remark 1, $\phi(S, \alpha(D)) = \phi(S, D)$ gives the desired conclusion.

Suppose first that F is *decomposable*. Let $F = \phi \circ \alpha$. Take any $D \in \mathbb{R}^{N \times N}$. Then for each $S \subseteq \mathbb{R}^N$ such that $(S, D) \in \mathcal{B}$, $\phi(S, \alpha(D)) = F(S, D)$. Since by the previous paragraph $\phi(S, \alpha(D)) = F(S, \alpha(D))$, we have $\alpha(D) \in \delta^F(D)$. Since $\delta^F(D) \neq \emptyset$ for each $D \in \mathbb{R}^{N \times N}$, F is *disagreement-simple*.

Conversely assume that F is *disagreement-simple*. Then for each $D \in \mathbb{R}^{N\times N}$, $\delta^F(D) \neq \emptyset$. Define the aggregator function $\alpha : \mathbb{R}^{N\times N} \to \mathbb{R}^N$ as a selection from δ^F, that is, for each $D \in \mathbb{R}^{N\times N}$, $\alpha(D) \in \delta^F(D)$. Let $\phi = F\mid_{\mathcal{B}_=}$. Then for each $(S,D) \in \mathcal{B}$, $F(S,D) = F(S,\alpha(D)) = \phi(S,\alpha(D))$. Thus $F = \phi \circ \alpha$, that is, F is decomposable.

■

Proof. (Corollary 7) It follows from Theorem 6 that a bargaining rule satisfies the given properties if and only if it is a monotone path rule F^p where the generator function p is *scale invariant*.

Let G_1 and G_2 be as defined in the statement of Corollary 7. Now let $(S,D) \in \mathcal{B}^2_>$ satisfy $D_i > D_j$. Let $\lambda_1(x) = \frac{x-D_{j1}}{D_{i1}-D_{j1}}$ and $\lambda_2(x) = \frac{x-D_{j2}}{D_{i2}-D_{j2}}$. Note that $\lambda(D_i) = \mathbf{1}$ and $\lambda(D_j) = \mathbf{0}$. Therefore, $\lambda(p(D)) = G_i$. Then by *scale invariance*, $F^p(S,D) = \lambda^{-1}(F^p(\lambda(S),\lambda(D))) = \lambda^{-1}(WPO(\lambda(S),\lambda(D)) \cap G_i)$ establishes the claim.

■

Proof. (Corollary 8) It follows from Theorem 6 that a bargaining rule satisfies the given properties if and only if it is a monotone path rule F^p where the generator function p is *scale invariant*.

Now let $(S,D) \in \mathcal{B}^2_\geq$ satisfy $D_{ik} > D_{jk}$ and for $l \neq k$, $D_{il} = D_{jl}$. Let $\lambda_k(x) = \frac{x-D_{jk}}{D_{ik}-D_{jk}}$ and for $l \neq k$, let $\lambda_l(x) = x - D_{il}$. Note that $\lambda(D_i) = e_k$ and $\lambda(D_j) = \mathbf{0}$. Therefore, $\lambda(p(D)) = G_{ik}$. Then by *scale invariance*, $F^p(S,D) = \lambda^{-1}(F^p(\lambda(S),\lambda(D))) = \lambda^{-1}(WPO(\lambda(S),\lambda(D)) \cap G_{ik})$ establishes the claim.

Let G_{11}, G_{12}, G_{21}, and G_{22} be as defined in the statement of Corollary 8. We next show that each G_{ik} is of the claimed form (see Figure 2). Note that G_{ik} is used for the matrix D^{ik} such that $D^{ik}_{ik} = 1$ and $D^{ik}_{jk} = D^{ik}_{il} = D^{ik}_{jl} = 0$. Thus $\overline{d}(D^{ik}) = e_k$. Also note that $y \in G_{ik}$ implies $y \geqq \mathbf{0}$.

If all $y \in G_{ik}$ satisfies $y_l = 0$, then for $c = 1$ and $m = k$, $G_{ik} = \{ce_k + re_m \mid r \in \mathbb{R}_+\}$ and the result holds. Alternatively assume there is $y \in G_{ik}$ such that $y_l > 0$. Suppose $y_k > 1$. Let $r \in \mathbb{R}_{++} \setminus \{1\}$, and define $\lambda^r \in \Lambda$ as $\lambda^r_k(x) = x$ and $\lambda^r_l(x) = rx$. Let $S \subseteq \mathbb{R}^N$ be such that $(S, D^{ik}) \in \mathcal{B}^2_\geq$ and $y \in PO(S, D^{ik})$. Then $F^p(S, D^{ik}) = y$ and by *scale invariance*, $F^p(\lambda^r(S), \lambda^r(D^{ik})) = \lambda^r(y)$. However, $\lambda^r(D^{ik}) = D^{ik}$. Thus, $F^p(\lambda^r(S), D^{ik}) = \lambda^r(y)$, by definition of F^p, implies $\lambda^r(y) \in G_{ik}$. But since $y_l > 0$, $\lambda^r(y) \neq y$. This and $y_k = (\lambda^r(y))_k >$

$1 = \overline{d}_k(D)$ contradict Condition (ii) of p. Therefore, the supposition is wrong and we have $y_k \leqq 1$.

Now $y \in G_{ik}$, $y_l > 0$, $y_k \leqq 1$, and for each $r \in \mathbb{R}_{++}$, $\lambda^r(y) \in G_{ik}$. Also $\lim_{r\to 0} \lambda^r(y) = y_k e_k$. These establish that for $c = y_k \in [0,1]$ and $m = l$, $G_{ik} = \{ce_k + re_m \mid r \in \mathbb{R}_+\}$.

■

Example 13 *(A nondecomposable monotone path rule F whose every ε neighborhood contains a decomposable monotone path rule F^ε) For $n = |N|$, let $x^* = \frac{1}{n}\mathbf{1} + \frac{1}{2n}(e_1 - e_2)$. For $k \in \mathbb{N}$, let $x_k = \frac{1}{n}\mathbf{1} + \frac{k}{2n(k+1)}(e_1 - e_2)$ and note that $\lim_{k\to\infty} x_k = x^*$. Let $[a,b] \subseteq \mathbb{R}^N$ represent the line segment that connects $a, b \in \mathbb{R}^N$. Let $D^* = \left[\begin{smallmatrix}-e_1\\-e_2\end{smallmatrix}\right]$. For each $D \in \mathbb{R}^{N\times N}$, let*

$$p(D) = \begin{cases} [d,\mathbf{0}] \cup [\mathbf{0}, x_k] \cup [x_k, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} & \text{if } D = d = -k\mathbf{1} \text{ for } k \in \mathbb{N}, \\ [\mathbf{0}, x^*] \cup [x^*, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} & \text{if } D = D^*, \\ \{\overline{d}(D) + r\mathbf{1} \mid r \geq 0\} & \text{otherwise.} \end{cases}$$

For each $(S,D) \in \mathcal{B}$, let $F(S,D) = WPO(S,D) \cap p(D)$. Since there is no $d \in \mathbb{R}^N$ such that $p(D^*) \subseteq p(d)$, by Remark 5, F is nondecomposable.

Given $\varepsilon > 0$, let $K(\varepsilon) \in \mathbb{N}$ be such that $K(\varepsilon) > \frac{2(1-(n+1)\varepsilon)}{n\varepsilon}$. For each $D \in \mathbb{R}^{N\times N}$, let

$$p^\varepsilon(D) = \begin{cases} [d,\mathbf{0}] \cup [\mathbf{0}, x^*] \cup [x^*, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} & \text{if } D = d = -K(\varepsilon)\mathbf{1}, \\ [d,\mathbf{0}] \cup [\mathbf{0}, x_k] \cup [x_k, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} & \text{if } D = d = -k\mathbf{1} \text{ for } k \in \mathbb{N}\setminus\{K(\varepsilon)\}, \\ [\mathbf{0}, x^*] \cup [x^*, \mathbf{1}] \cup \{r\mathbf{1} \mid r > 1\} & \text{if } D = D^* \\ \{\overline{d}(D) + r\mathbf{1} \mid r \geq 0\} & \text{otherwise.} \end{cases}$$

and for each $(S,D) \in \mathcal{B}$, let $F^\varepsilon(S,D) = WPO(S,D) \cap p^\varepsilon(D)$. Now $p(D^*) \subseteq p(-K(\varepsilon)\mathbf{1})$ and thus, F^ε is decomposable. Furthermore, by choice of $K(\varepsilon)$

$$\begin{aligned} \mu^{\mathcal{F}}(F, F^\varepsilon) &= \max\{\frac{\|x-y\|}{1+\|x-y\|} \mid x \in p(-K(\varepsilon)\mathbf{1}),\ y \in p^\varepsilon(-K(\varepsilon)\mathbf{1}),\ x \not> y,\ y \not> x\} \\ &= \frac{\frac{2}{n(K(\varepsilon)+2)}}{1 + \frac{2}{n(K(\varepsilon)+2)}} < \varepsilon. \end{aligned}$$

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