Pairing Games and Markets*

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Abstract

Pairing Games or *Markets* studied here are the non-two-sided NTU generalization of assignment games. We show that the *Equilibrium Set* is nonempty, that it is the set of *stable* allocations or the set of *semistable* allocations, and that it has has several notable structural properties. We also introduce the solution concept of *pseudostable* allocations and show that they are in the Demand Bargaining Set. We give a dynamic *Market Procedure* that reaches the Equilibrium Set in a bounded number of steps. We use elementary tools of graph theory and a representation theorem obtained here.

Keywords : Stable Matching, Competitive Equilibrium, Market Design, NTU Assignment Game, Roommate Problem, Coalition Formation, Bargaining Set, Bilateral Transaction, Gallai Edmonds Decomposition

1 Introduction

Matching models in economics mostly have a *two-sided* structure, e.g., workers and firms, buyers and sellers. In this paper we study *Pairing Games* or *Pairing Markets* where an arbitrary set of players partition into pairs and singletons. Each pair of players has a continuum of activities to jointly choose from if they form a pair - call it a *partnership* or a *bilateral transaction*. We are interested in outcomes that are stable or in competitive equilibrium and in designing a procedure to achieve them.

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Our model is a generalization of the assignment game (Shapley and Shubik (1972)) in two ways. First, players are not a priori partitioned into two sides. Second, utility realizations permit income effects and are not restricted to the quasilinear, i.e., *transferable utility (TU)* domain.

The assignment game has been very fruitful in modelling a wide range of economic situations, e.g., markets with indivisibles, marriage, fair allocations, principal-agent matching.[1] An important property of the assignment game is the existence and coincidence of pairwise stable and competitive equilibrium allocations. Also, two sidedness has permitted the design of rather simple *coordinated* market procedures[2] for attaining desired outcomes, and the results carry over to more general cases. For example, players' preferences may belong to the general *nontransferable utility* domain[3], players on one side may have *multiple partners* if preferences satisfy *gross substitutability*[4], and players on both sides may have multiple partners if preferences are *additive separable.*[5]

Yet many markets are not two-sided : For example many mergers occur among firms that are alike. Likewise, acquisitions and joint ventures.[6] Various swap markets are example to the multiple partners version of our model.[7] So are organized markets for bilateral contracting in electricity where some players are seller to one partner and buyer to another.[8] It is only recently on the other hand that Pairing Games and Markets are being explored.[9]

One reason why non-two-sided models have not been much considered is the possible nonexistence of stable or competitive equilibrium allocations. This possibility is not uncommon. For example, in the three-player game where two players may share a cake and none of the cakes is sufficiently large in comparison to the other two cakes, the odd-man-out will be able to lure away one of the partners in any pair that forms. So there is no stable allocation or equilibrium in partnership prices.

In this paper we offer a comprehensive analysis for pairing games. We use elementary tools from graph theory and and a representation theorem that we obtain here. We address *Existence*

[1]As in Beckmann and Koopmans (1957), Becker (1973), Alkan, Demange and Gale (1991), Dam and Perez-Castrillo (2006) respectively.

[2]The multi-item auctions in Crawford and Knoer (1981), Demange, Gale and Sotomayor (1986), Perez-Castrillo and Sotomayor (2002).

[3]Alkan (1989,1992,1997), Alkan and Gale (1990).

[4]Kelso and Crawford (1982), Gul and Stachetti (2000), Milgrom (2009).

[5]Sotomayor (1992,2009).

[6]Gong et al (2007) report that most joint ventures especially those succesful are bilateral.

[7]Our main results in this paper would carry over to the multiple partners model under additive separability.

[8]As in the Free Contract Market ACL in Brazil.

[9]This is in contrast to the discrete counterpart of our model, the *roommate problem* (Gale and Shapley (1962)), which has a fairly substantial literature including the interesting application for kidney exchange - e..g., Irving (1985), Tan (1990,1991), Diamantoudi, Miyagawa and Xue (2004), Inarra, Larrea and Molis (2008), Klaus, Klijn and Walzl (2011), and Roth, Sonmez and Unver (2005).

of Solutions, Structural Properties, Bargaining Aspects, and *Procedure Design*. The reason for including all in one paper is the common underlying mathematics.

One of our interests is to address what may happen when stable allocations do not exist. To this end, we first consider *half-partnerships* and allow a player to have two half-partners as an alternative to having one full partner. We call an allocation with no blocking pair *stable* if only full-partnerships can form and *semistable* if both full-partnerships and half-partnerships can form. We then show that there is an *Equilibrium Set* always nonempty (Theorem 1) and that this set is the set of stable allocations or the set of semistable allocations (Theorem 2.) Stable and semistable allocations are competitive equilibrium allocations when players' utilities are interpreted as partnership prices.

The structural properties of the Equilibrium Set do not depend on whether it consists of stable or semistable allocations. The reason for this is that the variable part of the Equilibrium Set is always associated with full-partnerships (Proposition 2.) Then, on the TU domain, the players under full-partnership endogenously partition into *two sides* (Proposition 3) ; hence, the Equilibrium Set of a pairing game is essentially identical to the Equilibrium Set of an assignment game, in particular, it has a lattice structure and admits a median allocation. This may be somewhat surprising but, not surprisingly, is not true on the NTU domain. We also show that the Equilibrium Set has a median property and is a virtually convex set (Propositions 4 and 5.)

Then we consider what would happen if stable allocations do not exist and half-partnerships are not viable. Specifically we look at Bargaining Set allocations where no player joins a blocking pair if she sees disadvantageous counterblocking. Interestingly the Equilibrium Set enters the scene again. We show that each payoff vector in the Equilibrium Set generates a set of *maximum-stable* allocations (Proposition 6) and a particular subset of these - which we call *pseudostable* - is contained in the Demand Bargaining Set[10] (Theorem 3.) While they pertain to different institutional environments, semistable and pseudostable allocations are closely related. To illustrate, in the three-player game, the allocation where each player is half-partner to the other two players and the cakes are shared "equally" is semistable. And each of the three allocations where two players share the cake "equally" and the third player gets nothing is pseudostable.[11]

Another important part of our work is the *Market Procedure* for reaching the Equilibrium Set. It is a non-two-sided and NTU generalization of the Demange Gale Sotomayor (1986) auction. The NTU aspect is based on the key lemma behind Theorem 1. The non-two-sidedness aspect utilizes -

[10]Introduced for TU games by Morelli and Montero (2003) as a refinement of the Zhou Bargaining Set (Zhou (1994)).

[11]The three-player game was taken up by Binmore (1985) for a study of bargaining with pair formation. The three-player game is of course special. Binmore remarked that "the four-player game is less easily dealt with" citing "combinatorial difficulties intrinsic to the problem."

one of our main results in the paper - the representation theorem (Theorem 5) already mentioned.

We present the Market Procedure at two levels. We first show that it reaches the Equilibrium Set in a bounded number of steps (Theorem 4.) Then we spell out (in Appendix D) how the computations can be done *recursively* at each step. Thus the dynamics are specified at a basic level and respect computational efficacy.

In addition to our main results summarized above, we show that stable allocations exist when there are an even number of players in each type (Proposition 1). This result generalizes to the NTU domain the main result of Chiappori, Galichon and Salanie (2012). We also give a characterization of the Demand Bargaining Set for pairing games (Proposition 7.)

The organization of the paper is as follows : In the first subsection below, we review the existing literature on pairing games and point out our contributions ; in the second subsection, we give an analytical overview and describe Theorem 5. Section 2 gives our model and basic definitions. Sections 3 to 6 contain the results mentioned above. Appendix A presents the mathematical tools we use and Theorem 5. Appendix B contains proofs for Sections 3 and 4. Appendix C is an addendum to Section 5 and Appendix D to Section 6.

1.1 Current Literature and Summary of Contributions

The existing literature on pairing games or markets consists of a small number of papers and they are all on the TU domain. The earliest one is by Eriksson and Karlander (2001) ; they give a characterization for stable allocations - at a given matching - that is similar to the characterization for roommate problems by Tan (1991) and then use linear programming duality for optimal matchings. Talman and Yang (2011) also give a characterization that uses linear programming duality. Sotomayor (2005) has a characterization that is based on "simple outcomes" and is self-contained but of a nonconstructive nature. Our Theorems 1 and 2 generalize these results by offering a complementary solution concept - semistable allocations - for when stable allocations do not exist and by covering the NTU domain. Our approach is self-contained and constructive.

Chiappori, Galichon and Salanie (2012), as already mentioned, consider games with player types and show that stable allocations exist for populations with an even number of players in each type. We infer this result for the NTU domain from our Theorems 1 and 2.

More recently, Biro et al (2012) have given an algorithm that finds a stable allocation via satisfying blocking pairs, but not in a genuine sense, as it makes use of a preconceived target stable allocation, and Andersson et al (2013) a market procedure that finds a stable allocation - via equal-surplus-division allocations at overdemanded sets - without, however, addressing bounded convergence.

Our Market Procedure is a genuine procedure that converges in a (polynomially) bounded number of steps and is on the NTU domain (Theorem 4.) It has moreover a recursive basic-level specification through the Algorithm we describe in Appendix D.

Our results on the properties of the Equilibrium Set (Propositions 2 to 6) are entirely new. So is the Bargaining Set analysis we offer, in particular, our result (Theorem 3) on the stability of pseudostable allocations - a solution concept introduced here - and the characterization of the Demand Bargaining Set for pairing games (Proposition 7.)

As one of the major contributions of our paper, lastly, we mention our "demand analysis" and the representation theorem (Theorem 5) in the Mathematical Section in Appendix A. These we describe in the subsection below.

1.2 The Analytical Aspect

We make no interpersonal comparison of utility : We work with *aspirations* that are payoff vectors which assign a maximum-utility to each player that she can achieve given the maximum-utilities of other players.[12] These utilities can be seen as *prices* that players ask for entering into partnership. At an aspiration, a player may find herself indifferent among a number of players for forming a partnership, thus have a non-singleton *demand set.* We typically deal with aspirations where many players have non-singleton demands sets.

At an aspiration, we look for demand-compatible matchings that leave a minimum number of *active* players unmatched - "active" meaning "above reservation utility". We call these matchings *active-minimum.*

Aspirations are of two types : At any aspiration, either there is a subset of players that partition into two sides, with an *excess* of (say) "buyers" over "sellers", in which case we say there is a *seller-market* - a definition we introduce here[13] - or there is no such subset of players. We call an aspiration of the latter type *balanced.* If it is possible to match all active players at a balanced aspiration then that aspiration is a stable allocation. Otherwise it is a semistable allocation. The Equilibrium Set is the set of all balanced aspirations.

The Market Procedure starts from any aspiration, traces a path of aspirations with seller-markets, and eventually reaches a balanced aspiration. It is fundamental - in this Procedure as well as in all our basic results - what properties seller-markets have, in particular, how they can be identified. A relevant fact here is that union of seller-markets need not be a seller-market. But there are *unitary* seller-markets - that we introduce - and their union is a seller-market. We define

[12] Aspirations in cooperative games go back to Cross (1967), Albers (1974), Bennett (1983).

[13] That is closely related to the definition of an *overdemanded set* in Demange, Gale and Sotomayor (1986).

the union of all unitary seller-markets to be *the Seller-Market* at an aspiration which coincides with the minimum-size maximum-excess seller-market. The Market Procedure is a Seller-Market tracing procedure that minimizes excess.

Our main result (Theorem 5) in the Mathematical Section says that the Seller-Market can be identified by a particular class of matchings. These are active-minimum matchings where the number of active unmatched players who do not belong to an *odd-cycle* with three or more players - therefore stand "solitary" - is minimum. We call them *solitary-minimum* matchings.[14]

Theorem 5 gives a representation for Seller-Markets via solitary-minimum matchings. We exploit this fact in designing the Market Procedure as well as in getting other essential results. The recursive *Seller-Market Algorithm* in Appendix D, for example, involves a judicious selection of successive solitary-minimum matchings along the Procedure Path. In particular, it is on the basis of Theorem 5 that we are able to specify the Market Procedure at a basic level and - a separate matter - prove that it converges in a bounded number of steps. As another example, our results on semistable allocations and pseudostable allocations use the fact that a balanced aspiration that is not a stable allocation admits a *solitary-null* matching.

2 Model and Basic Definitions

A *pairing game* is a triplet (N, r, f) where N is a finite set of *players*, the vector $r = (r_i)$ gives the *stand alone* utilities of players, and the array $f = (f_{ij})$ consists of *partnership* functions for pairs of players : $f_{ij}(u_j)$ is the utility u_i which i achieves as partner of j when j achieves the utility u_j. In particular

$$f_{ij} = f_{ji}^{-1}.$$

We assume f_{ij} are continuous decreasing functions and $f_{ij}(r_j) < \infty$. In the special class of *TU* or *quasilinear* games, $u_i = f_{ij}(u_j) = c_{ij} - u_j$ and $c_{ij} = c_{ji}$.

[14]Active-minimum matchings are essentially *maximum-cardinality* matchings and our work is closely related to the Gallai-Edmonds Decomposition Theorem (1963,1964,1965) although we nowhere use it explicitly. This Theorem says that, in any graph, players partition into three types - let us say, "independent", "central", "substitutable" - such that (i) every maximum-cardinality matching pairs an independent player with an independent player, a central player with a substitutable player, and leaves unmatched only a subset of the substitutable players, and that (ii) each unmatched player resides in an odd-cycle defined with respect to the matching. An odd-cycle may be a singleton. (In the Gallai Edmonds Theorem there is no distinction of active vs nonactive players : A solitary player is an active singleton player.) We do not know whether singleton-minimum matchings have been utilized.

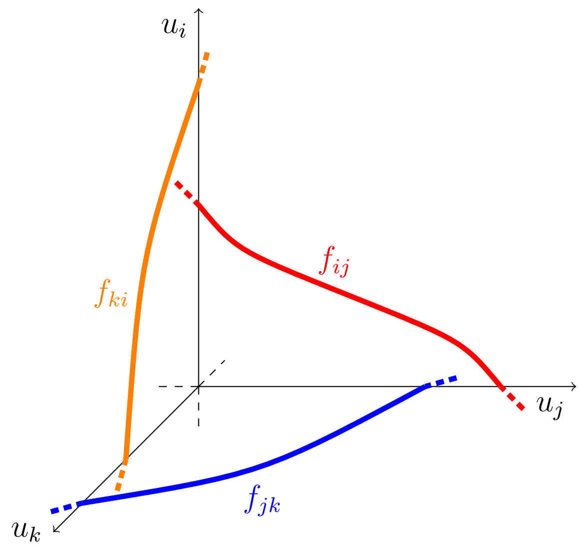


Figure 1: Partnership Functions

2.1 Stable Allocations

A *payoff* is a vector $u \in R^N$ that assigns a utility to each player. A pair ij is said to *block* a payoff u if there exists $(u_i', u_j') > (u_i, u_j)$ satisfying $u_i' = f_{ij}(u_j')$. A payoff is *stable* if it cannot be blocked by any pair. We throughout restrict attention to individually rational payoffs $u \geq r$.

A *matching* is a set of pairs where each player is in at most one pair. Given a matching μ, a payoff $u \geq r$ is *realizable* by μ if

$$u_i = f_{ij}(u_j) \text{ for } ij \in \mu$$

and $u_i = r_i$ for i unmatched. An *allocation* is a payoff that is realizable by some matching. We also give an allocation u in the form (u, μ) when u is realizable by μ. A *stable allocation* is a stable payoff that is an allocation.

An *aspiration* is a stable payoff u that is *individually feasible* in the sense that $u_i = r_i$ or $u_i = f_{ij}(u_j)$ for some j for every i. An aspiration is equivalently a payoff u where u_i is

$$\max\left\{r_i, \max_j f_{ij}(u_j)\right\},$$

namely, the maximum-utility that player i can achieve, through some partnership or by standing alone, given all the other maximum-utilities.[15]

Remark 1 *The aspiration utility of a player may be seen as her individually feasible* price *for entering into partnership. A pairing game then is equivalently a* pairing market *where a* competitive

[15]One can construct an aspiration in $|N|$ simple steps : Order the players in any way and let N_k be the top k players in that order. Let u_1 be the stand alone utility r_1 of the first player and step by step let $u_k = \max\{r_k, \max_{j \in N_k} f_{ij}(u_j)\}$ for the remaining players.

equilibrium *allocation is a list of prices or an aspiration that is realizable. Thus stable allocation, competitive equilibrium allocation and realizable aspiration are equivalent.*

We let $r = 0$ with no loss of generality and regard (N, f) as describing a pairing game or market fixed in the rest of the paper.

2.2 Demand at Aspirations and Seller-Markets

Let u be an aspiration. Define $D_i(u) = \{j | u_i = f_{ij}(u_j)\}$. We say i *demands* $j \in D_i(u)$ and call $D_i(u)$ the *demand set* of i. The set of all pairs ij who demand each other, $\mathcal{D}(u)$, is the *demand graph.* For $S \subset N$, $\mathcal{D}_S(u) = \{ij \in \mathcal{D}(u) | i \in S\}$.

A matching

$$\mu \subset \mathcal{D}(u)$$

is said to be *demand-compatible* or a matching *at* u. A player set $S \subset N$ is *matchable into* $T \subset N$ if there is a demand-compatible matching μ such that, for every $i \in S$, there is a pair $ij \in \mu$ with $j \in T$.

A player i is *active* if $u_i > 0$ and *nonactive* if $u_i = 0$. Note that u is realizable - hence, a stable allocation - if and only if the set of all active players is matchable into N.

We call a pair of player sets (B, S) a *submarket* at u if

(i) B consists of active players,

(ii) the demand set of every B-player is in S, and

(iii) S is matchable into B.

By (iii), the *excess* $|B| - |S|$ is nonnegative.

Our interest is in *bipartite* submarkets (B, S) where $B \cap S$ is empty.[16] We refer to B-players and S-players as *buyers* and *sellers* respectively.

Definition 1 *A* seller-market *at u is a bipartite submarket with positive excess and a* balanced-market *is a bipartite submarket with zero excess.*

If there is a seller-market at u, we say that u *has* a seller-market or that u is an aspiration *with* a seller-market. It is clear that if u has a seller-market then u is not a stable allocation. As we will show, on the other hand, a seller-market at u points the way to an aspiration with no seller-market.

Let us consider the three-player games to illustrate our basic definitions :

[16]Note that a bipartite submarket is not exactly a “two-sided buyers-and-sellers” market because a seller may demand a seller.

Example 1 *Let* $N = \{\mathbf{1}, \mathbf{2}, \mathbf{3}\}$*. Consider the TU games where* $f_{ij}(u_j) = c_{ij} - u_j$ *and say* $c_{ij} = 1$ *for every* i, j*. Take any aspiration* $u = (u_1, u_2, u_3)$*. There are two cases. Case 1:* $u_i = 1/2$ *for every* i*. Then* $D_i(u) = N - i$ *for every* i *and there is no submarket, in particular,* N *is not matchable into* N*. Case 2: wlog* $u_1 = \min\{u_1, u_2, u_3\}$ *and* $u_1 < 1/2$*. Then* $u_2 = u_3 > 1/2, D_1(u) = \{\mathbf{2}, \mathbf{3}\}, D_2(u) = D_3(u) = \{\mathbf{1}\}$ *and* u *has the seller-market* $(B = \{\mathbf{2}, \mathbf{3}\}, S = \{\mathbf{1}\})$*. No stable allocation exists.*

In general wlog $c_{12} = 1, c_{13} \leq 1, c_{23} \leq 1$*. It is straightforward to show that, when player* $\mathbf{3}$ *is "small" in the sense that* $c_{13} + c_{23} \leq 1$*, there are aspirations where player* $\mathbf{3}$ *is nonactive and any such aspiration is a stable allocation. When* $c_{13} + c_{23} > 1$*, on the other hand, (Case 1)* $D_i(u) = N - i$ *for every* i *at the aspiration* $u = (u_1, u_2, u_3) = 1/2(1 + c_{13} - c_{23}, 1 - c_{13} + c_{23}, -1 + c_{13} + c_{23})$*, (Case 2) every aspiration* $u' \neq u$ *has a seller market, therefore no stable allocation exists. The same goes for NTU three-player games as well.*

2.3 An Extension : Half-Partnerships and Semistable Allocations

Stable or competitive equilibrium allocations do not necessarily exist as in Example 1. Here we give an extension of our model where they exist and are equivalent.

The extension is in the notion of an allocation or realizability : We allow a player to have *two* half-partners as an alternative to *one* full-partner, understanding that half-partnership is reciprocal, namely, a player i is half-partner to j if and only if j is half-partner to i. We actually assume that a pair of players i, j can achieve the "half-partnership utilities" $(v_{ij}, v_{ji}) = (h_{ij}(v_{ji}), h_{ji}(v_{ii}))$ through the "half-partnership functions" h_{ij} that satisfy

$$h_{ij}(z) = f_{ij}(2z)/2 \text{ for all } z$$

(*constant-returns-to-scale*) and that the utility of a player with two half-partners is the sum of her half-partnership utilities (*separability.*) Under these assumptions, when $|D_i(u)| \geq 2$, a player i is indifferent between any player in $D_i(u)$ as a full-partner and any two players in $D_i(u)$ as half-partners.

We will show that if there is no stable allocation then there is a "stable" allocation where every player has one full-partner, two half-partners or no partner. In three-player games for instance, when there is no stable allocation, there is an aspiration where $D_i(u) = N - i$ for each player i, and each player fulfils her demand by having the other two players as half-partners.

Formally, a *half-matching* χ is a set of pairs where each player in χ belongs to two pairs, in other words, has two distinct *half-partners*. Let us note that any half-matching is a disjoint union of *cycles* where each half-partnerhip in a cycle shares one player with each of its two neighbors in the cycle. A *semi-matching* is a pair (μ, χ) where μ is a matching, χ is a nonempty half-matching and μ, χ have no player in common.

A payoff $u \geq r$ is *realizable* by a semi-matching (μ, χ) if there is an array (v_{ij}) of half-partnership utilities such that

$$u_i = f_{ij}(u_j) \text{ for } ij \in \mu,$$

$$u_i = h_{ij}(v_{ji}) + h_{ij'}(v_{j'i}) \text{ for } ij, ij' \in \chi,$$

and $u_i = 0$ otherwise. An *allocation* now is a payoff u that is realizable by a matching or by a semi-matching.

Definition 2 *We call a stable payoff* semistable *if it is realizable by a semi-matching but not realizable by a matching.*

Let us note that a semistable allocation u is a competitive equilibrium allocation where u_i is the price of player i for full-partnership and $u_i/2$ for half-partnership.

Remark 2 *By our definition, a matching is not a semi-matching and a stable allocation not a semistable allocation. On the other hand, a payoff may be realizable both by a matching and by a semi-matching : For example, in a four-player game with $N = \{1, 2, 3, 4\}$ and an aspiration u where $D_i(u) = \{(i-1) \bmod N, (i+1) \bmod N\}$, u is realizable by the half-matching $\{12, 23, 34, 41\}$ - a cycle of even cardinality, namely, an* even-cycle *- as well as by the matching $\{12, 34\}$. In general, if u is an aspiration that is realizable by a semi-matching (μ, χ) and η is any even-cycle in χ, then there is a demand-compatible matching ν that covers the η-players, so that u is realizable by the semi-matching $(\mu \cup \nu, \chi - \eta)$. In particular, a semistable allocation is always realizable by a semi-matching that contains* odd-cycles *only. In Section 4, we make use of "essential" semi-matchings that contain a minimum number of odd-cycles.*

3 Existence of Stable and Semistable Allocations : The Equilibrium Set

We call an aspiration that has no seller-market a *balanced* aspiration. Our first theorem says that there always exists a balanced aspiration. Let U be the set of all balanced aspirations.

Theorem 1 *U is nonempty.*

The proof of this result is of a constructive nature, based on the *Direction Lemma* which says that, an aspiration with a bipartite submarket (B, S) can be altered to an aspiration with higher S-utilities and lower B-utilities at which (B, S) is still a bipartite submarket.

We state here a key lemma that is the analog of the *Decomposition Lemma* in the two-sided matching literature. For any two aspirations u, u', consider the disjoint player sets

$$N^+_{uu'} = \{i|u_i > u'_i\}, N^-_{uu'} = \{i|u_i < u'_i\}.$$

Note $N^+_{u'u} = N^-_{uu'}$.

Lemma 1 *$(N^+_{uu'}, N^-_{uu'})$ is a balanced-market at u for every $u, u' \in U$.*

Our second main result says that a balanced aspiration is a stable allocation or a semistable allocation. More precisely :

Theorem 2 *U is the set of all stable allocations or the set of all semistable allocations.*

We call a matching at an aspiration *active-minimum* if the number of active players it leaves unmatched is minimum among all matchings at that aspiration. The underlying fact behind Theorem 2 is that, at any balanced aspiration, there is an active-minimum matching with the following *odd-cycle property* : every active unmatched player i belongs to a distinct cycle - with at least three players - in which every player demands her two neighbors and every player other than i is matched to a neighbor. So any balanced aspiration that is not realizable by a matching is realizable by a semi-matching. The proof then follows from the fact (based on Lemma 1) that if a balanced aspiration is not realizable by a matching then no balanced aspiration is.

Let us recall that aspiration utilities can be seen as prices and stable or semistable allocations as competitive equilibrium allocations. We call U the *Equilibrium Set.* In the next section we consider the structural properties of U. Below we give a sufficient condition for the existence of a stable allocation.

Players i and i' are of the same *type* if

$$f_{ij} = f_{i'j}$$

for all j other than i, i'.[17]

Proposition 1 *U is the set of all stable allocations if there are an even number of players in each type.*

[17] This condition allows $f_{ii'}$ to be any partnership function for i, i' of the same type. If there are more than two players in their type, however, it is easy to show that $f_{ii'}$ is neccessarily "symmetric" with respect to equal utility realization.

Our demonstration is based on the fact that, when there are an even number of players in each type, a balanced aspiration cannot admit any odd-cycle of half-partnerships, therefore, cannot be a semistable allocation.[18]

Remark 3 *It is easily seen that, if there is an even number of players in each type, then there is a stable allocation where same-type players have same-utility. This is not true at every stable allocation, though, as is evident upon considering a two-player game. If there is an even number but more than two players in each type, however, it is easily shown that same-type players have same-utility at every stable allocation.*

4 Structural Properties of the Equilibrium Set

In this section we look at situations where the Equilibrium Set U is not a singleton and consider its structural properties. As pointed out in Example 1, U is a singleton for three-player games that admit a semistable allocation, and as another example, U is a singleton for four-player games that admit a stable allocation where each player demands each of the other players. In general though U is not a singleton. Our first result below shows that the structural properties of U do not depend on whether U consists of stable or semistable allocations.

4.1 The Variable Set and Stable Bipartitions : TU vs NTU

Given a half-matching χ, let $o(\chi)$ be the number of cycles in χ. Let u be a semistable allocation and (u, μ, χ) a realization of u. We say that (u, μ, χ) is *essential* if $o(\chi) \leq o(\chi')$ for every realization (u, μ', χ') of u.

Let us call a player in $X = \{i \in N | u_i = u'_i \text{ for all } u, u' \in U\}$ a *constant* player and a player in $Y = N - X$ a *nonconstant* player. We show below that any player who has a half-partner at some essential semistable allocation is necessarily a constant player, more generally, that a nonconstant player is always full-partner with a nonconstant player.

For any matching μ and player set S, let $\mu(S) = \{j | ij \in \mu, i \in S\}$.

Proposition 2 *$\mu(Y) = Y$ at every stable allocation (u, μ) and $\mu(Y) = Y$ at every essential semistable allocation (u, μ, χ).*

[18] Proposition 1 can be gotten in two other ways : One involves the fact that the stable allocations of a two-fold pairing market coincide with the stable allocations of the two-sided market which has one copy of each type. The other way is to set up a similar equivalence in our extended model with half-partnerships.

Let us call $V = \{u_Y | u \in U\}$ - the projection of U to Y - the *Variable* Equilibrium Set. Proposition 2 says that payoffs in V are realizable (only) by matchings in $Y \times Y$.[19,20] For further insight, we ask whether Y partitions into two sides anywhere in V :

We say that (i) (Y_1, Y_2) is a *stable bipartition at* $v \in V$ if $Y = Y_1 \cup Y_2, Y_1 \cap Y_2 = \varnothing$ and

$$\mu(Y_1) = Y_2$$

for every matching μ by which v is realized, and that (ii) (Y_1, Y_2) is a *stable bipartition over* V if (Y_1, Y_2) is a stable partition at every v in V.

Our finding is that, while a stable bipartition exists in general at every $v \in V$ (see Lemma 11), a stable partition over V exists for TU but not necessarily for NTU games.

Proposition 3 *Let (N, f) be a TU game. (i) If some $v \in V$ is realizable by a matching μ, then every $v \in V$ is realizable by μ. (ii) There is a stable bipartition over V.*

This result says that, on the TU domain, V has essentially the same properties as the Equilibrium Set of a TU assignment game. In particular, with respect to a stable partition (Y_1, Y_2) of the nonconstant player set Y,[21] V has a lattice structure and a Y_1-optimal allocation that is Y_2-pessimal. Moreover, with reference to Schwarz and Yenmez (2011), we can conclude that V - therefore U - has a unique median allocation.

Below is a *heterogenously linear* game where there is no stable bipartition over $V = U$.

Example 2 *There are six players in $N = \{\boldsymbol{1},\boldsymbol{2},\boldsymbol{3},\boldsymbol{4},\boldsymbol{5},\boldsymbol{6}\}$. The partnership functions are*

$$u_i = f_{ij}(u_j) = c_{ij} - q_{ij}u_j$$

where the pair (c_{ij}, q_{ij}) is equal to

$$(15, 2) \text{ for } ij \in \{\boldsymbol{12}, \boldsymbol{23}, \boldsymbol{31}\} \text{ and } (15/2, 1/2) \text{ for } ij \in \{\boldsymbol{21}, \boldsymbol{32}, \boldsymbol{13}\}$$

[19] Recall the players labelled *independent* in the Gallai-Edmonds Decomposition Theorem (Footnote 14.) Every nonconstant player is an independent player except possibly at u on the boundary of U.

[20] It is worthwhile to add the following observation : Consider the game restricted to constant players, i.e., (X, f). It is easily seen that the Equilibrium Set of (X, f) is identical to $U_X = \{u_X | u \in U\}$. The Equilibrium Set of (Y, f) on the other hand is in general a superset of $U_Y = V$. For example, when $N = \{\mathbf{1},\mathbf{2},\mathbf{3}\}$ and the worth of a pair is 3 for $\{\mathbf{1},\mathbf{2}\}$ and 1 otherwise, $U = \{x, 3 - x, 0\}$ where $1 \leq x \leq 2$. $Y = \{\mathbf{1},\mathbf{2}\}$ and the Equilibrium set of (Y, f) is $\{x, 3 - x\}$ where $0 \leq x \leq 3$.

[21] It is clear that there is a *unique* stable bipartition over V unless V is a "product" : In general, let $V^1 \times ... \times V^K$ be the factorization of V where $Y^1 \cup ... \cup Y^K$ is the finest partition of Y such that (i) $V^k \subset R^{Y^k}$ and (ii) if $ij \in \mathcal{D}_Y(u)$ for some $u \in U$ then $i, j \in Y^k$ for some k. Then there is a unique stable bipartition over each V^k.

$$(30,10) \textit{ for } ij \in \{\boldsymbol{46},\boldsymbol{65},\boldsymbol{54}\} \textit{ and } (3,1/10) \textit{ for } ij \in \{\boldsymbol{64},\boldsymbol{56},\boldsymbol{45}\}$$

$$(10,1) \textit{ for } ij \in \{\boldsymbol{14},\boldsymbol{41},\boldsymbol{25},\boldsymbol{52},\boldsymbol{36},\boldsymbol{63}\}$$

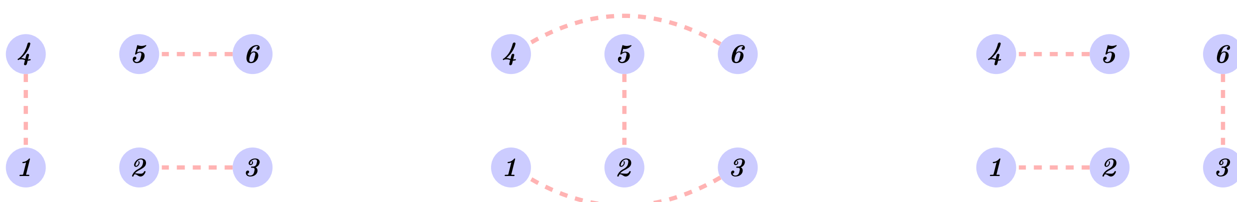
and $(0,0)$ *otherwise. It is straightforward to check that the demand graphs at the three allocations*

$$u = [7,9,3,3,2,10], u' = [3,7,9,10,3,2], u'' = [9,3,7,2,10,3]$$

are

$$\mathcal{D}(u) = \{(\boldsymbol{14}),(\boldsymbol{23}),(\boldsymbol{56})\}, \mathcal{D}(u') = \{(\boldsymbol{13}),(\boldsymbol{25}),(\boldsymbol{46})\}, \mathcal{D}(u'') = \{(\boldsymbol{12}),(\boldsymbol{45}),(\boldsymbol{36})\}$$

respectively (see Figure 2) and that each allocation is realizable by a unique matching. It is easily seen that N *has no partition to two sides such that each of these matchings matches one side to the other.*



(a) $\mathcal{D}(u)$ (b) $\mathcal{D}(u')$ (c) $\mathcal{D}(u'')$

Figure 2: Example 2

4.2 Median Property and Virtual Convexity

Let $K = \{u^k\}$ be any finite collection of payoff vectors. Let $m = |K|/2$ for $|K|$ even and $m = (|K|+1)/2$ for $|K|$ odd. For every player i, let K_i be any nondecreasing ordering of $\{u_i^k\}$. Let u_i^* be the mth payoff in K_i, namely, the *median* of K_i for $|K|$ odd and the *lower median* of K_i for $|K|$ even. Let $u_i^{**} = u_i^*$ for $|K|$ odd and u_i^{**} be the $m+1$st payoff or *upper median* of K_i for $|K|$ even.

We define

$$med\{K_i\} = [u_i^*, u_i^{**}]$$

and say that U has the *median* property if, for every finite collection $K = \{u^k\}$ with $u^k \in U$, there is a $u \in U$ with $u_i \in med\{K_i\}$ for every i.

Proposition 4 *U has the median property.*

Proposition 4 is a generalization to pairing games of the median property Schwarz and Yenmez (2011) have shown for TU assignment games and of the median property Eriksson and Karlander (2001) have shown for TU pairing games when $|K| = 3$.[22]

Clearly U is a closed bounded set. Our next result says that U is akin to a convex set. Say that a vector $\overline{z}$ is *between* two vectors z, z' if

$$\overline{z}_i \in (\min\{z_i, z_i'\}, \max\{z_i, z_i'\})$$

in case $z_i \neq z_i'$ and $\overline{z}_i = z_i = z_i'$ otherwise. Call any set Z in R^N *virtually convex* if for every z, z' in Z there is a $\overline{z} \in Z$ that is between z, z'.

Proposition 5 *U is a virtually convex set.*

It can be shown that a virtually convex set is equivalently a set Z such that (i) any pair z, z' in Z can be connected by a continuous "*monotone*" path in Z or (ii) any z not in Z can be *separated* from Z by an *orthant*. See Alkan and Gale (1990). It is a straightforward conclusion that U is a convex polyhedral set when the partnership functions f_{ij} are linear.

5 Pseudostable Allocations

In this section we consider what may happen in a pairing game when no stable allocation exists and half-partnerships are not viable.

This question has been taken up in the context of the roommate problem by Tan (1990) who offered *maximum stable matchings* as a solution concept, namely, the matchings that leave a minimum number of players unmatched and are stable when the unmatched players are excluded.[23] Below we first introduce the analogous concept of *maximum-stable allocations* and show that every balanced aspiration generates a set of maximum-stable allocations. There are, however, non-balanced aspirations that generate maximum-stable allocations as well. Our primary interest in this section is to introduce *pseudostable allocations* - a further refinement of maximum-stable allocations - as a solution concept for pairing games. Our main result (Theorem 3) shows that pseudostable allocations have a bargaining-set stability property.

[22] For even collections K, Schwarz and Yenmez (2011) show the stronger result that U contains the upper median and the lower median of K.

[23] See Inarra, Larrea and Molis (2008) for another soultion concept.

5.1 Maximum-Stable Allocations and Balanced-Aspiration-Allocations

For any payoff z and $T \subset N$, let $z_T = (z_i)_{i \in T}$. For any allocation (v, μ) and $T \supseteq \mu(N)$, consider the "restricted" game (T, f) and note that (v_T, μ) is an allocation for (T, f); in particular $v_i = 0$ for i in $T - \mu(N)$.

We say that an allocation (v, μ) is *restricted-stable* if there is a player set $T \supseteq \mu(N)$ such that (v_T, μ) is a stable allocation in (T, f). Given a restricted-stable allocation (v, μ), let T^* be the largest $T \supseteq \mu(N)$ such that (v_T, μ) is stable in (T, f) and call the players in $N - T^*$ the *outcasts* of (v, μ).

Definition 3 *A* maximum-stable *allocation is a restricted-stable allocation with a minimum number of outcasts.*

For any payoff z and matching μ, we denote z^μ the payoff where

$$z_i^\mu = z_i \text{ for } i \in \mu(N) \text{ and } z_i^\mu = 0 \text{ for } i \notin \mu(N).$$

Definition 4 *An* aspiration-allocation *is an allocation (v, μ) such that $v = u^\mu$ for an aspiration u.*

A straighforward observation, which we state without proof, is that an allocation is restricted-stable if and only if it is an aspiration-allocation. We call an aspiration-allocation u^μ *balanced* if u is balanced and *maximum* if μ is active-minimum.

Proposition 6 *Every maximum balanced-aspiration-allocation is maximum-stable.*

Remark 4 *When no stable allocation exists, there exist - a plethora of - maximum-stable allocations that are not balanced-aspiration-allocations : This may be seen by considering any aspiration $u = (u_1, u_2, u_3)$ in a three-player game where each player is active and demand is $\{12, 13\}$. Clearly $(u_1, u_2, 0)$ is a maximum-stable allocation and u is not balanced. This shows that maximum-stability may have less appeal as a solution concept for pairing games than for roommate problems.*

5.2 Pseudostable Allocations and Bargaining Set Stability

Recall that a three-player pairing game with no stable allocation has a unique balanced aspiration $u = (u_1, u_2, u_3)$ and so - the null allocation $(0, 0, 0)$ aside - the balanced-aspiration-allocations

$$(u_1, u_2, 0), (u_1, 0, u_3), (0, u_2, u_3).$$

Binmore (1985) showed that (u_1, u_2, u_3) is the only mutually consistent endogenous *outside-option* vector - when any two players may bargain and the outside player is a potential partner in case they cannot agree - and argued that the three allocations above is the "stable set" of the game.[24] Another argument to support this view is the following two-step farsighted stability or Bargaining Set argument : Each of the three allocations would survive - because a *prudent* player would not be lured into forming a blocking pair with the odd-man-out since he could in turn become the odd-man-out - and every other allocation would be blocked even under prudence.

Here we consider whether there is a natural generalization of the "stable set" above to pairing games with any number of players. We give a partial answer. We show that there is a particular subset of maximum balanced-aspiration-allocations - that we call pseudostable - which is always in an "exclusive" Bargaining Set of the game.

Let us recall that, at any balanced aspiration, there is a *particular* active-minimum matching with the following odd-cycle property : Each active unmatched player i belongs to a distinct cycle - with at least three players - in which every player demands her two neighbors and every player other than i is matched to a neighbor ; namely, no active unmatched player is "solitary". We call such a matching *solitary-null*.

Definition 5 *We call a balanced-aspiration-allocation u^{μ}* pseudostable *if μ is solitary-null.*

Remark 5 *Let u be any balanced aspiration. Recall that the set of maximum-stable allocations generated by u is $BAA(u) = \{u^{\mu} | \mu \text{ active-minimum at } u\}$. The set of pseudostable allocations generated by u is*

$$PSA(u) = \{u^{\mu} | \mu \text{ solitary-null}\} \subset \{u^{\mu} | \mu \text{ active-minimum}\} = BAA(u).$$

Not surprisingly, pseudostable and semistable allocations are closely related : Let (u, μ, χ) be any essential semistable allocation at u and $H \subset N$ be the players in the half-matching χ. As previously noted, H partitions into k odd-cycles C_i each associated with an active unmatched player i. In fact, μ' is a solitary-null matching at u iff

$$\mu' = \mu \cup \mu_1 ... \cup \mu_k$$

where μ_i is a matching in C_i that leaves any one *player in C_i unmatched. So there is a solitary-null matching μ' at u for every selection of k players from $C_1 \times ... \times C_k$. Associated with each essential semistable allocation (u, μ, χ) then, we obtain a set of $|C_1| \times ... \times |C_k|$ solitary-null matchings or pseudostable allocations. $PSA(u)$ is their union over all the essential semistable allocations at u.*

[24] Binmore also showed that each allocation in the "stable set" is the unique subgame perfect equilibrium of a sequential bargaining game. Bennett (1997) has shown for a general class of cooperative games that there always exist "consistent endogenous outside-options", that they are aspirations, and that a *large* set of aspirations turn out as the SPE outcomes of sequential bargaining games.

Having defined pseudostable allocations, let us define a Bargaining Set. There are several definitions and variants. We employ the Demand Bargaining Set proposed by Morelli and Montero (2003) which is a refinement of the well-known Bargaining Set proposed by Zhou (1994)). The latter has the following definition :

Let υ be an allocation. An *objection* against υ is a pair (T, υ') where $T \subset N$ and υ' is an allocation for the restricted game (T, f) such that

$$\upsilon_i' > \upsilon_i \text{ for } i \in T.$$

A *counterobjection* to (T, υ') is a pair (Q, υ'') where $Q \subset N$ and υ'' is an allocation for the restricted game (Q, f) such that

$$Q - T \neq \varnothing, T - Q \neq \varnothing, T \cap Q \neq \varnothing,$$

$$\upsilon_i'' \geq \upsilon_i \text{ for } i \in Q - T \text{ and } \upsilon_i'' \geq \upsilon_i' \text{ for } i \in T \cap Q.$$

An objection against υ is *justified* if there is no counterobjection to it. An allocation is in the *Zhou Bargaining Set* $\boldsymbol{Z}$ if there is no justified objection against it.

It is well known that a Bargaining Set - $\boldsymbol{Z}$ included - is typically "large" and not sufficiently exclusive in describing bargaining outcomes. Our main reason in employing the *Demand Bargaining Set* $\boldsymbol{D}$ is that $\boldsymbol{D}$ is more exclusive than $\boldsymbol{Z}$. We give the definition of $\boldsymbol{D}$ by stating the differences it has with the definition of $\boldsymbol{Z}$.[25] There are four differences :

(i) the allocation υ is an aspiration-allocation, let $\upsilon = u^{\mu}$,[26]

(ii) $\upsilon_i'' = u_i$ for $i \in Q$,

(iii) $\upsilon_i'' > \upsilon_i'$ for $i \in T \cap Q$,

(iv) $Q - T$ or $T - Q$ may be empty.[27]

Remark 6 *There will in general be many allocations that belong to* $\boldsymbol{Z}$ *but not to* $\boldsymbol{D}$. *This is because* $\boldsymbol{D}$ *admits aspiration-allocations only and because - primarily by condition (ii) - counterobjection in* $\boldsymbol{D}$ *is highly restricted than in* $\boldsymbol{Z}$.

In Appendix C we give a characterization for the Demand Bargaining Set from which follows our main result :

[25] The definition of $\boldsymbol{D}$ in Morelli and Montero (2003) is for TU games.

[26] Morelli and Montero (2003) allow more general allocations but then show that the Demand Bargaining Set consists of aspiration-allocations.

[27] Note that if condition (iv) were excluded then $\boldsymbol{D}$ would be a subset of $\boldsymbol{Z}$. We show in Appendix C (Lemma 12) that condition (iv) is in fact vacuous and $\boldsymbol{D}$ is a subset of $\boldsymbol{Z}$. Morelli and Montero (2003) show the same for TU games.

Theorem 3 *Pseudostable allocations are in the Demand Bargaining Set.*

Theorem 3 says that pseudostable allocations are "stable" from an "exclusive" Bargaining Set perspective. By the same perspective, on the other hand, there may be other "stable" allocations. Examples in Appendix C show that these may in fact be various not fitting into a classification at hand. The Demand Bargaining Set may actually contain a non-balanced aspiration-allocation even when there is a competitive equilibrium at some other aspiration.[28] In the next section, we give a *coordinated* Market Procedure that always arrives at a balanced aspiration.

6 Market Procedure

Here we give a Procedure for finding a balanced aspiration. For simplicity, we restrict our presentation to *heterogenously linear* partnership functions that have the form $f_{ij}(u_j) = c_{ij} - q_{ij}u_j$.[29] The Procedure starts from any aspiration, generates a piecewise linear path of aspirations, and stops in a bounded number of steps at a balanced aspiration.

Here is a preview : The Procedure is coordinated by a Center that displays an *aspiration* at each moment and players register their demand sets at that aspiration. (Since demand is reciprocal, i registers j if and only if j registers i.) The Center observes all demand and stops if there is no seller-market. Otherwise, the Center identifies a set of players who constitute a seller-market and alters the aspiration continously along a suitable *direction*. The direction is reset when the seller-market changes.

The Center can actually choose any seller-market. In the Procedure we present here, it is the "grand" *Seller-Market* - the union of all "*unitary*" seller-markets - that is chosen at each aspiration. The Center is able to identify the Seller-Market continuously by a simple recursive algorithm that we spell out in the subsection below. This is based on the characterization of Seller-Markets via "*solitary-minimum*" matchings that we give in Theorem 5 in Appendix A.

There is a single criterion for *admitting* a direction d at any aspiration u on the path, namely the requirement that the Seller-Market at $u+\lambda d$ be identical to the Seller-Market at u for all sufficiently small $\lambda > 0$. When the Seller-Market changes at an aspiration and d is no longer "Seller-Market-Preserving" (definition below), the Center needs to find a new Seller-Market-Preserving direction. On the quasilinear domain, the direction that has the entry $+1$ for every Seller, -1 for every Buyer and 0 for all other players ensures this. On the heterogeneously linear domain, the Center determines

[28]This may be compared with Klijn and Masso (2003) who show that the core in the discrete two-sided case is essentially equivalent to the Zhou Bargaining Set.

[29]The extension to games with piecewise linear partnership functions can be carried by adaptation from Alkan (1992,1997).

differential rates by interacting with the Sellers and Buyers about their "marginal" demand sets. This is described in the *Direction Procedure* below.

To conclude the preview, there is actually one other situation where the Center has to reset the direction. This occurs when the path arrives at an aspiration where the Seller-Market has not changed but would change for any continuation along the current direction. Such a situation may arise only when a new demand is registered by a Buyer Seller pair. This does not occur on the quasilinear domain.

Formally, let u be an aspiration and d be a feasible direction (namely, a vector in R^N such that $u + \lambda d$ is an aspiration for $\lambda > 0$ sufficiently small.) By linearity of the partnership functions, the demand graph

$$\mathcal{D}(u + \lambda d) = \{ij | j \in D_i(u + \lambda d)\}$$

is identical for all sufficiently small $\lambda > 0$. We denote this graph

$$\mathcal{D}^+(u, d)$$

and call it the *outgoing directional demand graph* at u in the direction d. We will say that d is *Seller-Market-Preserving* at u if the Seller-Market at u is identical to the Seller-Market in $\mathcal{D}^+(u, d)$.

Similarly $\mathcal{D}(u - \lambda d) = \{ij | j \in D_i(u - \lambda d)\}$ is identical for all sufficiently small $\lambda > 0$ which we denote

$$\mathcal{D}^-(u, d)$$

and call the *incoming directional demand graph*. Likewise the set of active players $A(u - \lambda d)$ is identical for all sufficiently small $\lambda > 0$ which we denote

$$A^-(u, d).$$

Clearly, demand *changes* at u if and only if

$$\mathcal{D}(u) \neq \mathcal{D}^-(u, d) \text{ or } A(u) \neq A^-(u, d).$$

It is important to note this may happen finitely often and when it does

$$\begin{aligned} \mathcal{D}(u) &\supset \mathcal{D}^-(u, d), \\ A(u) &\subset A^-(u, d). \end{aligned}$$

Market Procedure

Step 0 : Take any aspiration $u = u^0$.

Step t : End if there is no seller-market at u^t. Otherwise, find a Seller-Market-Preserving direction d^t by the Direction Procedure below. Then, display the aspiration

$$u^t + \lambda d^t$$

as λ increases above 0 and let the Buyers in the Seller-Market register the changes in their demand sets or indicate whether they become nonactive. Stop at the smallest $\lambda = \lambda^*$ such that d^t is not Seller-Market-Preserving at $u^t + \lambda^* d^t$. Set

$$u^{t+1} = u^t + \lambda^* d^t.$$

Let us suppress reference to u^t and write $\mathcal{D} = \mathcal{D}(u^t), \mathcal{D}^+(e) = \mathcal{D}^+(u^t, e)$. Let (B^*, S^*) be the Seller-Market at u^t. The Procedure below utilizes the information

$$f'_{ij} = -q_{ij}.$$

The Direction Procedure

Step 0 : Set the initial direction to be the vector e^0 where e_i^0 is equal to 1 if $i \in S^*$, $\min_{j \epsilon D_i} \{q_{ij}\}$ if $i \in B^*$, and 0 otherwise.

Step k : End if the Seller-Market (B^*, S^*) in the demand graph $\mathcal{D}$ is the Seller-Market in the directional demand graph $\mathcal{D}^+_{B^*}(e^k)$ and set $d^t = e^k$. Otherwise, find the Seller Set S^k in $\mathcal{D}^+_{B^*}(e^k)$ and set $e_i^k(\delta)$ equal to

$$(1 + \delta)e_i^k \text{ for } i \epsilon S^k,$$

$$e_i^k \text{ for } i \epsilon (S^* - S^k),$$

$$\min_{j \epsilon D_i} \{q_{ij} e_j^k\} \text{ for } i \epsilon B^*.$$

Then alter the direction $e^k(\delta)$ by increasing δ continuously above 0 to δ^* where a new pair joins $\mathcal{D}^+_{B^*}(e^k(\delta))$. Set $e^{k+1} = e^k(\delta^*)$.

The Direction Procedure[30] finds a Seller-Market preserving direction at u^t : This can be seen in the proof of Lemma 7 in Appendix B.

[30]Has been adapted from Alkan (1992,1997) where it is given for arbitrary piecewise linear partnership functions. On the present domain, it is a "multiplicative" analog of the well-known DGS auction (1986) and has identical convergence properties.

Theorem 4 *The Market Procedure reaches a balanced aspiration in a bounded number of steps.*

It is immediate from the stopping rule that the Market Procedure ends at a balanced aspiration. We state below the main reason why it converges in a bounded[31] number of steps : Three attributes of the Seller-Market - the excess in the Seller-Market, the number of Sellers, the Seller-Market itself - are lexicographically monotone along the Procedure Path.

Lemma 2 *Let (u^t) be any sequence of aspirations generated by the Market Procedure, (B_t, S_t) be the Seller-Market at u^t and $a_t = |B_t| - |S_t|, b_t = |S_t|$. Then, for all t,*

$$a_{t+1} \leq a_t,$$

and if $a_{t+1} = a_t$ then

$$b_{t+1} \geq b_t,$$

moreover if $a_{t+1} = a_t$ and $b_{t+1} = b_t$ then

$$(B_{t+1}, S_{t+1}) = (B_t, S_t).$$

We prove Lemma 2 and Theorem 4 in Appendix D.

Remark 7 *The Market Procedure is based on identifying the Seller-Market at aspirations on the Procedure Path where demand changes. It would be computationally demanding if this had to done from scratch each time. This is not the case. In Appendix D, we give a simple Algorithm that generates successive solitary-minimum matchings and solitary-player sets* - recursively - *at aspirations where demand changes. By Theorem 5, then, the Seller-Market is identifiable recursively. In result, the Algorithm and the Direction Procedure together specify a "dynamic" Market Procedure that is computationally efficacious.*

7 Concluding Remarks

We have given a comprehensive analysis for pairing games or markets which are the non-two-sided *and* NTU generalization of assignment games. The complexity that these two aspects bring remain separate : In fact the Direction Lemma (Appendix B) captures nearly all that is essential in our

[31]It can be shown by an argument similar to the one given in Alkan (1992,1997) that, in fact, the number of steps is polynomially bounded.

treatment of the NTU aspect[32] and a substantial part of our work would have to be carried in nearly the same way if we had stayed on the TU domain - e.g, our characterization of the Seller-Market and its identification on the Procedure Path or our Bargaining Set analysis. Also, there are significant differences between what results hold on the TU vs NTU domains but these are not so surprising. The sharpest difference we have pointed out is the fact that the Equilibrium Set does not have the stable bipartition property on the NTU domain that it has on the TU domain.

We have looked at pairing games from both a coalitional game and a market equilibrium perspective. In our context *essential* blocking coalitions are pairs. Relatedly, stable and competitive equilibrium allocations coincide when they exist. In fact, in our first solution concept extension - half-partnerships and semistable allocations - stable and competitive equilibrium allocations do coincide.

In the second extension - Bargaining Set stability and pseudostable allocations - coalitions of all sizes may be essential, coincidence breaks down and "market forces" may be ineffective. The exclusive Demand Bargaining Set (Morelli and Montero 2003) we have employed here is in a way market-based because it admits aspiration-allocations only and aspirations are market-prices.[33] Still, as we have shown, it may contain allocations "distant" to market equilibrium. It would be of interest what refinement of the Demand Bargaining Set would still contain pseudostable allocations or what additional criteria characterize them. Pairing games are surely a relatively tractable class of coalitional games. Our work here shows that they are at the same time an interesting class for reviewing the various Bargaining Set solution concepts.

Finally a remark about a limiting case of our model : The partnership functions in our model do not allow "flats" that arise under budget constraints for example. The broader model that allows for flats can be uniformly approximated by our model and existence results would carry over. On the other hand, some of our results on the properties of the Equilibrium Set do not and designing a Market procedure appears more involved.[34]

[32] We have adapted the Direction Lemma from our earlier work on the NTU assignment game where it has sometimes been referred to as the Perturbation Lemma : Alkan (1989,1992,1997),, Alkan and Gale (1990), Alkan, Demange and Gale (1991). There are only a few other papers on the NTU assignment game : Moldovanu (1990), Kucuksenel (2011).

[33] Bennett (1983,1997) and Bennett and Zame (1988) have elaborated on the market aspect of aspirations in their work on general coalitional games.

[34] Alkan, Anbarci and Sarpça (2012) is an exercise in this domain.

A Mathematical Section

Let (N, f) be a pairing game and u be any aspiration. In this section we use elementary notions from graph theory to make some observations about the demand graph $\mathcal{D}(u)$. The main tool is a "maximum-cardinality" matching that we call *active-minimum*. Our main objective is to define a unique *Seller-Market* at u and to identify it via certain active-minimum matchings.

A.1 Definition : the Seller-Market is the Union of all Unitary Markets

Definition 6 *A* unitary market *at u is a seller-market (B, S) where $|B| - |S| \ = 1$ and S is matchable into $B - i$ for every $i \in B$.*

It is in general not true that the union of two seller-markets is a seller-market : For example, suppose $\mathcal{D}(u) = \{13, 23, 34, 45\}$ among five active players. Both $(\{1, 2, 4\}, \{3, 5\})$ and $(\{1, 2, 5\}, \{3, 4\})$ are seller-markets but not their union $(\{1, 2, 4, 5\}, \{3, 4, 5\})$. Note that each of the two seller-markets here contains the unitary seller-market $(\{1, 2\}, \{3\})$.

The importance of unitary markets is that union of unitary markets is a seller-market. We omit the straightforward proof. We call the union of all unitary markets the *Seller-Market* at u.

A.2 Active-Minimum Matchings and μ^i−Markets

Let μ be a matching at u. A player i is *active-unmatched* if i is active at u and not matched by μ. Let $A(\mu)$ be the set of active-unmatched players at μ.

Let $i \in A(\mu)$. The following are standard definitions : We say j is *μ-reachable from i* if there is a sequence of distinct players

$$i_0, i_1, ..., i_{n-1}; j_1, ..., j_n$$

where $i_0 = i, j_n = j \neq i, i_{k-1}j_k \in \mathcal{D}(u)$, and

$$i_k j_k \in \mu,$$

for every $k \leq n - 1$. Let $i_0, i_1, ..., i_{n-1}; j_1, ..., j_n$ be such a sequence from $i = i_0$. If j_n is unmatched, then μ can be *augmented* to the matching that contains the pairs $i_{k-1}j_k$ - instead of the pairs $i_k j_k$ - and matches at least one more active player. If j_n is matched with a nonactive player, then μ can be *alternated* to the matching that contains the pairs $i_{k-1}j_k$ - instead of the pairs $i_k j_k$ and $j_n\mu(j_n)$ - and matches one more active player.

Definition 7 *A matching μ is* active-minimum *if $|A(\mu)| \leq |A(\mu')|$ for every matching μ' at u.*[35],[36]

Let μ be active-minimum and $i \in A(\mu)$.

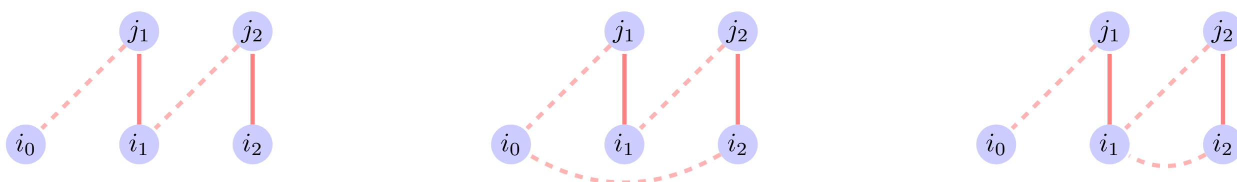
We refer to the sequence $i_0, i_1, ..., i_n; j_1, ..., j_n$ - where $i_0 = i, j_n = j, i_{k-1}j_k \in \mathcal{D}(u), i_k j_k \in \mu$ - as a *μ-sequence* from i ; we say it is *cyclic* or a *μ^i-cycle* if

$$i_0 i_n \in \mathcal{D}(u),$$

and call it *cycle-free* if there is no player i_m such that

$$i_m i_n \in \mathcal{D}(u).$$

We say player i is *$\mu-$cyclic* if there is a *μ^i-cycle.*



(a) cycle-free (b) cyclic (c) not cycle-free

Figure 3: μ-sequences

Definition 8 *Let (I, J) be the pair of player sets where J is the set of all μ-reachable players from i and $I = i \cup \mu(J)$. We call (I, J) the μ-*market *from i or the μ^i-*market *at u.*

Note that, in a μ^i-market (I, J), I consists of active players for otherwise μ could be alternated to match an additional active player. Also, by "reachability", the demand sets of I-players are in J and $\mu(J) \subset I$. Thus a μ^i-market is a submarket at u. It need not be bipartite.

Example 3 *Suppose there are three players all active at u and $\mathcal{D}(u) = \{(1,3),(2,3)\}$. Consider the matching $\mu = \{2,3\}$. The μ-sequence $1,2;3$ reaches 3 from 1. The μ^1-market is $(I, J) = (\{1,2\}, \{3\})$ and bipartite. Now suppose $\mathcal{D}(u) = \{(1,2),(1,3),(2,3)\}$. In this case, the μ^1-market is $(\{1,2,3\}, \{2,3\})$ and not bipartite.*

[35] A matching at u has *maximum-cardinality* if it contains a maximum number of pairs. A characterization statement for active-minimum matchings, similar to the characterization for maximum cardinality matchings by Berge (1957), would say : A matching μ is active-minimum if and only if every μ-reachable player from an active-unmatched player is matched with an active player.

[36] An active-minimum matching has maximum-cardinality *unless* it can be augmented to a matching that contains *two* additional *nonactive* players. There is always an active-minimum matching which has maximum-cardinality.

The following is a straightforward observation.

Lemma 3 *A μ^i-market is bipartite if and only if every μ-sequence from i is cycle-free.*

Proof. The "only if" part is clear from Example 3. For the "if" part, note that since μ is active-minimum, the demand set of every I-player is in J, so it remains to show $I \cap J = \emptyset$. Suppose not : Then there are two μ-sequences $i_0, i_1, ..., i_n; j_1, ..., j_n$ and $i'_0, i'_1, ..., i'_m; j'_1, ..., j'_m$ from $i_0 = i'_0 = i$ and a smallest index k such that (say)

$$j_k = i'_{k'}$$

for some $1 \leq k \leq n$ and $1 \leq k' \leq m$. Then $k \neq 1$ for otherwise the μ-sequence $i'_0, i'_1, ..., i'_{k'}; j'_1, ..., j'_{k'}$ is cyclic. For $k \geq 2$, the μ-sequence $i''_0, i''_1, .., i''_{n''}; j''_1, ..., j''_{n''}$ where $i''_0, i''_1, .., i''_{k'} = i'_0, i'_1, .., i'_{k'}$ and $j''_1, ..., j''_{k'} = j'_1, ..., j_{k'}$ also $i''_{k'+1}, .., i''_{n''} = j_{k-1}, ..., j_1$ and $j''_{k'+1}, ..., j''_{n''} = i_{k-1}, .., i_1$ is cyclic because $i''_{n''} = j_1$. Contradiction. ■

Remark 8 *A bipartite μ^i-market (I, J) is a unitary market : To see this, note first that (I, J) is a seller market with unit excess, and that, for any $i' \in I - i$, since $\mu(i')$ is μ-reachable from i, μ can be alternated to a matching that matches J to $I - i'$.*

Below we introduce a class of active-minimum matchings μ for which μ-markets from solitary players are bipartite.

A.3 Solitary-Minimum Matchings and Solitary-Player Markets

Let μ be an active-minimum matching. We distinguish between the players in $A(\mu)$ according to whether they are μ-cyclic : We call a μ-cyclic player *nonsolitary* and a non-μ-cyclic player *solitary*.

Lemma 4 *Let μ, μ' be any two active-minimum matchings at u. A player who is nonsolitary (solitary) at μ is either matched or nonsolitary (solitary) at μ'.*

Proof. If a player i is nonsolitary at μ, then there is a μ-cycle $i_0, i_1, ..., i_n; j_1, ..., j_n$ from $i_0 = i$. Let ν be an active-minimum matching where i_n is unmatched. Suppose i is unmatched at μ' and consider the μ'-sequence $C = i'_0, i'_1, ..., i'_m; j'_1, ..., j'_m$ from $i'_0 = i$ with $j'_k = \upsilon(i'_{k-1})$ and maximum length m. If i'_m is matched in ν, say $\nu(i'_m) = j$, then either μ' is not active-minimum (when j is unmatched in μ') or m is not maximum length. Therefore, i'_m must be unmatched in ν and then $i'_m = i_n$ (otherwise i'_m is ν-reachable from i_n and so ν is not active-minimum). Hence, C is cyclic and so i is nonsolitary at μ'. ■

We now give our main definition : Let $Z(\mu)$ denote the set of all solitary players at μ.

Definition 9 *An active-minimum matching μ is* solitary-minimum *if $|Z(\mu)| \leq |Z(\mu')|$ for every active-minimum matching μ' at u. A μ^i-market is a* solitary-player market *if i is solitary and μ is solitary-minimum.*

(a) not solitary-minimum

(b) solitary-minimum

Figure 4: Active-Minimum Matchings

Lemma 5 *A solitary-player market is bipartite.*

Proof. Suppose to the contrary that i is a solitary player at μ and the μ^i-market is not bipartite. By Lemma 3, there is a μ-sequence $i_0, i_1, ..., i_n; j_1, ..., j_n$ from $i_0 = i$ such that

$$i_m i_n \in \mathcal{D}(u)$$

for some player $i_m \neq i_0$. Alternate μ to μ' which matches i_0 but not i_n. Now i_n is nonsolitary at μ'. This is because the μ'-sequence $i'_0, i'_1, ..., i'_{n-m}; j'_1, ..., j'_{n-m}$ from $i'_0 = i_n$ where $i'_{n-m} = i_m$ is cyclic. Note that, except for i_0 and i_n, the unmatched players at μ and μ' are identical, hence by Lemma 4, except i, the solitary players at μ and μ' are identical. Then μ' has one less solitary player than μ. This contradicts the fact that μ is solitary-minimum. ■

It follows from Remark 8 that a solitary-player market is a unitary market. The converse is not true : Consider an aspiration u where demand consists of $\{(1,4),(2,4),(3,4)\}$ among four active players. Then $(B = \{1,2\}, S = \{4\})$ is a unitary market but not a solitary-player market at the solitary-minimum matching $\mu = \{(3,4)\}$. Still, the union of all solitary-player markets gives the union of all unitary markets.

A.4 Main Result : The Seller-Market is the Union of all Solitary-Player Markets

For any solitary-minimum matching μ, let S^μ be the set of all μ-reachable players from players in $Z(\mu)$ and $B^\mu = Z(\mu) \cup \mu(S^\mu)$. Note that (B^μ, S^μ) is the union of all solitary-player markets at μ. By Remark 8, (B^μ, S^μ) is a seller-market.

Theorem 5 *Let μ be any solitary-minimum matching at u. The Seller-Market at u is the union of all solitary-player markets, i.e., $(B^*, S^*) = (B^\mu, S^\mu)$.*

Lemma 6 *Let (B, S) be any bipartite submarket and μ be an active-minimum matching at u. Then there is (i) no unmatched player in S and (ii) no nonsolitary player in B.*

Proof. Suppose there is an unmatched player in S. By definition of a submarket, there is a matching ν that matches S into B. The matching μ' that agrees with ν for S-players and with μ for other players matches more active players than μ does. But then μ is not active-minimum. Contradiction.

Suppose there is a nonsolitary player $i \in B \cup S$. Then $i \in B$ by (i) and there is a μ-cycle $i_0, i_1, ..., i_n; j_1, ..., j_n$ from $i_0 = i$ with $j_1 \in S$. But then, by alternation, there is an active-minimum matching that leaves j_1 unmatched. This contradicts (i). ■

Proof. (*Theorem* 5) By Lemma 5 and Remark 8, $(B^\mu, S^\mu) \subset (B^*, S^*)$. We complete the proof by showing that (B^μ, S^μ) contains every unitary seller-market.

Let (B, S) be any unitary seller-market at u, B_0 be the set of all B-players unmatched at μ, S' be the set of all μ-reachable players from B_0-players and $B' = B_0 \cup \mu(S')$. By Lemma 6(ii) $B_0 \subset Z(\mu)$ so $(B', S') \subset (B^\mu, S^\mu)$. We will show $(B, S) = (B', S')$.

By construction, no B'-player has demand for any player in $S - S'$ (since $S - S'$ is "unreachable" from B_0). Also $\mu(B - B') \subset (S - S')$ since $\mu(S') \subset B'$. Therefore μ matches $B - B'$ to $S - S'$ (otherwise S is not matchable into B.) Then $B - B'$ and $S - S'$ must be empty because (B, S) is unitary (otherwise S is not matchable into $B - i$ for some $i \in B - B'$.) ■

The Market Procedure we give in Section 6 is a Seller-Market tracing procedure. We use the following fact in proving its convergence.

Corollary 1 *The excess in the Seller-Market at u is equal to the number of solitary players at any solitary-minimum matching at u.*

We refer to a solitary-minimum matching with no solitary player as a *solitary-null* matching.

Corollary 2 *An aspiration is balanced if and only if it admits a solitary-null matching.*

The statement below gives a characterization for solitary-minimum matchings : The "only if" part follows from the proof of Lemma 5 and the "if" part follows from Theorem 5.

Corollary 3 *An active-minimum matching μ is solitary-minimum if and only if all the μ-sequences from solitary players are cycle-free.*[37]

[37] Corollary 3 is the counterpart of the characterization for active-minimum matchings in Footnote 35 and would be used to find a solitary-minimum matching in any demand graph.

Remark 9 *It also follows from Theorem 5 that the Seller-Market is the* maximum-excess bipartite submarket *with* minimum size*. In particular, maximum-excess bipartite submarkets are closed under intersection hence there is a unique minimum-size maximum-excess bipartite submarket, namely, the Seller-Market.*

B Proofs for Sections 3 and 4

The proof of Theorem 1 uses the key result below. We say (N, f) is *piecewise linear* if every f_{ij} is piecewise linear.

Recall that a *feasible direction* at an aspiration u is a nonzero vector $d \in R^N$ such that $u + \lambda d$ is an aspiration for all sufficiently small $\lambda > 0$. Given a feasible direction d at u, if f_{ij} are piecewise linear, the demand graph $\mathcal{D}(u + \lambda d)$ is identical for all sufficiently small $\lambda > 0$ which we denote $\mathcal{D}^+(u, d)$.

Lemma 7 *(*Direction Lemma*) Let (N, f) be piecewise linear. If (B, S) is a bipartite submarket at an aspiration u, then there is a feasible direction d with*

$$d_i < 0 \textit{ for } i \in B,$$

$$d_i > 0 \textit{ for } i \in S,$$

$$d_i = 0 \textit{ for } i \in N - B \cup S$$

such that (B, S) is a bipartite submarket at $u + \lambda d$ for all sufficiently small $\lambda > 0$.

Proof. Let f'_{ij} denote the right-hand derivative of f_{ij}. Take any $d \in R^N$ such that

$$d_i > 0 \text{ for } i \in S,$$

$$d_i = \max_{j \in D_i(u)} \{f'_{ij}(u_j) d_j\} \text{ for } i \in B,$$

$$d_i = 0 \text{ for } i \in N - B \cup S.$$

Clearly, d is a feasible direction at u. Let ν be any active-minimum matching in $\mathcal{D}^+_B(u, d)$. If ν matches every S-player, then (B, S) is a bipartite submarket at $u + \lambda d$ for all sufficiently small $\lambda > 0$. Therefore, suppose ν does not match every S-player. Let B' be the set of all unmatched B-players and S' be the set of all S-players which are ν-reachable from B'-players in $\mathcal{D}^+_B(u, d)$.

We claim that there is a feasible direction d^* such that $\nu \subset \mathcal{D}^+_B(u, d^*)$ and (i) $\mathcal{D}^+_B(u, d^*)$ contains a matching of greater cardinality than ν *or else* (ii) the set of all S-players, say S^*, which are

ν-reachable from B'-players in $\mathcal{D}_B^+(u,d^*)$ has a greater cardinality than S'. By recursion, this will prove the lemma since S is a finite set.

Set $d_i(\delta)$ equal to

$$(1+\delta)d_i \text{ for } i \in S' \text{ and } d_i \text{ for } i \in S - S',$$
$$\max_{j \in D_i(u)} \{f'_{ij}(u_j)d_j(\delta)\} \text{ for } i \in B,$$
$$0 \text{ for } i \in N - B \cup S.$$

Alter the direction $d(\delta)$ by increasing δ continuously above 0 up to δ^* where a new pair ij joins $\mathcal{D}_B^+(u,d(\delta))$. Set $d^* = d(\delta^*)$. Note that $\mathcal{D}_B^+(u,d) \subset \mathcal{D}_B^+(u,d^*)$ and hence $\nu \subset \mathcal{D}_B^+(u,d^*)$. Let $\overline{B} = B' \cup \nu(S')$. See that $(i,j) \in \overline{B} \times (S - S')$. Therefore, player j is ν-reachable from B', i.e., $j \in S^*$. If j is unmatched at ν, then ν is not active-minimum at $\mathcal{D}_B^+(u,d^*)$, in which case claim (i) holds. Otherwise, S^* has a greater cardinality than S' since $j \in S^* - S'$ and $S' \subset S^*$. In this case, claim (ii) holds. End of claim. ■

PROOF OF THEOREM 1 : Suppose (N,f) is piecewise linear. For any aspiration u and any seller-market (B,S) at u, let $g_S(u)$ be the sum of u_i for $i \in S$, and let $g(u)$ be the maximum of $g_S(u)$ over all seller-markets at u. Since the set of aspirations is nonempty and closed, there is an aspiration u^* such that $g(u^*)$ is maximum among all aspirations. Then u^* has no seller-market, for otherwise by the Direction Lemma, there is an aspiration u' with $g(u') > g(u^*)$ contradicting maximality of u^*. So there exists a balanced aspiration for every piecewise linear (N,f) and by uniform approximation for (N,f). ■

Lemma 8 *The demand set of every $N^+_{uu'}$-player at u is in $N^-_{uu'}$.*

Proof. If i demands j at u and u' is an aspiration with $u'_i < u_i$, then $u'_j \geq f_{ji}(u'_i) > f_{ji}(u_i) = u_j$. ■

Lemma 9 *Let u be a balanced aspiration and u' be any aspiration. Then $N^+_{uu'}$ is matchable into $N^-_{uu'}$ at u.*

Proof. Every player in $N^+_{uu'}$ is active at u (otherwise $u'_i < 0$ for some $i \in N^+_{uu'}$ hence u' is not an aspiration.) Suppose $N^+_{uu'}$ is not matchable into $N^-_{uu'}$ at u. Let μ be a matching at u that matches a maximum number of players in $N^+_{uu'}$ and let i be a player unmatched. Let (B,S) be the μ^i-market at u. By Lemma 8 and maximality of μ, using induction, $S \subset N^-_{uu'}$ and $\mu(S) \subset N^+_{uu'}$. But then (B,S) is a seller-market at u. Contradiction. ■

PROOF OF LEMMA 1 : By Lemma 9, $N^+_{uu'}$ is matchable into $N^-_{uu'}$ at u and symmetrically $N^+_{u'u}$ is matchable into $N^-_{u'u}$ at u'. Then, $(N^+_{uu'}, N^-_{uu'})$ and $(N^+_{u'u}, N^-_{u'u})$ are bipartite submarkets at u and u' respectively, so $\left|N^+_{uu'}\right| = \left|N^-_{uu'}\right|$, therefore they are balanced-markets. ■

Lemma 10 *Let u, u' be any two balanced aspirations. Then u is a stable allocation if and only if u' is a stable allocation.*

Proof. Suppose (u, μ) is a stable allocation. Then μ matches $N^+_{uu'}$ and $N^-_{uu'}$ to each other (otherwise μ leaves a player i in $N^+_{uu'}$ unmatched, which is not possible, because i is active). So

$$\mu(N^0_{uu'}) \subset N^0_{uu'},$$

where $N^0_{uu'} = N - (N^+_{uu'} \cup N^-_{uu'})$. Let μ_0 be the set of all pairs $ij \in \mu$ with $i, j \in N^0_{uu'}$. By Proposition 1, there is a matching ν at u' that matches $N^+_{uu'}$ and $N^-_{uu'}$ to each other. The matching that agrees with μ_0 for $N^0_{uu'}$-players and with ν otherwise is u'-compatible and leaves no active player unmatched. So u' is a stable allocation. ■

PROOF OF THEOREM 2 : Case (i) : There is a stable allocation. By Lemma 10, U is the set of all stable allocations.

Case (ii) : There is no stable allocation. Take any $u \in U$. Let μ be a solitary-minimum matching at u. By Corollary 2, every active-unmatched player is nonsolitary. For every nonsolitary i, pick a μ-cycle $C_i = i_0, i_1, ..., i_n, j_1, ..., j_n$. Let μ_i and ν_i respectively be the matchings that consist of all pairs $i_k j_k$ and $i_{k-1} j_k$ with $j_k \in C_i$. Let $\nu'_i = \nu_i \cup i_0 i_n$ and μ^+, ν^+ respectively be the union of μ_i, ν'_i over all unmatched i. Denote ν the matching $\mu - \mu^+$ and χ the half-matching $\mu^+ \cup \nu^+$. Thus the semi-matching (ν, χ) leaves no active player unmatched. Hence u is a semistable allocation. ■

PROOF OF PROPOSITION 1 : By Theorem 1, there is a balanced aspiration, say u. We will show that u is a stable allocation. Suppose not.

Let μ be a solitary-minimum matching at u. Since u is nonrealizable, μ leaves an active player unmatched, say i. Note that i is *nonsolitary* at μ, because otherwise by Lemma 5 u has a seller-market and is not balanced.

Let C be a μ-cycle from i which consists of a maximum number of players. Since there are an *odd* number of players in C, there must be two same-type players, say j, j', such that j is in C and j' is not in C. We claim

$$u_j = u_{j'}.$$

If not, then $u_j < u_{j'}$ (otherwise no C-player demands j at u contradicting $j \in C$). Then no player other than j demands j' at u. So $D_{j'}(u) = \{j\}$ (otherwise $D_{j'}(u)$ is empty but j' is active at u). Hence j' is unmatched at u. But then μ is augmentable contradicting the fact that μ is active-minimum. End of claim.

Therefore there is a C-player who demands j' at u. Then j' must be matched at u (otherwise μ is augmentable therefore not active-minimum). But then there is a μ-cycle from i (obtained by "adding" the pair $(j', \mu(j'))$ to C) which contains a greater number of players. Contradiction. ■

PROOF OF PROPOSITION 2 : Let i be a nonconstant player and let u be any aspiration in U. Take any $u' \in U$ such that $u_i \neq u'_i$. By Lemma 1, $(N^+_{uu'}, N^-_{uu'})$ is a balanced-market at u.

Case (i) : Let (u, μ) be a stable allocation. Every $N^+_{uu'}$-player is active at u so in $\mu(N)$. Then μ matches $N^+_{uu'}$ to $N^-_{uu'}$. Recall i is in $N^+_{uu'} \cup N^-_{uu'}$. So $\mu(i)$ is also in $N^+_{uu'} \cup N^-_{uu'}$ hence a nonconstant player.

Case (ii) : Let (u, ν, χ) be a semistable allocation. Any player with whom an $N^-_{uu'}$-player is in half-partnership or full-partnership must be in $N^+_{uu'}$, for otherwise by balancedness there would be an $N^+_{uu'}$-player unmatched or single-half-matched contradicting (u, ν, χ) is semistable. In particular $\nu(N^-_{uu'}) \subset N^+_{uu'}$. Also no $N^-_{uu'}$-player is in half-partnership because otherwise there would be an even half-partner cycle in $N^+_{uu'} \cup N^-_{uu'}$ contradicting (u, ν, χ) is essential. Then $\nu(N^-_{uu'}) = N^+_{uu'}$ because otherwise an $N^+_{uu'}$-player is unmatched contradicting (u, ν, χ) is semistable. Thus $\nu(i)$ is in $N^+_{uu'} \cup N^-_{uu'}$ hence a nonconstant player. ■

Lemma 11 *There is a stable bipartition at any υ in V.*

Proof. Suppose there is no stable bipartition at some υ in V. Let υ' be any payoff in V where $N^+_{\upsilon\upsilon'} \cup N^-_{\upsilon\upsilon'}$ contains a maximum number of players. Then, there exist a nonconstant player i such that $\upsilon_i = \upsilon'_i$ (otherwise $(N^+_{\upsilon\upsilon'}, N^-_{\upsilon\upsilon'})$ is a stable bipartition at υ by Lemma 1). Let υ'' be a payoff in V such that $\upsilon''_i \neq \upsilon'_i$. By Lemma 1, $(N^+_{\upsilon'\upsilon''}, N^-_{\upsilon'\upsilon''})$ is a bipartite market at υ'. By Lemma 7, pick a payoff υ^* in V that is sufficiently close to υ' such that $\upsilon'_i \neq \upsilon^*_i$. Then, $\upsilon^*_j \neq \upsilon_j$ for every j in $N^+_{\upsilon\upsilon'} \cup N^-_{\upsilon\upsilon'}$ since $\upsilon'_j \neq \upsilon_j$ for every j in $N^+_{\upsilon\upsilon'} \cup N^-_{\upsilon\upsilon'}$. Also, $\upsilon_i \neq \upsilon^*_i$ (since $\upsilon'_i = \upsilon_i$ and $\upsilon'_i \neq \upsilon^*_i$). Contradiction. ■

PROOF OF PROPOSITION 3 : *(i)* Let υ, υ' any payoffs in V and suppose υ and υ' are realizable respectively by μ and μ'. We claim $\mu \subset \mathcal{D}(\upsilon')$ which completes the proof.

By Lemma 1, μ, μ' both match $N^+_{\upsilon\upsilon'}$ and $N^-_{\upsilon\upsilon'}$ to each other. Suppose the claim is not true. Then there is a pair $(i, j) \in (N^+_{\upsilon\upsilon'}, N^-_{\upsilon\upsilon'})$ such that $ij \in \mu - \mathcal{D}(\upsilon')$. Let $i_1 = i$ and $I = \{i_1, ..., i_n\}, J = \{j_1, ..., j_n\}$ be the player sets defined recursively by setting $j_k = \mu(i_k)$ and $i_{k+1} = \mu'(j_k)$. Then

$$(i_{n+1}, j_{n+1}) = (i_1, j_1).$$

Clearly $I \subset N^+_{\upsilon\upsilon'}$ and $J \subset N^-_{\upsilon\upsilon'}$ (since $\mu(N^+_{\upsilon\upsilon'}) = \mu'(N^+_{\upsilon\upsilon'}) = N^-_{\upsilon\upsilon'}$.) Then $c_{i_k j_{k-1}} - \upsilon'_{j_{k-1}} \geq c_{i_k j_k} - \upsilon'_{j_k}$ since $i_k j_{k-1} \in \mathcal{D}(\upsilon')$ and $c_{i_k j_k} - \upsilon_{j_k} \geq c_{i_k j_{k-1}} - \upsilon_{j_{k-1}}$ since $i_k j_k \in \mathcal{D}(\upsilon)$ for all k. So

$$\upsilon'_{j_{k-1}} - \upsilon_{j_{k-1}} \leq \upsilon'_{j_k} - \upsilon_{j_k}$$

for all k. But then $\upsilon'_{j_1} - \upsilon_{j_1} = \upsilon'_{j_n} - \upsilon_{j_n}$. Therefore $i_1 j_1 \in \mathcal{D}(\upsilon')$ (since $i_1 j_1 \in \mathcal{D}(\upsilon)$, $i_1 j_n \in \mathcal{D}(\upsilon')$). Contradiction.

(ii) By Lemma 11, let (Y_1, Y_2) be a stable bipartition at some υ in V. By Lemma 3, (Y_1, Y_2) is a stable bipartition at V. ■

PROOF OF PROPOSITION 4 : Consider any piecewise linear game. Let K be any finite collection of balanced aspirations. Let u be any balanced aspiration in U at which $u_i \in med(K)_i$ for a maximum number of players. We claim $u_i \in med(K)_i$ for every i. Suppose not. Then the sets $B = \{i \in N|\ u_i^{**} < u_i\}$ and $S = \{i \in N|\ u_i < u_i^*\}$ cannot both be empty. Note that players in B and S are nonconstant players.

Define $U' = \{u' \in U | u_i^{**} \leq u_i'$ for $i \in B$, $u_i' \leq u_i^*$ for $i \in S$, and $u_i' = u_i$ for $i \notin B \cup S\}$. Clearly U' is nonempty and closed. So there is a $\upsilon \in U'$ such that $\sum\limits_{i \in B} \upsilon_i \leq \sum\limits_{i \in B} u_i'$ for every $u' \in U'$. If $\upsilon_i = u_i^{**}$ for some $i \in B$ or $\upsilon_i = u_i^*$ for some $i \in S$, then there would be an additional player i with $\upsilon_i \in med(K)_i$. Contradiction. So $B = \{i \in N|\ u_i^{**} < \upsilon_i\}$ and $S = \{i \in N|\ u_i^* < \upsilon_i\}$. By Proposition 2, there is a matching at υ, say μ, that matches all the nonconstant players (among each other.)

Let $n = m$ when $|K|$ is odd and $n = m + 1$ when $|K|$ is even.

Let i be any player in S and $j = \mu(i)$. By Lemma 1, $u_j' < \upsilon_j$ for every $u' \in U$ such that $u_i' > \upsilon_i$. Since at least n elements of K give a higher payoff to i than υ, at least n elements of K give a lower payoff to j than υ. Hence $j \in B$. Thus S is matchable into B at υ.

Let i be any player in B and $j \in D_i(\upsilon)$. By Lemma 8, $u_j' > \upsilon_j$ for any $u' \in U$ such that $u_i' < \upsilon_i$. Since at least n elements of K give a lower payoff to i than υ, at least n elements of K give a higher payoff to j than υ. Hence $j \in S$ and the demand set of every B-player is in S at υ.

Thus (B, S) is a balanced-market at υ. But then, by Lemma 7, there exists $\upsilon^* \in U'$ such that $\sum\limits_{i \in B} \upsilon_i^* < \sum\limits_{i \in B} \upsilon_i$. Contradiction. This proves our claim and the Proposition 4 for any piecewise linear game. Proposition 4 holds for any game by uniform approximation. ■

PROOF OF PROPOSITION 5 : Suppose U is the Equilibrium Set of a piecewise linear game. Take any u, u' in U. By Proposition 1 the pair of player sets $(N_{uu'}^+, N_{uu'}^-)$ is a balanced-market and by Lemma 7 there exist payoffs between u, u' that belong to U. By uniform approximation, the result holds for any game. ■

C Addendum to Section 5

PROOF OF PROPOSITION 6 : Let u be a balanced aspiration and u' be any aspiration. Let μ' be any matching at u' and

$$\mu_+' = \left\{ ij \in \mu' | i \in N_{uu'}^+ \right\}.$$

By Lemma 8 the demand set of every $N^-_{uu'}$-player is in $N^+_{uu'}$ at u'. So the matching $\mu'_0 = \mu' - \mu'_+$ contains only players in $N^0_{uu'} = N - (N^+_{uu'} \cup N^-_{uu'}) = \{i \in N | u_i = u'_i\}$. Using Lemma 9, let μ_+ be a matching at u that matches $N^+_{uu'}$ into $N^-_{uu'}$. It is clear that

$$\mu = \mu'_0 \cup \mu_+$$

is a matching at u. We show below that the number of active-unmatched players that μ' leaves at u' is more than that of μ leaves at u.

Since the demand set of every $N^-_{uu'}$-player at u' is in $N^+_{uu'}$,

$$\mu'(N^-_{uu'}) \subset N^+_{uu'}.$$

Let A be the set of all players in $N - N^0_{uu'}$ who are unmatched at μ and active at u. By definition of μ, $A \subset N^-_{uu'} - \mu(N^+_{uu'})$. Hence

$$\left|N^-_{uu'}\right| \geq \left|\mu(N^+_{uu'})\right| + |A| = \left|N^+_{uu'}\right| + |A|.$$

Let A^0 be the set of $N^0_{uu'}$-players who are unmatched at μ and active at u but matched at μ'. Then $\mu'(A^0) \subset N^+_{uu'}$. Therefore

$$\left|\mu'(N^-_{uu'})\right| \leq \left|N^+_{uu'}\right| - \left|A^0\right|.$$

So $\left|N^-_{uu'}\right| - \left|\mu'(N^-_{uu'})\right| \geq |A| + |A^0|$. Recall that $N^-_{uu'}$-players are active at u'. ■

C.1 A Characterization of the Demand Bargaining Set

Let $\boldsymbol{D}$ be the Demand Bargaining Set at an aspiration u. Consider the set of active-unmatched players $A(\mu)$. We say that an aspiration-allocation u^μ is *maximal* if there is no $ij \in \mathcal{D}(u)$ with $i, j \in A(\mu)$.

Proposition 7 *An aspiration-allocation u^μ is in $\boldsymbol{D}$ if and only if u^μ is maximal and u has no balanced market (B, S) with $B \subset A(\mu)$.*

Proof. ($\Rightarrow$) Let u^μ be a maximal aspiration-allocation and suppose there is no balanced-market (B, S) at u such that $B \subset N - \mu(N)$. Suppose to the contrary that there is a justified objection (T, u') to u^μ.

Let (u', μ') be an allocation for the restricted market (T, f). Let $B = \{i \in T | u'_i < u_i\}$. Then $u_i > u'_i > u^\mu_i$ for all $i \in B$, so $B \subset A(\mu)$.

We claim $|T - B| \leq |B|$. Suppose not. Since (u', μ') is an allocation for (T, f), there is a pair $ij \in \mu'$ with $i, j \in T - B$. So $(u'_i, u'_j) \geq (u_i, u_j)$. Since u is an aspiration, $(u'_i, u'_j) = (u_i, u_j)$. Therefore $ij \in \mathcal{D}(u)$ and $(u_i, u_j) > (u^\mu_i, u^\mu_j)$, saying of u^μ is not maximal. End of claim.

Now let S be the union of the demand sets of B-players. Suppose $S \subsetneq T$. Then, there is a pair $ij \in \mathcal{D}(u)$ with $i \in B$ and $j \in S - T$. So $((i,j),(u_i,u_j))$ is a counterobjection to (T,u'). Contradiction. Therefore, $S \subset T$. By maximality of u^μ, $S \cap A(\mu) = \varnothing$. Hence $S \subset T - B$. So from the claim above $|S| \leq |B|$.

Finally let ν be any active-minimum matching in $\mathcal{D}_B(u)$. It is not possible that ν matches B into S for otherwise (B,S) would be a balanced-market at u with $B \subset A(\mu)$. Therefore, since $|S| \leq |B|$ as shown above, it must be that ν leaves a player i in B unmatched. Let (B_i, S_i) be the ν^i-market. Then $(B_i,S_i) \subset (B,S)$ and $(B_i - i, S_i)$ is a balanced-market at u with $B_i - i \subset A(\mu)$. Contradiction.

($\Leftarrow$) Suppose u^μ is not maximal. Then there is a pair $ij \in \mathcal{D}(u)$ with $i,j \in A(\mu)$. Then $((i,j),(u_i,u_j))$ is a justified objection to u^μ. Therefore $u^\mu \notin \boldsymbol{D}$.

Consider any piecewise linear game. Suppose there is a balanced-market (B,S) at u wth $B \subset A(\mu)$. By Lemma 7, let $u' = u + \lambda d$, where d is a feasible direction such that $d_i > 0$ for all $i \in S$ and $d_i < 0$ for all $i \in B$. It is clear that $(B \cup S, u')$ is a justified objection to u^μ for sufficiently small $\lambda > 0$. Hence $u^\mu \notin \boldsymbol{D}$. By uniform approximation, the same holds for any game. ■

C.2 Proof of Theorem 3, Inclusion Lemma and Examples

PROOF OF THEOREM 3 : Suppose u^μ is a pseudostable allocation not in $\boldsymbol{D}$. By Proposition 7, there is a balanced market (B,S) at u such that $B \subset N - \mu(N)$. But then, by Corollary 2, B-players are nonsolitary at μ. This contradicts with Lemma 6(ii). ■

Theorem 3 says that pseudostable allocations are "stable" from an "exclusive" Bargaining Set perspective. It is on the other hand true that the Demand Bargaining Set may contain non-pseudostable allocations. We show in the examples below that these may in fact be various not fitting into a classification at hand. First, we show that the Demand Bargaining Set is a refinement of the Zhou Bargaining set :

Lemma 12 $\boldsymbol{D} \subset \boldsymbol{Z}$.

Proof. Suppose to the contrary that there is an aspiration-allocation u^μ in $\boldsymbol{D}$ but not in $\boldsymbol{Z}$. Then, there is an objection (T,u') to u^μ such that any counterobjection (Q,u) to (T,u') satisfies either $Q \subset T$ or $T \subset Q$.

There can be no counterobjection (Q,u) to (T,u') with $Q \subset T$: Otherwise $u_i > u'_i > u^\mu_i$ for $i \in Q$ so (Q,u) is a justified objection to u^μ, implying $u^\mu \notin \boldsymbol{D}$. Contradiction.

Consider now any counterobjection (Q,u) to (T,u') with $T \subset Q$. Then $u_i > u'_i > u^\mu_i$ for $i \in T$. Now let (u,μ') be any allocation for the restricted game (Q,f) and i be any player in T. If $\mu'(i) \in T$, then $((i,j),(u_i,u_j))$ is a justified objection to u^μ, implying $u^\mu \notin \boldsymbol{D}$. Contradiction. If $\mu'(i) \notin T$, then $((i,\mu'(i)),(u_i,u_{\mu'(i)}))$ is a counterobjection to (T,u'), but then it is not true that $T \subset \{i,\mu'(i)\}$ since there are at least two players in T. Contradiction. ■

Example 4 shows that $\boldsymbol{D}$ may contain some *non-pseudostable* maximum balanced-aspiration-allocations but not all. Example 5 shows that $\boldsymbol{D}$ may contain maximum *non-balanced*-aspiration-allocations. Example 5 also displays *dominated* allocations that are in $\boldsymbol{Z}$ and not in $\boldsymbol{D}$. It is worth adding that the null-allocation is not in $\boldsymbol{D}$ but may be in $\boldsymbol{Z}$, for instance, in any three-person game where there is no stable allocation.

Example 4 *There are five players in* $N = \{\boldsymbol{1,2,3,4,5}\}$. *The worth of a partnership is* 2 *for the pairs in*

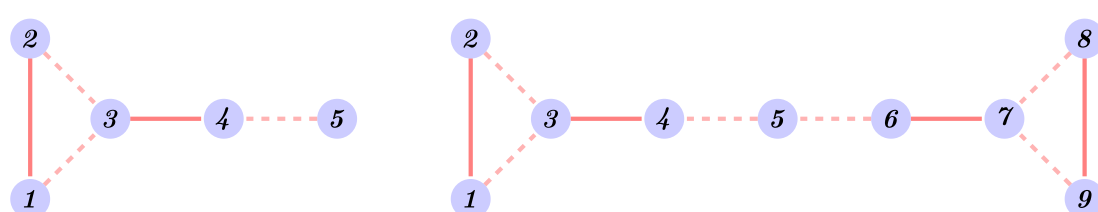
$$\{\boldsymbol{12,13,23,34,45}\}$$

and 0 *otherwise. Let* $u = (1,1,1,1,1)$ *and* $\mu = \{\boldsymbol{12,34}\}$ *(see Figure 6a). Clearly* u^μ *is a maximum balanced-aspiration-allocation but* μ *is not solitary-null at* u. *Let* $T = \{\boldsymbol{4,5}\}$ *and* $u' = (u'_{\boldsymbol{4}}, u'_{\boldsymbol{5}}) = (1+\epsilon, 1-\epsilon)$ *where* $0 < \epsilon < 1$. *It is easily checked that* (T,u') *is a justified objection to* u^μ *so* u^μ *is not in* $\boldsymbol{Z}$ *and therefore not in* $\boldsymbol{D}$.

Now consider the extended game with four additional players $\{\boldsymbol{6,7,8,9}\}$ *where the worth of a partnership is* 2 *for the pairs in*

$$\{\boldsymbol{12,13,23,34,45,56,67,78,79,89}\}$$

and 0 *otherwise. Let* $u = (1,1,1,1,1,1,1,1,1)$ *and* $\mu' = \mu \cup \{\boldsymbol{67,89}\}$ *(see Figure 6b). Clearly, again,* $u^{\mu'}$ *is a maximum balanced-aspiration-allocation but* μ' *is not solitary-null at* u. *Using Proposition 7, it is easy to see that* $u^{\mu'}$ *is in* $\boldsymbol{D}$ *and therefore in* $\boldsymbol{Z}$.



(a) (b)

Figure 5: Example 4

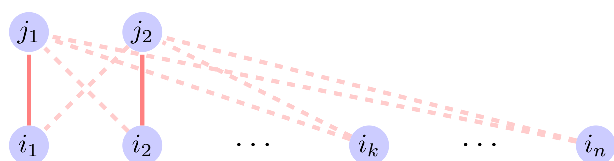
Example 5 *There are two sets - I and J - of same-type players where* $|I| = n \geq 3$ *and J consists of two players say* j, j'. *The worth of a pair with one player from each set is* 2 *and with both players from I is* $2 - 2\epsilon$ *(where* $0 < \epsilon < 1$.*) The players* j, j' *cannot form a pair with each other.*

The demand is equal to $\{ij, ij'|i \in I\}$ *for* $\delta < \epsilon$ *(see Figure 7a) and equal to* $\{ij, ij'|i \in I\} \cup \{ii'|i, i' \in I\}$ *for* $\delta = \epsilon$ *(see Figure 7b). The payoff* u *where* u_i *is equal to* $1 - \delta$ *for* $i \in I$ *and* $1 + \delta$ *for* $j \in J$ *is an aspiration for every* $\delta \leq \epsilon$*. (It is a non-balanced aspiration for every* $\delta < \epsilon$ *and a balanced aspiration for* $\delta = \epsilon$*.) Let* μ *be any matching that consists of two pairs* ij *and* $i'j'$ *where* $i, i' \in I$ *and* $j, j' \in J$*.*

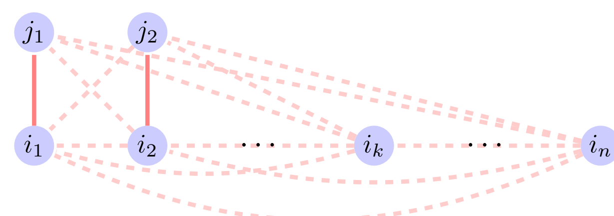
It is easily seen that u^μ *is in* $\boldsymbol{Z}$ *for any* odd n *for all* $\delta \leq \epsilon$*. Note that* u^μ *is dominated for* $n \geq 5$ *: There are* $n - 2$ unmatched I*-players all but one of whom can form a pair with another unmatched* I*-player and achieve a payoff equal to* $1 - \epsilon$ *strictly above her stand alone utility.*

On the other hand, by Proposition 7, u^μ *is not in* $\boldsymbol{D}$ *for* $n > 3$ *for any* $\delta \leq \epsilon$*. To see this, let* $I_0 = \{i, i'\} \subset I$ *be any two unmatched players. In case* $\delta < \epsilon$*,* (I', J) *is a balanced-market with* $I_0 \subset N - \mu(N)$*, and in case* $\delta = \epsilon$*,* u^μ *is not maximal, so in both cases* u^μ *is not in* $\boldsymbol{D}$ *by Proposition 7.*

It is easily checked that u^μ *is in* $\boldsymbol{D}$ *for* $n = 3$ *for any* $\delta \leq \epsilon$*.*



(a) $\delta < \epsilon$



(b) $\delta = \epsilon$

Figure 6: Example 5

D Identifying the Seller-Market Recursively along the Market Procedure Path and Proof of Convergence

Here we present the *Seller-Market Algorithm* by which the Seller-Market can be identified recursively along the Procedure Path.

Let u be any aspiration on the (piecewise linear) Path and consider the demand graphs $\mathcal{D}, \mathcal{D}', \mathcal{D}''$ where $\mathcal{D}$ is the incoming directional demand graph, $\mathcal{D}'$ the demand graph and $\mathcal{D}''$ the outgoing directional demand graph at u. Likewise consider the respective sets of active players A, A', A'' at u. (We will say that $(\mathcal{D}, A), (\mathcal{D}', A')$ are *successive* and so are $(\mathcal{D}', A'), (\mathcal{D}'', A'')$.) For all but a finite number of aspirations on the Path, $(\mathcal{D}, A) = (\mathcal{D}', A') = (\mathcal{D}'', A'')$.

Our main task here is to identify the Seller-Market $(B^{*\prime}, S^{*\prime})$ in $(\mathcal{D}', A')$ given the Seller-Market (B^*, S^*) in $(\mathcal{D}, A)$ when $(\mathcal{D}', A') \neq (\mathcal{D}, A)$. From Theorem 5, this is equivalent to finding a solitary-minimum matching μ' and the set of solitary players $Z(\mu')$ in $(\mathcal{D}', A')$ given a solitary-minimum matching μ and the set of solitary players $Z(\mu)$ in $(\mathcal{D}, A)$.

Let $(\mathcal{D}, A), (\mathcal{D}', A')$ be successive and recall

$$\mathcal{D} \subset \mathcal{D}', A \supset A'.$$

The Seller-Market Algorithm consists of the *Solitary-Minimal-Matching Routine* (*SMMR*) and the *Solitary-Player Set Routine*. The first finds a solitary-minimum matching μ' in $(\mathcal{D}', A')$ given a solitary-minimum matching μ in $(\mathcal{D}, A)$. The second identifies $Z(\mu')$ given $\mu, Z(\mu)$ and μ' found by SSMR.

Remark 10 *There may be* several *changes between* $(\mathcal{D}, A), (\mathcal{D}', A')$. *Our Algorithm takes the changes in* $\mathcal{D}' - \mathcal{D}$ *and* $A - A'$ *one at a time in any order. Below we assume that* ***either*** $A - A'$ *consists of a single player and* $\mathcal{D} = \mathcal{D}'$ ***or*** $\mathcal{D}' - \mathcal{D}$ *consists of a single pair and* $A = A'$.

Solitary-Minimal Matching Routine

Let μ be a solitary-minimum matching in $(\mathcal{D}, A)$.

Step 1 : If μ is not active-minimum in $(\mathcal{D}', A')$ then augment/alter μ to an active-minimum matching μ_1. Otherwise, let $\mu_1 = \mu$.

Step 2 : If μ_1 is not solitary-minimum in $(\mathcal{D}', A')$ then alter μ_1 to a solitary-minimum matching μ_2. Otherwise, let $\mu_2 = \mu_1$.

Then let $\mu' = \mu_2$.

The Routine above finds a solitary-minimum matching μ' in $(\mathcal{D}', A')$. It is an elementary recursion[38] and we omit the straightforward proof. It remains to identify $Z(\mu')$.

Lemma 13 *If* (B, S) *is a unitary market in* $(\mathcal{D}', A')$, *then* (B, S) *is a seller-market in* $(\mathcal{D}, A)$.

Proof. Since $A' \subset A$ and $\mathcal{D} \subset \mathcal{D}'$, we only need to show that S is matchable into B in $\mathcal{D}$. Let ν be a matching that matches S into B in $\mathcal{D}'$. In all cases except when the pair ij in $\mathcal{D}' - \mathcal{D}$ is in ν

[38] Take any active-unmatched player. If an unmatched player is μ-reachable, then augment μ. If a nonactive matched player is μ-reachable, then alter μ. Since $(\mathcal{D}', A')$ differs from $(\mathcal{D}, A)$ by a singleton, the active-unmatched player and the reachability sequence can be selected judiciously. Step 1 thus can be carried out in a simple way. In particular, μ_1 differs from μ by at most a single pair. Similar comments go for Step 2 and μ_2 differs from μ_1 by at most a single pair.

and $i \in B, j \in S$, it is clear that ν also matches S into B in $\mathcal{D}$. In the remaining case, let ν' be a matching that matches S into $B - i$ in $\mathcal{D}'$, then ν' is in $\mathcal{D}$. ■

It is worth pointing out that the result above holds neither for successive unitary markets nor for successive seller-markets. We use it in proving our key lemma below which says that successive solitary-player sets $Z(\mu'), Z(\mu)$ are nested when μ' is *selected* by SMMR.

Let Z^0 be the set of all players in $Z(\mu)$ who are in A' and not matched by μ'.

Lemma 14 $Z(\mu') \subset Z^0 \subset Z(\mu)$.

Proof. A player $i \in Z(\mu')$ is in A' and therefore in A. Also, i is not matched by the matching μ_1 that is constructed in SMMR (for otherwise i would not be in $Z(\mu')$). Therefore i is not matched by μ (otherwise i would not be in A'). By Theorem 5, then, i belongs to a unitary market in $(\mathcal{D}', A')$, and therefore by Lemma 13, to a seller-market in $(\mathcal{D}, A)$. Therefore $i \in Z(\mu)$ by Lemma 6(ii). ■

Solitary-Player Set Routine

Let μ be a solitary-minimum matching and $Z(\mu)$ the solitary-player set in $(\mathcal{D}, A)$. Let μ' be the solitary-minimum matching in $(\mathcal{D}', A')$ found by the Solitary-Minimum Matching Routine. If

$$\mu' = \mu \text{ and } \mathcal{D}' - \mathcal{D} = b_1 b_2 \text{ for some } b_1, b_2 \in B^*$$

then b_1, b_2 are in a μ'-cycle $C \subset \mathcal{D}'$, the unmatched player b in C is nonsolitary at μ', and

$$Z(\mu') = Z(\mu) - b,$$

otherwise

$$Z(\mu') = Z^0.$$

Let $Z(\mu) = Z, Z(\mu') = Z'$. The Routine above merely involves checking which of two cases occur and updating the solitary-player set accordingly. One of these cases is very *particular*. Apart from this case, Z' is equal to Z^- : In words, the solitary players at μ' consist of the solitary players at μ excluding (naturally) those who have become matched or nonactive. In the remaining case, one solitary player at μ - say b - becomes nonsolitary at μ' and all the other solitary players (if any) remain solitary. Thus $Z' = Z - b$. As we show below, this particular case occurs when the "new" demand in $\mathcal{D}' - \mathcal{D}$ is a pair $b_1 b_2$ where b_1, b_2 are two Buyers in $(\mathcal{D}, A)$ *and* SMMR finds μ' to be the same as μ. (So $b_1 b_2 \notin \mu'$.) Then, it is a fact that, there is a μ'-cycle in $\mathcal{D}'$ to which b_1, b_2 belong and whose unmatched player is $b = Z - Z'$. To be precise, b is the player b_1 if $b_1 \in Z, b_2 \in (B^* - Z)$; otherwise both $b_1, b_2 \in (B^* - Z)$ and b is a player other than b_1, b_2. We prove these assertions below.

Proposition 8 *The Solitary-Player Set Routine finds the solitary set Z' in $(\mathcal{D}', A')$.*

Proof. Suppose Z' is not equal to Z^0. Then, by Lemma 14, Z' is contained in but not equal to Z^0. Let i be any player in $Z^0 - Z'$ and (B_i^*, S_i^*) be the μ^i-market in $\mathcal{D}$. We will show that (Claim 1) $\mathcal{D}' - \mathcal{D} = b_1 b_2$ where $b_1, b_2 \in B_i^*$, (Claim 2) $\mu = \mu'$ and $Z' = Z - i$.

Note i is nonsolitary at μ' in $(\mathcal{D}', A')$ (since i is in $Z^0 - Z'$), so there is a μ'-cycle $C = i_0, i_1, ..., i_n, j_1, ..., j_n$ in $(\mathcal{D}', A')$ from $i_0 = i$. Since $\mathcal{D}' - \mathcal{D}$ is at most a singleton, $i_0 j_1$ or $i_0 i_n$ is in $\mathcal{D}$. Then j_1 or i_n is in S_i^* (since $i \in B_i^*$). Say $j_1 \in S_i^*$. Let ν be an active-minimum matching in $\mathcal{D}'$ that leaves j_1 unmatched.

Claim 1 is true, because otherwise the demand set of every B_i^*-player except possibly one (say player k) in $\mathcal{D}'$ would be in S_i^*, implying $(B_i^* - k, S_i^*)$ is a bipartite submarket in $(\mathcal{D}', A')$ (since S_i^* is matchable into $B_i^* - k$ in $\mathcal{D}$ and so in $\mathcal{D}'$), and contradicting the fact that ν is active-minimum by Lemma 6(i) in $\mathcal{D}'$.

We prove Claim 2 in two steps :

Step (i) μ is active-minimum in $(\mathcal{D}', A')$: Otherwise, since μ is active-minimum in $(\mathcal{D}, A)$ and $b_1 b_2$ is the only demand in $\mathcal{D}' - \mathcal{D}$, any matching that is active-minimum in $(\mathcal{D}', A')$ would necessarily contain $b_1 b_2$. But consider the active-minimum matching ν constructed above and let $\nu_{S_i^*}, \mu_{S_i^*}$ be the restriction of ν, μ respectively to the pairs that have a player in S_i^*. Note that the matching $(\nu - (\nu_{S_i^*} \cup b_1 b_2)) \cup \mu_{S_i^*}$ is active-minimum (because $\nu_{S_i^*} \cup b_1 b_2$ and $\mu_{S_i^*}$ have equal cardinality and contain an equal number of active players) but does not contain $b_1 b_2$. Contradiction.

Step (ii) $\mu = \mu'$ and $Z' = Z - i$: $A' = A$ since $\mathcal{D} \neq \mathcal{D}'$. Then, any μ-cycle in $\mathcal{D}$ is also a μ-cycle in $\mathcal{D}'$ because $\mathcal{D} \subset \mathcal{D}'$. Therefore, Z contains the solitary set at μ in $(\mathcal{D}', A')$. By Lemma 4, i is nonsolitary at μ in $(\mathcal{D}', A')$ since i is nonsolitary at μ' in $(\mathcal{D}', A')$. Therefore there is a μ-cycle C in $\mathcal{D}'$ from i. Note that $b_1 b_2 \in C$ (otherwise C is in $\mathcal{D}$ and so $i \notin Z$). Take any $i' \in Z - i$ and let C' be any μ-sequence from i' in $\mathcal{D}'$. Then, $b_1 b_2 \notin C'$ (since $C \cap C' = \varnothing$ by the fact that μ is active-minimum in $(\mathcal{D}', A')$) and so C' is in $\mathcal{D}$. Then, C' is cycle-free (otherwise $\mu^{i'}$-market is not bipartite by Lemma 3 in $(\mathcal{D}, A)$ and so $i' \notin Z$ by Lemma 5). Then, $Z - i$ is the solitary set at μ in $(\mathcal{D}', A')$ and by Corollary 3 μ is solitary-minimum in $(\mathcal{D}', A')$. So, $\mu' = \mu$ (recall SMMR) and $Z' = Z - i$.

$$\text{Case 1} : b_1 \in Z, b_2 \in (B^* - Z),$$

$$\text{Case 2} : b_1, b_2 \in (B^* - Z).$$

Note that, in Case 2, both b_1, b_2 are matched in μ and there exists a player

$$b_3 \in Z$$

such that b_1 and b_2 are both μ'-reachable in $\mathcal{D}'$ from only b_3 in Z. Now we are able to state the two cases:

Case 1) If i is b_1 or b_2, say b_1. Then, $Z' = Z - b_1$.

Case 2) Otherwise, say i is b_3. Then, $Z' = Z - b_3$. ■

Thus the Seller-Market Algorithm finds a solitary-minimum matching μ' and the set of solitary players $Z(\mu')$ in $(\mathcal{D}', A')$ given a solitary-minimum matching μ and the set of solitary players $Z(\mu)$ in $(\mathcal{D}, A)$.

Remark 11 *It remains to specify a solitary-minimum matching μ'' and the solitary-player set $Z(\mu'')$ in $(\mathcal{D}'', A'')$. This is routine : Let d be the Seller-Market preserving direction selected by the Direction Procedure at u, ν^{new} be any matching that matches $S^{*\prime}$ into $B^{*\prime}$ in $\mathcal{D}^{+}_{B^{*\prime}}(d)$ and $\nu^{old} = \{ij \in \mu' | i \in B^{*\prime}\}$. Define*

$$\mu'' = (\mu' \cup \nu^{new}) - \nu^{old} \text{ and } Z(\mu'') = B^{*\prime} - \nu^{new}(S^{*\prime}).$$

In particular, $(\mathcal{D}', A'), (\mathcal{D}'', A'')$ have the same Seller-Market and $|Z(\mu'')| = |Z(\mu')|$.

This completes the description of how solitary-minimum matchings and solitary player sets - therefore the Seller-Markets - can be generated recursively along the Procedure Path.

D.1 Proofs of Lemma 2 and Theorem 4

Lemma 15 *(i) $Z' \subset Z$. (ii) If $Z' = Z$ then $(B^*, S^*) \subset (B^{*\prime}, S^{*\prime})$. (iii) If $Z' = Z$ and $S^{*\prime} = S^*$ then $(B^{*\prime}, S^{*\prime}) = (B^*, S^*)$.*

Proof. *(i)* $Z' \subset Z$ by Lemma 14.

(ii) Suppose $Z' = Z$ and $(B^*, S^*) \subsetneq (B^{*\prime}, S^{*\prime})$. Let $B = B^* - B^{*\prime}$ and $S = S^* - S^{*\prime}$.

The demand set of every $B^{*\prime}$-player in $\mathcal{D}$ is in $S^{*\prime}$ since $\mathcal{D} \subset \mathcal{D}'$. The set B cannot be empty, because otherwise the demand set of every B^*-player in $\mathcal{D}$ is in $S^{*\prime}$, implying $S^* \subset S^{*\prime}$ and contradicting with $(B^*, S^*) \subsetneq (B^{*\prime}, S^{*\prime})$.

No player in $B^* - B$ demands an S-player in $\mathcal{D}'$ and so in $\mathcal{D}$ since $\mathcal{D} \subset \mathcal{D}'$. Therefore, μ matches S into B since $\mu(S^*) \subset B^*$ and in particular $|S| \leq |B|$. If $|S| = |B|$, then μ matches S to B and $S^* - S$ into $B^* - B$. Then, $Z \subset B^* - B$ and so any μ-market from any player in Z is in $(B^* - B, S^* - S)$ since the demand set of each player in $B^* - B$ in $\mathcal{D}$ is in $S^* - S$ and $\mu(S^* - S) \subset B^* - B$. By Theorem 5 B is empty. Contradiction. Thus, $|S| < |B|$.

Using $Z' = Z$, it must be that $|B^{*\prime}| - |S^{*\prime}| = |B^*| - |S^*|$. Then, $|S^{*\prime} - S^*| < |B^{*\prime} - B^*|$ since $|S| < |B|$. The demand set of each player in $B^{*\prime} - B^*$ in $\mathcal{D}'$ is in $S^{*\prime}$. By using $\mathcal{D} \subset \mathcal{D}'$, the demand set of each player in $B^{*\prime} - B^*$ in $\mathcal{D}$ is in $S^{*\prime}$. Then by using the fact that μ matches S^* into B^*, there is a player $i \in B^{*\prime} - B^*$ unmatched at μ since $|S^{*\prime} - S^*| < |B^{*\prime} - B^*|$. Player i is in A since $A' \subset A$. By Theorem 5 player i is in a unitary seller-market in $(\mathcal{D}', A')$ and then in a seller-market in $(\mathcal{D}, A)$ by Lemma 13. Therefore $i \in Z$ by Lemma 6(ii). By Theorem 5, $i \in B^*$. Contradiction.

(iii) If $Z' = Z$ and $S^{*\prime} = S^*$, then $|B^*| - |S^*| = |Z| = |Z'| = |B^{*\prime}| - |S^{*\prime}|$. Then, $B^* = B^{*\prime}$ since $|S^*| = |S^{*\prime}|$ and $B^* \subset B^{*\prime}$. ■

PROOF OF LEMMA 2 : By Corollary 1 and Lemma 15*(i)*, the Seller-Market excess in $(\mathcal{D}', A')$ is not greater than the Seller-Market excess in $(\mathcal{D}, A)$. By Remark 11, the same is true for $(\mathcal{D}'', A''), (\mathcal{D}', A')$. By succession, $a_t = |B_t| - |S_t| \geq |B_{t+1}| - |S_{t+1}| = a_{t+1}$ on the Procedure path. This gives the first assertion in Lemma 2. The remaining two assertions follow, similarly, from Corollary 1, Lemma 15*(ii)* and *(iii)* respectively, and Remark 11. ■

PROOF OF THEOREM 4 : Clearly (a_t) is bounded below and (b_t) bounded above. Therefore Theorem 4 would fail to hold only if there is a T such that $a_t = a_{t+1}$ and $b_t = b_{t+1}$ for all $t \geq T$. In that case, the Seller-Market remains unaltered while only the direction changes for all $t \geq T$. But by linearity there are only a finite number of directions that can be encountered for all $t \geq T$. Therefore it must be that $d^\tau = d^{\tau'}$ at two distinct steps $\tau < \tau'$. However, this is impossible because then Step τ need not have stopped at $u^{\tau+1}$. ■

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