

# Eigenvalue Formulation of Quantum Mechanics Near Closed Timelike Curves

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**Abstract.** Einstein's field equations of gravitation are known to admit closed timelike curve (CTC) solutions. Deutsch approached the problem from the quantum information point of view and proposed a self-consistency condition. In this work, the Deutsch equation is formulated as an eigenvalue problem. The disappearance of entanglement between two qubits in an Einstein-Podolsky-Rosen (EPR) state near a CTC is demonstrated. The method is utilized to analyze the discontinuous evolution of two chronology respecting (CR) qubits near a CTC.

## 1 Introduction

Gödel pointed out the possibility of closed timelike curve (CTC) solutions for the Einstein's field equations of gravitation [1]. Several models and calculations gave evidence for existence of CTCs [2,3,4,5,6,7,8,9,10]. Because CTCs allow a time traveling particle to go back in time and interact with its own past, one can end up with paradoxes. Deutsch analyzed the problem from the quantum information point of view and proposed a self-consistency condition involving the density matrices of the chronology respecting (CR) and CTC qubits undergoing a unitary interaction [11]. Deutsch showed that the self-consistency equation implies a non-unitary and nonlinear evolution for the CR components. Ralph introduced a toy model to show that unitarity can be recovered [12].

Consequences of existence of CTCs for quantum computation have been examined. Brun argued that using CTCs, composite numbers can be factorized efficiently with the help of a classical computer [13]. Bacon demonstrated that nonlinearity of Deutsch-like evolution can be used to efficiently solve computational problems by reducing NP-hard problems to P [14]. It has also been claimed that an observer with access to CTCs can perfectly distinguish nonorthogonal quantum states [15]. However, there are both counter and supporting arguments for this claim [16,17].

An alternative formulation of the quantum mechanics near CTCs is via post-selection [18,19]. Lloyd *et al.*, developed a model, which is phys-

ically inequivalent to Deutsch's, based on combining quantum teleportation with post-selection. Unlike Deutsch's mixed state generating CTCs, post-selected CTCs always send pure states to pure states, and hence, they do not create entropy.

Pati *et al.*, showed that in contrast to ordinary quantum mechanics where any mixed state can be purified by bringing an ancillary system, the states of quantum systems traveling in CTCs cannot be purified [20]. In other words, the CTC system interacting with the CR system cannot be viewed as a part of a larger CTC system in a pure entangled state in an enlarged Hilbert space. A practical consequence of this result is that in general it is not possible to simulate CTC qubits satisfying the Deutsch equation in laboratory, because arbitrary CTC states cannot be constructed in a consistent way.

In this study, after introducing an eigenvalue formulation of the Deutsch equation, the method is used to demonstrate that the entanglement between two qubits in Einstein-Podolsky-Rosen (EPR) state disappears when one of the two qubits interacts with a CTC qubit. The same formulation is utilized to examine the discontinuous evolution of CR qubit states, first pointed out by DeJonghe *et al.* [21].

## 2 The Deutsch Equation

Deutsch's model involves a unitary interaction  $U$  of a CR system with another system that traverses a CTC. The equation is a result of the kinematic self consistency condition, which implies that the density matrix of the CTC system after it interacts with the CR system is the same as that of the CTC system before the interaction. For each state of the CR register described by the density matrix  $\rho_{CR}$ , CTC register is postulated to find a fixed point such that

$$Tr_{CR} \left( U \rho_{CR} \otimes \rho_{CTC} U^\dagger \right) = \rho_{CTC}, \quad (1)$$

where  $\otimes$  is tensor product and  $U^\dagger$  is the hermitian conjugate of  $U$ . Here, it is assumed that CR and CTC system are initially in a separable state. Deutsch showed that there is always a density matrix  $\rho_{CTC}$  which satisfies the above self-consistency condition by using the compactness of the space of density operators. In fact, there are in general many solutions, and one might need additional assumptions to decide which  $\rho_{CTC}$  to choose [11,14,21,22]. One of the immediate consequences of the equation is that the universe (i.e., the space of CR registers) may evolve from a pure state

to a mixed state. Nonlinear and noncontractive evolutions are among the other unusual effects [11,23,14].

Equation (1) implies that  $\rho_{CTC}$ , and hence, its von Neumann entropy

$$S(\rho_{CTC}) = -Tr(\rho_{CTC} \ln \rho_{CTC}) \quad (2)$$

remains the same after CR and CTC systems interact. Therefore, the entanglement of  $\rho_{CTC}$  with the rest of the the enlarged CTC Hilbert space would remain the same for any local interaction if it were possible to distill  $\rho_{CTC}$ , which runs counter to observation of Pati *et al.* [20]. However, for arbitrary local operations, the entropy does not remain constant but instead decreases. That is why, in spite of its simplicity, the demonstration of the Deutsch equation in laboratory is not a trivial task. One needs to find a two-particle scattering event, where the density matrix of one of the particles remains the same after the scattering.

### 3 Eigenvalue Formulation

Let CR and CTC registers be single qubits. For a pure state  $\rho_{CR} = |\psi\rangle\langle\psi|$ , without loss of generality  $\rho_{CR}$  can be assumed to be given by  $|0\rangle\langle 0|$  since  $\tilde{U} = U(V \otimes I)$ , where  $|\psi\rangle = V|0\rangle$ , is also unitary. For  $\rho_{CR} = |0\rangle\langle 0|$  and

$$\rho_{CTC} = \begin{pmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{pmatrix}, \quad (3)$$

inserting matrix elements of  $\rho_{CR}$  and  $\rho_{CTC}$ , and comparing the entries of its left and right hand sides, Eq. (1) can be written as  $M\mathbf{v} = \mathbf{v}$  where  $M$  is given in terms of the matrix elements  $u_{ij}$  of unitary operator  $\tilde{U}$ , and their complex conjugates  $u_{ij}^*$ , as

$$M = \begin{pmatrix} u_{11}u_{11}^* + u_{31}u_{31}^* & u_{11}u_{12}^* + u_{31}u_{32}^* & u_{12}u_{11}^* + u_{32}u_{31}^* & u_{12}u_{12}^* + u_{32}u_{32}^* \\ u_{11}u_{21}^* + u_{31}u_{41}^* & u_{11}u_{22}^* + u_{31}u_{42}^* & u_{12}u_{21}^* + u_{32}u_{41}^* & u_{12}u_{22}^* + u_{32}u_{42}^* \\ u_{21}u_{11}^* + u_{41}u_{31}^* & u_{11}u_{12}^* + u_{41}u_{32}^* & u_{22}u_{11}^* + u_{42}u_{31}^* & u_{22}u_{12}^* + u_{42}u_{32}^* \\ u_{21}u_{21}^* + u_{41}u_{41}^* & u_{21}u_{22}^* + u_{41}u_{42}^* & u_{22}u_{21}^* + u_{42}u_{41}^* & u_{22}u_{22}^* + u_{42}u_{42}^* \end{pmatrix} \quad (4)$$

and  $\mathbf{v} = (\rho_{11}, \rho_{12}, \rho_{21}, \rho_{22})^T$ . Here,  $\mathbf{v}$  is normalized so as to satisfy  $\rho_{11} + \rho_{22} = 1$ . Therefore, the Deutsch equation can be interpreted as a diagonalization problem  $M\mathbf{v} = \lambda\mathbf{v}$ , where one looks for  $\lambda = 1$  eigenvalues and vectors  $\mathbf{v}$  whose entries define a valid density matrix.

Although Deutsch gave a general proof for existence of a solution, it is worth rederiving the same result using the current formalism due to its

simplicity. Using the unitarity of  $\tilde{U}$ , it is seen that the matrix  $M$  is of the form

$$M = \begin{pmatrix} a & b & b^* & c \\ d & e & f^* & g \\ d^* & f & e^* & g^* \\ 1 - a & -b & -b^* & 1 - c \end{pmatrix}. \quad (5)$$

Adding the last row of the matrix  $M - \lambda I$  ( $I$  being the identity matrix) to its first row, one can immediately observe that  $1 - \lambda$  is always a factor of the characteristic equation  $\det(M - \lambda I)$ . Therefore, there is at least one solution. If  $\lambda = 1$  is two-fold degenerate, any convex linear combination,  $\alpha \mathbf{v}_1 + (1 - \alpha) \mathbf{v}_2$  with  $\alpha \in [0, 1]$ , of the corresponding eigenvectors  $\mathbf{v}_1$  and  $\mathbf{v}_2$  is also a solution. As shall be seen below, there can be cases where  $\mathbf{v}_2$  does not correspond a valid density matrix, while its convex linear combinations with  $\mathbf{v}_1$  are proper solutions.

A practical way to construct  $M$  is to write it as  $M = A_{11} \otimes A_{11}^* + A_{21} \otimes A_{21}^*$  where  $A_{11}$  and  $A_{21}$  are  $2 \times 2$  matrices in

$$\tilde{U} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \quad (6)$$

and  $A_{ij}^*$  denotes complex conjugation of  $A_{ij}$ .

Generalization to  $n$  CR qubits near a CTC qubit problem is straightforward. In this case,

$$M = \sum_{i=1}^{2^n} A_{i1} \otimes A_{i1}^*, \quad (7)$$

where  $A_{ij}$  are  $2 \times 2$  matrices making up the  $2^{n+1} \times 2^{n+1}$  unitary matrix  $\tilde{U}$  as in Eq. (6). Even though the eigenvalue formalism has been presented for pure a  $\rho_{CR}$ , it can easily be generalized to mixed states. In this case  $M$  matrix is slightly more complicated, and it contains all four entries of  $\rho_{CR}$ .

## 4 Disappearance of Entanglement Near a CTC

An interesting problem involving entanglement near a CTC is the behavior of an EPR pair. In order to illustrate Deutsch's model, Bennett *et al.*, considered half of a maximally entangled state, i.e., an EPR pair  $\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ , and put it into a CTC [16]. Using both the single and multiple universe pictures, the authors showed that the joint state at

any time after the interaction between CR and CTC systems is a product state. In the eigenvalue formulation introduced in Sec. IV, the total unitary interaction  $\tilde{U}$  involving three qubits (i.e., two CR qubits making up the EPR pair and one CTC qubit) can be written as  $\tilde{U} = (I_2 \otimes E) (CNOT \otimes I_2) (H \otimes I_4)$  where  $H$  and  $CNOT$  denote the Hadamard and controlled-NOT operations, respectively. The EPR state is obtained by acting the operator  $(CNOT \otimes I_2) (H \otimes I_4)$  on the initial CR two qubit product state  $|00\rangle$ , where  $I_2$  and  $I_4$  denote  $2 \times 2$  and  $4 \times 4$  unit matrices, respectively. Here,

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (8)$$

denotes the exchange operation, where a qubit emerges from the CTC and half of the EPR pair is put into the CTC.

Diagonalization of the corresponding  $M$  matrix gives the eigenvalues  $\lambda_1 = 1$ , and  $\lambda_2 = \lambda_3 = \lambda_4 = 0$ . Hence, the solution is unique, and the reduced density matrices for CR and CTC registers are  $I_4/4$  and  $I_2/2$ , respectively. Therefore, two CR qubits are not entangled anymore. Since the reduced density matrix of the CR qubits is found to be mixed after the interaction, this is an expected result. It is known that according to the monogamy property of entanglement, if two qubits are maximally entangled, they cannot be entangled with a third qubit [26].

## 5 Discontinuous Evolutions Near CTCs

DeJonghe *et al.*, demonstrated that Deutsch's equation can lead to discontinuities in the evolution of the CR systems [21]. The authors consider two CR qubits interacting with a CTC qubit via the unitary evolution

$$U = |000\rangle\langle 100| + |100\rangle\langle 000| + |010\rangle\langle 011| + |011\rangle\langle 010| \\ |101\rangle\langle 110| + |110\rangle\langle 101| + |001\rangle\langle 001| + |111\rangle\langle 111|. \quad (9)$$

It is also assumed that the density matrix of CR qubits before the interaction is of the form  $\rho_{CR} = \rho_1 \otimes \rho_2$ . For three distinct initial states  $(\rho_{CR}^A, \rho_{CR}^B, \rho_{CR}^C)$  which are infinitesimally close to each other, it has been shown that there is no choice of  $\rho_{CTC}$  which is continuous in the vicinity of  $\rho_{CR}^B$ . Since the CR and CTC density matrices are obtained from the same matrix by partial trace operations, the discontinuity in the CTC

state implies the behavior of the CR state. In the current formalism, starting again from CR state  $|00\rangle$ , the three pure states  $\rho_{CR}^A, \rho_{CR}^B, \rho_{CR}^C$  correspond to the following  $M$  matrices

$$M^A = \begin{pmatrix} 1 - \epsilon & 0 & 0 & \epsilon \\ 0 & 0 & \epsilon & 0 \\ 0 & \epsilon & 0 & 0 \\ \epsilon & 0 & 0 & 1 - \epsilon \end{pmatrix}, \quad (10)$$

$$M^B = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad (11)$$

$$M^C = \begin{pmatrix} 1 & 0 & 0 & \epsilon \\ 0 & \sqrt{\epsilon(1-\epsilon)} & 0 & 0 \\ 0 & 0 & \sqrt{\epsilon(1-\epsilon)} & 0 \\ 0 & 0 & 0 & 1 - \epsilon \end{pmatrix} \quad (12)$$

where  $\epsilon \in [0, 1]$ . Eigenvalues for three cases are given by  $\lambda_1^A = 1, \lambda_2^A = 1 - 2\epsilon, \lambda_3^A = -\epsilon, \lambda_4^A = \epsilon$  for initial state  $\rho_{CR}^A$ ,  $\lambda_1^B = \lambda_2^B = 1, \lambda_3^B = \lambda_4^B = 0$  for  $\rho_{CR}^B$ , and  $\lambda_1^C = 1, \lambda_2^C = 1 - \epsilon, \lambda_3^C = \lambda_4^C = \sqrt{\epsilon(1-\epsilon)}$  for  $\rho_{CR}^C$ . Any eigenvector with eigenvalue 1 is a solution of the Deutsch equation. Degeneracy of  $\lambda_1^B$  and  $\lambda_2^B$  indicate that for initial state  $\rho_{CR}^B$  there are infinitely many solutions which can be obtained by taking convex linear combinations of the two degenerate states. CTC state solutions, all of which are independent of  $\epsilon$ , for the three cases are given by

$$\rho_{CTC}^A = \begin{pmatrix} 1/2 & 0 \\ 0 & 1/2 \end{pmatrix}, \quad (13)$$

$$\rho_{CTC}^B = \begin{pmatrix} \beta & 0 \\ 0 & 1 - \beta \end{pmatrix}, \text{ for } \beta \in [0, 1], \quad (14)$$

$$\rho_{CTC}^C = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}. \quad (15)$$

Density matrices  $\rho_{CTC}^A$  and  $\rho_{CTC}^C$  can be obtained from  $\rho_{CTC}^B$  by choosing  $\beta = 1/2$  and  $\beta = 1$ , respectively. As  $\epsilon \rightarrow 0$ , all three initial states approach

each other, but  $\rho_{CTC}^A$  and  $\rho_{CTC}^C$ , being independent of  $\epsilon$ , are always different. Solving the Deutsch equation by a different method, DeJonghe *et al.* concluded that there is a discontinuity near  $\rho_{CTC}^B$  for the unitary interaction given above. Therefore, for finite  $\epsilon$  values, the eigenvalue formulation reproduces their result. However, as  $\epsilon \rightarrow 0$ , eigenvalues  $\lambda_2^A, \lambda_2^C \rightarrow 1$ , and hence the corresponding density matrices satisfy the Deutsch equation. At this limit, all three initial states lead to the same solution set, given by Eq. (14), for  $\rho_{CTC}$ . More explicitly, approximate solutions obtained by taking linear combinations of eigenvectors with eigenvalues  $\lambda_1$  and  $\lambda_2$  for the initial states  $\rho_{CR}^A$  and  $\rho_{CR}^C$  can be written as

$$\rho_{CTC}^A = \begin{pmatrix} 1/2 - \alpha & 0 \\ 0 & 1/2 + \alpha \end{pmatrix}, \text{ for } \alpha \in [-1/2, 1/2], \quad (16)$$

and

$$\rho_{CTC}^C = \begin{pmatrix} 1 - \gamma & 0 \\ 0 & \gamma \end{pmatrix}, \text{ for } \gamma \in [0, 1]. \quad (17)$$

For both  $\rho_{CR}^A$  and  $\rho_{CR}^C$ ,  $\lambda_2$  solution is an  $\epsilon$  independent vector  $\mathbf{v}_2 = (-1, 0, 0, 1)^T$  which gives a traceless  $2 \times 2$  matrix. Even though this solution cannot be associated with a density matrix, its linear combinations with  $\mathbf{v}_1$  (eigenvector with eigenvalue  $\lambda_1 = 1$ ), lead to proper vectors. Solution set for all three cases at  $\epsilon \rightarrow 0$  is the same. For  $\rho_{CR}^B$ , the expression in Eq. (14) is a solution for all  $\beta \in [0, 1]$ . However, in case of  $\rho_{CR}^A$  and  $\rho_{CR}^C$ , the only  $\epsilon$ -independent and exact solutions are  $\alpha = 0$  and  $\gamma = 0$ , respectively.

## 6 Conclusion

The consistency condition proposed by Deutsch to avoid paradoxes near a CTC has been transformed to an eigenvalue equation. The proposed approach is a systematic method that solves the Deutsch equation. Two problems, namely an EPR pair near a CTC and the discontinuous evolution of the CR-CTC system, have been reexamined. Even though the eigenvalue formulation has been demonstrated for initially pure CR states, it can easily be generalized to the mixed case.

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