Nonlinear quantum mechanics allows better cloning without signalling

Z. Gedik* and B. Çakmak

Faculty of Engineering and Natural Sciences,

Sabanci University, 34956 Tuzla, Istanbul, Turkey

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Abstract

We present a nonlinear generalization of quantum mechanics where one can copy arbitrary pure states with fidelities higher than those of the quantum cloners. Weinberg's nonlinear quantum mechanics has been criticized for its possible implications for superluminal communication. We demonstrate that universal symmetric cloners can make better copies than standard quantum ones and yet they do not violate the no-signalling principle. If linearity constraint is added, we show that for a given fidelity there is a unique cloner which can be written down explicitly.

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INTRODUCTION

Special relativity forbids superluminal influences. No-signalling (NS) principle is necessary for consistency of the theory of relativity and quantum mechanics. Any generalization of quantum theory, e.g., introducing nonlinear time evolution, must take the NS principle into account. It has been claimed that modification of the quantum theory by introducing nonlinear time evolution for pure states [1, 2], might lead to superluminal communication [3–6]. In this work, we show that universal and symmetric 1-to-M cloning of qubits, can be performed without violating the NS principle.

Impossibility of faster than light communication based on quantum correlations [7] led to the discovery of the no-cloning theorem [8, 9]. However, the NS principle leaves room for an approximate cloning. Imperfect or approximate 1-to-2 optimal quantum cloners have been shown to exist [10, 11] and the results have been generalized to 1-to-M cloning [12]. An expression for the maximum fidelity of N-to-M cloning of qudits has been found [13], and corresponding cloners have been obtained [14, 15] . However, to the best of our knowledge, there is no general expression for cloners with arbitrary fidelity. In this work, we obtain the universal symmetric quantum cloners for any given fidelity.

Gisin has analyzed universal symmetric 1-to-2 cloning under the NS principle, including nonlinear machines, and has shown that the optimal value is the same as that of the optimal quantum cloner [16]. Bruß et al. have pointed out that the set of local quantum-mechanical maps is just a subset of the local maps that do not allow superluminal communication [17]. In this work, we generalize Gisin's result to 1-to-M cloning. We find, surprisingly, that starting from the 1-to-3 cloning, the NS principle leads to larger fidelity values than the quantum ones. The formalism allows us to find out all possible cloners for a given value of fidelity, including the quantum one, if such a cloner exists. Furthermore, we construct the prime cloners which have the property that the fidelity of a 1-to-MN cloner can be obtained by the successive use of 1-to-M and 1-to-N cloners. For a given number of copies, we show that the quantum prime cloner is unique whereas there are infinitely many nonlinear prime cloners.

The article is organized as follows. We first introduce the pseudo-spin formalism for universal symmetric cloning. We examine the implications of the NS condition on the transformation, and hence explicitly obtain all possible values of fidelity. We then introduce the linearity condition to find the quantum cloners along with what we call *prime cloners.* We conclude with a discussion on possible applications of our results and implications for generalizing quantum mechanics by introducing nonlinear time evolution laws for pure states.

PSEUDO-SPIN FORMALISM

Symmetry of the output state, namely invariance of the wave function under the exchange operation, reduces the dimension of the Hilbert space from 2^M to $M+1$. Pseudo-spin formalism utilizes this dimensional reduction. Let $|\hat{n}\rangle$ be the state vector of the qubit to be cloned. In the so-called pseudo-spin representation, we treat the qubit as a spin-1/2 object, and thus $|\hat{n}\rangle$ corresponds to a spin-up state in the $\hat{n}-$direction. Then, symmetric $M-$qubit states can simply be represented by the total spin states with $j = M/2$. Therefore, we can use the states

$$|\hat{n}; jm\rangle = \begin{pmatrix} 2j \\ j+m \end{pmatrix}^{-1/2} \mathcal{P}\{\underbrace{|\hat{n}\rangle \otimes ... \otimes |\hat{n}\rangle}_{j+m} \otimes \underbrace{|-\hat{n}\rangle \otimes ... \otimes |-\hat{n}\rangle}_{j-m}\} \tag{1}$$

as the basis elements in the $M-$qubit symmetric space. Here, $\mathcal{P}$ denotes all possible permutations of the product state in the parenthesis and $m = -j, -j+1, ..., j-1, j$. Pseudo-spin formulation allows us to solve the problem by using the techniques of rotations in quantum mechanics.

Any quantum operation performed on qubits can be modeled as a unitary operation acting on the qubits plus an ancillary system. In the case of cloning, this system is called the cloning machine. We shall assume that the final states of the machine $|R_{jm}(\hat{n})\rangle$, are orthonormal, i.e., $\langle R_{jm}(\hat{n})|R_{jm'}(\hat{n})\rangle = \delta_{mm'}$. Orthonormality, which is guaranteed by a properly chosen ancillary, is the only constraint on the final states of the machine. After tracing out the states of the machine, we can formulate the problem in terms of the original state and its copies. Consequently, we do not need any additional assumptions or constraints (e.g., such as the relation of $|R_{jm}(\hat{n})\rangle$ and $|R_{jm}(-\hat{n})\rangle$ considered in [12] and [14]).

In the most general sense, the transformation for universal and symmetric pure state cloning is given by

$$|\hat{n}\rangle \otimes \underbrace{|0\rangle \otimes ... \otimes |0\rangle}_{M-1} \otimes |R\rangle \rightarrow \sum_{m=-j}^{j} a_{jm} |\hat{n}; jm\rangle \otimes |R_{jm}(\hat{n})\rangle \tag{2}$$

where $|0\rangle$ and $|R\rangle$ are blank copy and initial machine states, respectively. The normalization of the output state implies that $\sum_m p_{jm} = 1$, where $p_{jm} = |a_{jm}|^2$. Even though it is trace preserving, the mapping is not necessarily linear. Therefore, it includes possible nonlinear generalizations of quantum mechanics. We shall first introduce the NS principle and examine its consequences. Linearity, or in other words, operations in the domain of quantum mechanics, is an additional constraint which we shall impose later.

We can write the cloning transformation (2) in terms of the original state and the copies by tracing out the machine degrees of freedom. The reduced transformation becomes

$$T_j\left(|\hat{n}\rangle\langle\hat{n}|\right) = \sum_{m=-j}^{j} p_{jm}|\hat{n};jm\rangle\langle\hat{n};jm|. \tag{3}$$

We see that, due to the orthonormality of the final states of the machine, the output state of the cloning transformation is described by a diagonal density matrix. We assume that probabilities for possible final states p_{jm} are independent of $|\hat{n}\rangle$, and hence, the transformation described by Eq. (3) is universal in the sense that

$$T_j\left(U|\hat{n}\rangle\langle\hat{n}|U^\dagger\right) = U^{\otimes M} T_j\left(|\hat{n}\rangle\langle\hat{n}|\right)\left(U^\dagger\right)^{\otimes M}, \tag{4}$$

where U is any unitary transformation acting on a qubit.

Fidelity, the measure of the quality of cloning, is defined as the projection of the final single qubit state (obtained by tracing out the other $M-1$ qubits) onto the original state. Therefore, $(j-m)-$combinations of $2j-1$ elements of the sum in Eq. (3) contribute to the fidelity expression, and the resulting value is given by

$$F_j = \sum_{m=-j}^{j} \begin{pmatrix} 2j-1 \\ j-m \end{pmatrix} \begin{pmatrix} 2j \\ j-m \end{pmatrix}^{-1} p_{jm} = \frac{1}{2}\left(1 + \frac{1}{j}\sum_{m=-j}^{j} m p_{jm}\right). \tag{5}$$

Fidelity is a linear function of the expectation value of the $z-$component of the pseudo-angular momentum. If perfect cloning were possible, we would have $p_{jm} = \delta_{mj}$, which results in $F_j = 1$. In the next section, we shall evaluate upper and lower limits for F_j when the NS principle is taken as a constraint.

NO-SIGNALLING CONSTRAINT

The impossibility of superluminal communication implies that transforms of indistinguishable mixtures are also indistinguishable. For example, since $\left(|\hat{n}\rangle\langle\hat{n}| + |-\hat{n}\rangle\langle-\hat{n}|\right)/2$

is equivalent to the identity operator for any $\hat{n}$, all its images, i.e., M clones, should be invariant under changes in $\hat{n}$, too. Since, $|-\hat{n}; jm\rangle = |\hat{n}; j, -m\rangle$, the indistinguishability requirement states that

$$T_j\left(|\hat{n}\rangle\langle\hat{n}|\right) + T_j\left(|-\hat{n}\rangle\langle-\hat{n}|\right) = \sum_{m=-j}^{j}\left(p_{jm} + p_{j,-m}\right)|\hat{n}; jm\rangle\langle\hat{n}; jm| \tag{6}$$

is rotationally invariant in the pseudo-spin space, and thus the coefficients of expansion should be independent of m, i.e.,

$$p_{jm} + p_{j,-m} = \frac{2}{2j+1}. \tag{7}$$

Equation (7) is the fundamental relation to be satisfied by any universal symmetric cloner. The probabilities p_{jm} with negative index m are determined by their positive partners, which allows us to write the fidelity (5) as

$$F_j^{\mathrm{NS}} = \frac{1}{2} - \frac{1}{j(2j+1)}\sum_{m>0}^{j} m + \frac{1}{j}\sum_{m>0}^{j} m p_{jm}, \tag{8}$$

where $0 \le p_{jm} \le 2/(2j+1)$. The largest fidelity is given by the largest p_{jm}, i.e.,

$$\left(F_j^{\mathrm{NS}}\right)_{\max} = \frac{1}{2}\left(\frac{3\lfloor j\rfloor + 2}{2\lfloor j\rfloor + 1}\right) \tag{9}$$

where $\lfloor j\rfloor$ the greatest integer less than j. Since we can apply the flip operator to the output qubits, the presence of a maximum fidelity implies that there is also a minimum value given by $\left(F_j^{\mathrm{NS}}\right)_{\min} = 1 - \left(F_j^{\mathrm{NS}}\right)_{\max}$. These are the upper and the lower limits of the fidelity under the NS condition.

We observe that $\left(F_j^{\mathrm{NS}}\right)_{\max} > \left(F_j^{\mathrm{Q}}\right)_{\max}$ for $j \ge 3/2$ where $\left(F_j^{\mathrm{Q}}\right)_{\max} = (4j+1)/6j$ is nothing but the well-known optimal fidelity value for 1-to-$2j$ quantum cloning [12]. For 1-to-2 cloning, Gisin pointed out that quantum mechanics is right at the border line of contradicting the theory of relativity, but it does not cross it [16]. Starting from 1-to-3 cloning, we see, surprisingly, that quantum mechanics does not saturate the NS upper value. We can say it is the nonlinearity of time evolution which allows us to obtain better copies (while not violating the NS condition) than the quantum ones. However, we should be very careful in nonlinear generalizations of the quantum mechanics since this construction must be done in a consistent way [18].

For a given fidelity, the cloning map (3) is not unique because there are in general infinitely many maps satisfying Eq. (8). We can express the density of cloners (DOC) function in the form

$$\rho_j(\lambda) \propto \prod_{m>0} \int_0^{\frac{2}{2j+1}} dp_{jm} \delta \left(\lambda - \left(F_j^{\mathrm{NS}}\right)_{\min} - \frac{1}{j} \sum_{m>0}^{j} m p_{jm} \right). \tag{10}$$

DOC is the intersection of a hyper-cube (given by $[0, 2/(2j+1)]^{\lfloor j+1/2 \rfloor}$)with a hyper-plane (given by $\sum_{m>0}^{j} m p_{jm}$ = constant) and it changes as $\left[\left(F_j^{\mathrm{NS}}\right)_{\max} - \lambda\right]^{\lfloor j-1/2 \rfloor}$ when λ approaches $\left(F_j^{\mathrm{NS}}\right)_{\max}$. A similar expression holds as λ tends to $\left(F_j^{\mathrm{NS}}\right)_{\min}$. In the next section, we shall see that for a given fidelity there is only one linear machine since the fidelity determines all p_{jm} coefficients uniquely. Therefore, linear cloners constitute a set of measure zero embedded in nonlinear cloners.

LINEAR CLONING

Let $|\hat{n}\rangle$ and $|\hat{n}'\rangle$ be two arbitrary qubit states. The condition of linearity is

$$T_j\left(r|\hat{n}\rangle\langle\hat{n}| + (1-r)|\hat{n}'\rangle\langle\hat{n}'|\right) = rT_j\left(|\hat{n}\rangle\langle\hat{n}|\right) + (1-r)T_j\left(|\hat{n}'\rangle\langle\hat{n}'|\right), \tag{11}$$

where $0 \leq r \leq 1$. We note that the NS principle is a special case of the linearity condition with $|\hat{n}'\rangle = |-\hat{n}\rangle$ and $r = 1/2$. In other words, quantum mechanics satisfied the NS principle due to its linearity. The density matrix $r|\hat{n}\rangle\langle\hat{n}| + (1-r)|\hat{n}'\rangle\langle\hat{n}'|$ is diagonal for some $|\hat{m}\rangle$, and hence, it can be written as $s|\hat{m}\rangle\langle\hat{m}| + (1-s)|-\hat{m}\rangle\langle-\hat{m}|$ with $0 \leq s \leq 1$. We can choose our coordinate axes so that $\hat{m} = \hat{z}$, and $r|\hat{n}\rangle\langle\hat{n}| + (1-r)|\hat{n}'\rangle\langle\hat{n}'|$ becomes

$$\frac{1}{2}\left(1 + \frac{\sin(\theta+\theta')}{\sin\theta + \sin\theta'}\right)|\hat{z}\rangle\langle\hat{z}| + \frac{1}{2}\left(1 - \frac{\sin(\theta+\theta')}{\sin\theta + \sin\theta'}\right)|-\hat{z}\rangle\langle-\hat{z}|, \tag{12}$$

where θ (θ') is the angle between $\hat{z}$ and $\hat{n}$ ($\hat{n}'$) and we observe that $r = \sin\theta'/(\sin\theta + \sin\theta')$. Therefore, the linearity condition takes the form

$$\sum_{m=-j}^{j} \left(c_+ p_{jm} + c_- p_{j,-m}\right)|\hat{z}; jm\rangle\langle\hat{z}; jm| = \sum_{m=-j}^{j} p_{jm}\left(\sin\theta'|\hat{n}; jm\rangle\langle\hat{n}; jm| + \sin\theta|\hat{n}'; jm\rangle\langle\hat{n}'; jm|\right), \tag{13}$$

where $2c_\pm = \sin\theta + \sin\theta' \pm \sin(\theta+\theta')$. We note that $|\langle\hat{z}; jm|\hat{n}; jm'\rangle| = |d_{mm'}^{(j)}(\theta)|$, where $d_{mm'}^{(j)}(\theta)$ are the elements of the reduced Wigner rotation matrix. Similarly, we have $|\langle\hat{z}; jm|\hat{n}'; jm'\rangle| = |d_{mm'}^{(j)}(\theta')|$. Therefore, Eq. (13) can be written as

$$c_+ p_{jm} + c_- p_{j,-m} = \sum_{m'=-j}^{j} \left(|d_{mm'}^{(j)}(\theta)|^2 \sin\theta' + |d_{mm'}^{(j)}(\theta')|^2 \sin\theta\right) p_{jm'}. \tag{14}$$

Finally, using the no-signalling constraint Eq. (7), the linearity condition can be written as an eigenvalue equation

$$\sum_{m'=-j}^{j} \left(|d^{(j)}_{mm'}(\theta)|^2 \sin\theta' + |d^{(j)}_{mm'}(\theta')|^2 \sin\theta - \frac{2c_-}{2j+1} \right) p_{jm'} = \sin(\theta+\theta') p_{jm}. \tag{15}$$

Since $|d^{(j)}_{mm'}(\theta)| = |d^{(j)}_{-m,-m'}(\theta)|$ when p_{jm} is a solution of Eq. (15), $p_{j,-m}$ is also a solution with the same eigenvalue. In other words, eigenvectors are (or, in case of degeneracy, can be chosen to be) either symmetric (even) or anti-symmetric (odd) in m.

Let us assume that p_{jm} can be written as an analytic function $f(m)$ of m. Since $|d^{(j)}_{mm'}(\theta)| = |d^{(j)}_{m'm}(\theta)|$, we have

$$\sum_{m'=-j}^{j} |d^{(j)}_{mm'}(\theta)|^2 f(m') = \sum_{m'=-j}^{j} \langle \hat{n}; jm' | f(J_z) | \hat{z}; jm \rangle \langle \hat{z}; jm | \hat{n}; jm' \rangle = f(m\cos\theta). \tag{16}$$

We see that $\sin(\theta+\theta')$ is a two-fold degenerate eigenvalue, and $p_{jm} = 1/(2j+1)$ is the only even solution whereas $p_{jm} = \pm m/j(2j+1)$ are the only possible odd solutions. The positivity of p_{jm}'s allows us to write two linearly independent solutions as $(j+m)/j(2j+1)$ and $(j-m)/j(2j+1)$. Hence, the most general solution becomes

$$p_{jm}(t) = t\frac{j+m}{j(2j+1)} + (1-t)\frac{j-m}{j(2j+1)}, \tag{17}$$

where $0 \le t \le 1$. The corresponding fidelity is given by

$$F^{\mathrm{Q}}_j(t) = \frac{2j-1+2(j+1)t}{6j}, \tag{18}$$

which has its maximum value at $(4j+1)/6j$ when $t=1$. This is the well known optimal quantum cloner fidelity [12]. In this case, p_{jm} coefficients become identical to optimal quantum machine coefficients. We observe that $p_{j,-j}$ vanishes only for the optimal cloner. Therefore, if we exclude the worst cloning case from the set of possible output states by assuming that $p_{j,-j} = 0$ (as it has been done in Refs. [12] and [14]), the optimal cloner is the only universal quantum cloning machine.

Equations (17) and (18) can be used to find the quantum cloner for a given fidelity F^{Q}_j in the allowed region $\left[1 - \left(F^{\mathrm{Q}}_j\right)_{\max}, \left(F^{\mathrm{Q}}_j\right)_{\max}\right]$. They can also be used to construct a quantum cloner satisfying some specific property. For example, let us consider the cloners where successive use of 1-to-M and 1-to-N cloners gives the same fidelity as a single 1-to-MN

cloner. We call such a cloner as *prime cloner* since it is enough to have 1-to-prime number cloners to construct any 1-to-M cloner. The fidelities of prime cloners should satisfy

$$F^{\mathrm{P}}_{M/2}F^{\mathrm{P}}_{N/2}+\left(1-F^{\mathrm{P}}_{M/2}\right)\left(1-F^{\mathrm{P}}_{N/2}\right)=F^{\mathrm{P}}_{MN/2}. \tag{19}$$

We find the coefficients of expansion as

$$p_{jm}=\frac{1}{2j+1}\left(1+\frac{3m}{2j(j+1)}\right). \tag{20}$$

We note that fidelity $F^{\mathrm{P}}_{j}=(2j+1)/4j$ tends to $F^{\mathrm{P}}_{\infty}=1/2$ as the number of copies goes to infinity. Therefore, fidelity values 3/4, 2/3, 1/2, 1/3 and 1/4 correspond to asymptotic fidelity values $\left(F^{\mathrm{NS}}_{\infty}\right)_{\max}$, $\left(F^{\mathrm{Q}}_{\infty}\right)_{\max}$, F^{P}_{∞}, $\left(F^{\mathrm{Q}}_{\infty}\right)_{\min}$, and $\left(F^{\mathrm{NS}}_{\infty}\right)_{\min}$, respectively. We can construct nonlinear prime cloners by substituting Eq. (8) (rather than Eq. (18)) into Eq. (19). However, in this case the cloner is not unique but the corresponding fidelity for a given number of copies is same as the quantum one.

CONCLUSION

We presented a method for constructing universal symmetric 1-to-M qubit cloners, including non-linear and linear evolutions, for any given fidelity value in an allowed region. In particular, we systematically derived the properties of universal symmetric quantum cloning machines instead of guessing or postulating them first and proving them afterwards. Our cloner construction method can find applications in quantum key distribution since cloning has an important role in eavesdropping strategies on a quantum channel [19]. We observe that the NS principle leads to a wider fidelity range beyond what quantum mechanics allows. This is consistent with the earlier observation that non-local *superquantum* correlations that preserve relativistic causality, can violate Bell-like inequalities more strongly than any quantum correlations [20]. Existence of nonlinear cloners not violating the NS principle is a direct evidence for possibility of generalizing quantum mechanics by introducing nonlinear time evolution laws for pure states. Furthermore, such machines can attain larger fidelities than those of linear quantum cloners.

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* gedik@sabanciuniv.edu

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