# Isomorphism of spaces of analytic functions on $n$-circular domains

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**Abstract**

The space $A(D)$ of all analytic functions in a complete $n$-circular domain $D$ in $\mathbb{C}^n$, $n \geq 2$, is considered with a natural Fréchet topology. Some sufficient conditions for the isomorphism of such spaces are obtained in terms of certain subtle geometric characteristic of domains $D$. This investigation complements essentially the second author's result [8] on necessary geometric conditions of such isomorphisms.

## 1 Introduction

By $A(D)$ we denote the Fréchet space of all analytic functions in a domain $D \in \mathbb{C}^n$ with the natural topology of the uniform convergence on compact subsets of $D$. We study the isomorphic classification of the spaces $A(D)$ with $D$ from the class $\mathcal{R}^n$ of all *complete logarithmically convex $n$-circular* (Reinhardt) domains in $\mathbb{C}^n$, $n \geq 2$ (see also, [1, 7, 8, 10]). We represent the system of monomials $z^k := z_1^{k_1} \cdot \cdots \cdot z_n^{k_n}$, $k = (k_1, \ldots, k_n) \in \mathbb{Z}_+^n$, which forms an absolute basis in each space $A(D)$, $D \in \mathcal{R}^n$, as a sequence

$$e_i(z) := z^{k(i)}, \quad i \in \mathbb{N}, \tag{1}$$

so that $|k(i)| := k_1(i) + \ldots + k_n(i)$ does not decrease. The characteristic function of a domain $D \in \mathcal{R}^n$: $h_D(\theta) := \sup\left\{\sum_{\nu=1}^{n} \theta_\nu \ln|z_\nu| : \ z = (z_\nu) \in D\right\}$, defined on the simplex $\Sigma := \left\{\theta = (\theta_\nu) \in \mathbb{R}_+^n : \ \sum_{k=1}^{n} \theta_k = 1\right\}$, is convex (hence continuous) on the convex set $\pi(D) := \{\theta \in \Sigma : h_D(\theta) < \infty\}$. It turns out that invariant properties of spaces $A(D)$ depend essentially on the topological behavior of the set $\pi(D)$, for example, $A(D)$ is not isomorphic to $A(G)$ if $\pi(D)$ is relatively open in $\Sigma$ but $\pi(G) \neq \Sigma$ is closed. In what follows we restrict ourselves to the class $\mathcal{R}_o^n$ of domains $D$ for which $\pi(D)$ is relatively open in $\Sigma$, $\pi(D) \neq \Sigma$ (if $\pi(D) = \Sigma$, then $A(D) \simeq A(\mathbb{U}^n)$ [1, 7]). In order to investigate the isomorphic classification for this class it is convenient to introduce the following geometric characteristic of those domains:

$$g(\alpha) := g_D(\alpha) := \left(\frac{n!\ \mathrm{mes}\,\Sigma}{\mathrm{mes}\,\pi(D)}\right)^{1/n} \chi^{-1}(\alpha), \quad 0 < \alpha \leq 1, \tag{2}$$

where $\chi(t) := \dfrac{\mathrm{mes}\{\theta \in \pi(D) :\ h_D(\theta) \geq\ t\}}{\mathrm{mes}\ \pi(D)}$, $t \geq t_0 := \min\limits_{\theta \in \pi(D)} \{h_D(\theta)\}$ and mes is the Lebesgue measure on $\Sigma$.

Using this characteristic, the following necessary condition for the isomorphism of spaces from the class $\mathcal{A}_o^n := \{A(D) : D \in \mathcal{R}_o^n\}$ was obtained in [8].

**Proposition 1** *Given domains* $D, \widetilde{D} \in \mathcal{R}_o^n$ *and* $A(D) \simeq A\left(\widetilde{D}\right)$, *then*

$$\exists c : \ \frac{1}{c} g_D(c\alpha) \leq g_{\widetilde{D}}(\alpha) \leq c g_D\left(\frac{\alpha}{c}\right),\ 0 < \alpha \leq \frac{1}{c}.$$

As a corollary, it was proved in [8] that there is a *continuum of pairwise nonisomorphic spaces* in $\mathcal{A}_o^n$ . Here we represent, *in terms of the same characteristic* (2), some sufficient conditions for the isomorphism of those spaces. A distinction must be made between two types of domains from $\mathcal{A}_o^n$, described by one of the conditions:

$$\text{(a)}\ \ \mathrm{mes}(\Sigma \setminus \pi(D)) = 0; \quad \text{(b)}\ \ \mathrm{mes}(\Sigma \setminus \pi(D)) > 0. \tag{3}$$

It turns out that the spaces $A(D)$ and $A\left(\widetilde{D}\right)$ are not isomorphic for domains of different type (see, Proposition 5 and Remark 6).

**Theorem 2** *Suppose* $D, \widetilde{D} \in \mathcal{R}_0^n$, $g(\alpha) := g_D(\alpha)$, $\widetilde{g}(\alpha) := g_{\widetilde{D}}(\alpha)$ *and* $\sigma : [0, q] \to [0, 1]$, $0 < q < 1$, *is the continuous increasing function, which is continuously differentiable on* $(0, q]$ *and satisfies the differential equation*

$$\sigma'(\alpha) = \left(\frac{\widetilde{g}(\sigma(\alpha))}{g(\alpha)}\right)^n, \quad 0 < \alpha \leq q, \tag{4}$$

*with the initial condition* $\sigma(0) = 0$. *If there is a constant* $L > 0$ *such that*

$$\frac{1}{L} \leq \sigma'(\alpha) \leq L, \quad 0 < \alpha \leq q, \tag{5}$$

*and both domains are of the same type (3) , then* $A(D)$ *is isomorphic to* $A\left(\widetilde{D}\right)$; *moreover, there is an isomorphism* $T : A(D) \to A\left(\widetilde{D}\right)$ *such that* $Te_i = t_i\ e_{\rho(i)}$, $i \in \mathbb{N}$, *where* $e_i$ *is the monomial basis (1),* $t_i$ *a scalar sequence and* $\rho : \mathbb{N} \to \mathbb{N}$ *a bijection.*

This theorem will be an immediate consequence of some more general result about the isomorphic classification on a certain class of Köthe spaces (see, Theorem 7 below).

## 2 Modeled Köthe spaces

Köthe space $K(A)$ defined by a Köthe matrix $A = (a_{i,p})_{i,p\in\mathbb{N}}$ (see, e.g., [4]) is the Fréchet space of all sequences $x = (\xi_i)_{i\in\mathbb{N}}$ such that $|x|_p := \sum_{i=1}^{\infty} |\xi_i| a_{i,p} < \infty$ for all $p \in \mathbb{N}$, equipped with the topology generated by these seminorms. An operator $T : K(A) \longrightarrow K\left(\widetilde{A}\right)$ is called *quasidiagonal* (with respect to the canonical bases $e_i := (\delta_{i,j})_{j=1}^{\infty}$, $i \in \mathbb{N}$) if $Te_i = t_i e_{\sigma(i)}$, where $(t_i)$ is a scalar sequence, $\sigma : \mathbb{N} \to \mathbb{N}$; if $T$ is an isomorphism we say that the spaces $K(A)$ and $K\left(\widetilde{A}\right)$ are *quasidiagonally isomorphic.* Given $(a_i) \in \omega^+$ (where $\omega^+$ is the set of all positive scalar sequences) and $\lambda = (\lambda_i)$, $\lambda_i \geq 1$, the space

$$F(\lambda, a) := K\left(\exp\left(\min\left\{p,\, \lambda_i - \frac{1}{p}\right\} a_i\right)\right), \tag{6}$$

is called *power Köthe space of second type* (in contrast to those spaces of first type *[8, 9]); it is *Montel* if and only if $a_i \to +\infty$.

A. Grothendieck considered ([3], II,p.122) the important special classes of Köthe spaces:

$$E_\alpha(a) := K(\exp(\alpha_p a_i)), \tag{7}$$

where $a = (a_i)_{i\in\mathbb{N}} \in \omega^+, \alpha_p \uparrow \alpha$, $-\infty < \alpha \leq +\infty$. We will call them *power Köthe spaces of finite type* (if $\alpha < \infty$) or *infinite type* (if $\alpha < \infty$) (*centers of Riesz scales* in [5] or *power series spaces* in [6]).

The space (6) is quasidiagonally isomorphic to (i) *the space (7) of finite type* if $\lambda_i$ is bounded, (ii) *the space (7) of infinite type* if $\lambda_i \to \infty$. Otherwise the space (6) is called *mixed* power Köthe space of second type*;* it is *essentially mixed* if it is not isomorphic quasidiagonally to a Cartesian product $E_0(b) \times E_\infty(c)$.

**Proposition 3** ([*9*], *Lemma* 2.3)*. Let $(t_i)$ be a scalar sequence and $\rho : \mathbb{N} \to \mathbb{N}$ a bijection. Then the rule $Te_i = t_i e_{\rho(i)}$, $i \in \mathbb{N}$, defines a quasidiagonal isomorphism from a Montel space $F(\lambda, a)$ onto a space $F\left(\widetilde{\lambda}, \widetilde{a}\right)$ if and only if the following assertions are valid:* ***(a)*** $a_i \asymp \widetilde{a}_{\rho(i)}$, *i.e.* $a_i/c \leq \widetilde{a}_{\rho(i)} \leq ca_i$, $i \in \mathbb{N}$, *with some constant* $c > 1$*;* ***(b)*** $-\Delta \leq \dfrac{\ln|t_i|}{a_i} \leq \Delta$, $i \in \mathbb{N}$, *with some constant* $\Delta > 0$*;* ***(c)*** *for any subsequence* $I \subset \mathbb{N}$, *such that* $\lambda_i \to l \in [1, \infty]$, $\widetilde{\lambda}_{\rho(i)} \to \widetilde{l} \in [1, \infty]$, $\dfrac{\widetilde{a}_{\rho(i)}}{a_i} \to \gamma$ *as* $i \to \infty$, $i \in I$,*either* $l = \widetilde{l} = \infty$ *or both of* $l$ *and* $\widetilde{l}$ *are finite and* $\lim \dfrac{\ln|t_i|}{a_i} = l - \widetilde{l}\gamma$.

The following fact (see, e.g., [9], Proposition 3.3) will be useful later.

**Proposition 4** *Let* $m_a(t) := |\{k : a_k \leq t\}|$, $m_b(t) := |\{k : b_k \leq t\}|$ *be the counting functions of non-decreasing positive sequences* $a = (a_i)$ *and* $b = (b_i)$. *If* $m_a(t) \leq m_b(Ct)$, $t > 0$, *with some constant* $C$, *then* $b_k \leq Ca_k$, $k \in \mathbb{N}$.

With an eye to spaces from the class $\mathcal{A}_o^n$ we deal with the following quite narrow subclass of power Köthe spaces of the second type dealing only with "thickly distributed" sequences $\lambda$: $\Phi^{(n)}(\varphi, g) := F\left((g(\varphi(i))), \left(i^{1/n}\right)\right)$,where $g:(0,1] \to \mathbb{R}_+$ is a continuous function such that $\lim_{\xi\to 0} g(\xi) = \infty$ and $\varphi : \mathbb{N} \to (0,1]$ is a function with *equidistributed values,* that is

$$\lim_{t\to\infty} \frac{|\{i \le t : \ c < \varphi(i) \le d\}|}{t} = d - c, \quad 0 \le c < d \le 1. \tag{8}$$

Given $D \in \mathcal{R}_o^n$ we divide the sequence $k(i)$ into two parts: the subsequence $l(i) = k(j_i)$ covering the set $\left\{k \in \mathbb{Z}_+^n : \dfrac{k}{|k|} \in \pi(D)\right\}$ and the complementary subsequence $m(i)$. By Lemma 2 from [8], certain asymptotics for the counting functions of the sequences $|k(i)|, |l(i)|, |m(i)|$ hold; from them, using Proposition 4, one can derive the asymptotics:

$$|k(i)| \sim (n!\, i)^{1/n}\ , \ |l(i)| \sim \left(\frac{n! \ \operatorname{mes}\Sigma}{\operatorname{mes}\pi(D)}\, i\right)^{1/n}, \ |m(i)| \asymp \ i^{1/d}, \quad i \to \infty, \tag{9}$$

where $d - 1 = \dim(\Sigma \setminus \pi(D))$. Define the function $\varphi = \varphi_D : \mathbb{N} \to (0,1]$ by the formula

$$\varphi(i) := \chi(h_D(\theta(i))), \quad i \in \mathbb{N}, \tag{10}$$

where $\theta(i) := \dfrac{l(i)}{|l(i)|}$, $i \in \mathbb{N}$. To prove that $\varphi$ is a function with equidistributed values we use the asymptotics ($\tau \to \infty$):

$$\left|\left\{i : |l(i)| \le \tau, \ \chi^{-1}(d) \le h_D(\theta(i)) \le \chi^{-1}(c)\right\}\right| \sim \frac{(d-c) \ \operatorname{mes}\pi(D) \ \tau^n}{n! \ \operatorname{mes}\Sigma},$$

which follows from [8], Lemma 2. Then, taking into account (9), (10) and putting $t = \dfrac{\operatorname{mes}\pi(D) \ \tau^n}{n! \ \operatorname{mes}\Sigma}$, we arrive at (8).

A space $A(D) \in \mathcal{A}_o^n$ is represented as a direct sum of two closed basis subspaces $L(D) := \overline{\operatorname{span}}\left\{z^{l(i)} : i \in \mathbb{N}\right\}$ and $M(D) := \overline{\operatorname{span}}\left\{z^{m(i)} : i \in \mathbb{N}\right\}$. Due to the asymptotics (9) for $|m(i)|$, the space $M(D)$ is isomorphic to the space $E_\infty\left(\ i^{1/d}\right)$. On the other hand, since by Proposition 3 $F(\lambda, ca) = F(c\lambda, a)$, $c > 0$, we obtain that the space $L(D)$ is isomorphic to the space $\Phi^{(n)}(\varphi, g)$ with $\varphi$ and $g$ defined in (10) and (2). Since the space $E_\infty\left(\ i^{1/d}\right)$ is contained in $\Phi^{(n)}(\varphi, g)$ as a basic subspace if $d < n$ (what is the same, if $\operatorname{mes}(\Sigma \setminus \pi(D)) = 0$) we obtain the following statement.

**Proposition 5** *Suppose $D \in \mathcal{R}_o^n$ and $\varphi, g$ are defined in (10), (2) . Then $A(D) \simeq \Phi^{(n)}(\varphi, g)$ if $\operatorname{mes}(\Sigma \setminus \pi(D)) = 0$ and $A(D) \simeq \Phi^{(n)}(\varphi, g) \times E_\infty\left(\left(i^{\frac{1}{n}}\right)\right)$, otherwise.*

***Remark 6*** *The spaces $\Phi^{(n)}(\varphi, g) \times E_\infty\left(\left(i^{\frac{1}{n}}\right)\right)$ and $\Phi^{(n)}(\varphi, \widetilde{g})$ are not quasi-diagonally isomorphic for any functions $g$, $\widetilde{g}$, because the second space contains* *no basic subspace isomorphic to* $E_\infty\left(\left(i^{\frac{1}{n}}\right)\right)$*. In fact, these spaces are not isomorphic ([2]), but the proof of this fact is not the aim of the present paper.*

Proposition 5 reduces Theorem 2 to the following more general result which will be proved in section 4.

**Theorem 7** *Suppose* $g(\alpha)$, $\widetilde{g}(\alpha)$ *are two continuous functions on* $(0,1]$ *tending to* $\infty$ *as* $\alpha\to 0$*;* $\varphi,\widetilde{\varphi}$ *are mappings from* $\mathbb{N}$ *onto* $(0,1]$ *with equidistributed values and* $\sigma:[0,q]\to[0,1]$, $0<q<1$, *is the continuous increasing function, which satisfies the differential equation (4) with the initial condition* $\sigma(0)=0$*. If the condition (5) holds, then the spaces* $\Phi^{(n)}(\varphi,g)$ *and* $\Phi^{(n)}(\widetilde{\varphi},\widetilde{g})$ *are quasidiagonally isomorphic.*

# 3 Main Lemma

**Lemma 8** *Let* $\alpha$, $\beta$ *be two functions from* $\mathbb{N}$ *to* $(0,1]$ *with equidistributed values. Let* $\sigma:[0,1]\to[0,1]$ *be an increasing continuous function, continuously differentiable on* $(0,1]$, *such that* $\sigma(0)=0$, $\sigma(1)=1$*. Suppose that the condition (5) is fulfilled with* $q=1$*. Then there exists a bijection* $\rho:\mathbb{N}\to\mathbb{N}$*, satisfying the conditions:* ***(i)*** $i\asymp\rho(i)$; ***(ii )*** $\beta(\rho(i_k))\to\sigma(a)$, $\dfrac{i_k}{\rho(i_k)}\to\sigma'(a)$ *for each* $a\in(0,1]$ *and any subsequence* $(i_k)$ *such that* $\alpha(i_k)\to a$ *.*

**Proof.** First we set $\alpha_\nu^{(s)}:=\dfrac{\nu}{2^s}$, $\beta_\nu^{(s)}:=\sigma\left(\alpha_\nu^{(s)}\right)$, $\nu=\overline{0,2^s}$, $s\in\mathbb{Z}_+$. By (5) we have

$$\frac{1}{L}\le d_\nu^{(s)}:=\frac{\beta_\nu^{(s)}-\beta_{\nu-1}^{(s)}}{\alpha_\nu^{(s)}-\alpha_{\nu-1}^{(s)}}\le L,\ \ \nu=\overline{1,2^s},\ s\in\mathbb{N}. \tag{11}$$

Take any sequence $\varepsilon_s\downarrow 0$ with $\varepsilon_1\le 1/6$. Since the functions $\alpha$ and $\beta$ are equidistributed, for each $s\in\mathbb{N}$ we find $T_s$ such that for $t\ge T_s$, $\nu=\overline{1,2^s}$, $s\in\mathbb{N}$, the counting functions $n_\nu^{(s)}(t):=\left|\left\{i\le t:\alpha_{\nu-1}^{(s)}<\alpha(i)\le\alpha_\nu^{(s)}\right\}\right|$, $m_\nu^{(s)}(t):=\left|\left\{i\le t:\beta_{\nu-1}^{(s)}<\beta(i)\le\beta_\nu^{(s)}\right\}\right|$ satisfy the estimates

$$\begin{array}{c} t(1-\varepsilon_s)\left(\alpha_\nu^{(s)}-\alpha_{\nu-1}^{(s)}\right)\le n_\nu^{(s)}(t)\le t(1+\varepsilon_s)\left(\alpha_\nu^{(s)}-\alpha_{\nu-1}^{(s)}\right), \\ t(1-\varepsilon_s)\left(\beta_\nu^{(s)}-\beta_{\nu-1}^{(s)}\right)\le m_\nu^{(s)}(t)\le t(1+\varepsilon_s)\left(\beta_\nu^{(s)}-\beta_{\nu-1}^{(s)}\right)\end{array} \tag{12}$$

Now introduce the sets $N_\nu^{(s)}=\left\{i\in\mathbb{N}:\alpha_{\nu-1}^{(s)}<\alpha(i)\le\alpha_\nu^{(s)},a_s<i\le a_{s+1}\right\}$, $\nu=\overline{1,2^{s-1}}$, $s\in\mathbb{Z}_+$, where the sequence $a_s$ is chosen so that

$$a_0=0,\ \ 2LT_s\le a_s\le\frac{\varepsilon_s\ a_{s+1}}{8L^2},\quad s\in\mathbb{N}, \tag{13}$$

and the sets $M_\nu^{(s)}=\left\{i\in\mathbb{N}:\beta_{\nu-1}^{(s)}<\beta(i)\le\beta_\nu^{(s)},b_{\zeta(\nu)}^{(s)}<i\le b_\nu^{(s+1)}\right\}$, $\nu=\overline{1,2^{s-1}}$, $s\in\mathbb{Z}_+$ where $\zeta(\nu)$ is equal to the integral part of $\dfrac{\nu+1}{2}$ and the parameters

$b_1^{(0)}=0,\ b_\nu^{(s)},\ \nu=\overline{1,2^{s-1}},\ \ s\in\mathbb{N},$ are chosen so that

$$\left|N_\nu^{(s)}\right|=\left|M_\nu^{(s)}\right|=:K(\nu,s),\ \nu=\overline{1,2^{s-1}},\ \ s\in\mathbb{Z}_+. \tag{14}$$

Represent the sets $N_\nu^{(s)},\ M_\nu^{(s)}$ in the form of increasing finite sequences: $i_k^{(\nu,s)}$ and $j_k^{(\nu,s)}$ with $k=\overline{1,K(\nu,s)}$ and construct the bijection $\rho:\mathbb{N}\to\mathbb{N}$ by the rule $\rho\left(i_k^{(\nu,s)}\right):=j_k^{(\nu,s)},\ \ k=\overline{1,K(\nu,s)},\ \ \nu=\overline{1,2^{s-1}},\ \ s\in\mathbb{Z}_+$. Let us show that this is the desired mapping. Using (13), (14), (12), (11), one can easily check by induction that

$$b_\nu^{(s)}\geq\frac{a_s}{2L},\quad \nu=\overline{1,2^{s-1}},\ \ s\in\mathbb{N}. \tag{15}$$

Let us check the conditions $(i),(ii)$. Setting $r_s:=\dfrac{1+\varepsilon_s}{1-2\varepsilon_s}$, and applying (14), (12), we obtain the inequalities

$$\frac{a_s}{r_{s-1}d_{\zeta(\nu)}^{(s-1)}}\leq b_\nu^{(s)}\leq\frac{r_{s-1}a_s}{d_{\zeta(\nu)}^{(s-1)}},\quad \nu=\overline{1,2^{s-1}},\quad s\in\mathbb{N}. \tag{16}$$

The counting functions for the finite sequences $i_k^{(\nu,s)}$ and $j_k^{(\nu,s)},\ k=\overline{1,K(\nu,s)}$ can be written in the following form

$$\begin{aligned} p_\nu^{(s)}(t) &= \max\left\{0,\min\left\{n_\nu^{(s)}(t)-n_\nu^{(s)}(a_s),K(\nu,s)\right\}\right\}\\ q_\nu^{(s)}(t) &= \max\left\{0,\min\left\{m_\nu^{(s)}(t)-m_\nu^{(s)}\left(b_{\zeta(\nu)}^{(s)}\right),K(\nu,s)\right\}\right\} \end{aligned} \tag{17}$$

Due to (17), (12), (16), we obtain, for $a_s<t\leq a_{s+1}$, the estimates

$$\begin{aligned} p_\nu^{(s)}(t) &\leq ((1+\varepsilon_s)t-(1-\varepsilon_s)a_s)\left(\alpha_\nu^{(s)}-\alpha_{\nu-1}^{(s)}\right)\\ &\leq \left(\frac{(1+\varepsilon_s)t}{d_\nu^{(s)}}-\frac{(1-\varepsilon_s)b_{\zeta(\nu)}^{(s)}d_{\zeta(\nu)}^{(s-1)}}{r_{s-1}d_\nu^{(s)}}\right)\left(\beta_\nu^{(s)}-\beta_{\nu-1}^{(s)}\right)\\ &\leq m_\nu^{(s)}\left(\frac{h_\nu^{(s)}t}{d_\nu^{(s)}}\right)-m_\nu^{(s)}\left(b_{\zeta(\nu)}^{(s)}\right)=q_\nu^{(s)}\left(\frac{h_\nu^{(s)}t}{d_\nu^{(s)}}\right), \end{aligned} \tag{18}$$

where

$$h_\nu^{(s)}=\frac{(1+\varepsilon_s)r_{s-1}+2L\left|(1-\varepsilon_s)d_{\zeta(\nu)}^{(s-1)}-(1+\varepsilon_s)r_{s-1}d_\nu^{(s)}\right|}{(1-\varepsilon_s)r_{s-1}d_\nu^{(s)}}. \tag{19}$$

Analogously, we obtain the estimate:

$$q_\nu^{(s)}(t)\leq p_\nu^{(s)}\left(g_\nu^{(s)}d_\nu^{(s)}t\right), \tag{20}$$

with

$$g_\nu^{(s)}=\frac{(1+\varepsilon_s)r_{s-1}d_{\zeta(\nu)}^{(s-1)}d_\nu^{(s)}+2L\left|(1-\varepsilon_s)d_\nu^{(s)}-(1+\varepsilon_s)r_{s-1}d_{\zeta(\nu)}^{(s-1)}\right|}{(1-\varepsilon_s)r_{s-1}d_{\zeta(\nu)}^{(s-1)}}. \tag{21}$$

By Lemma 4 and (18), (20) we have

$$\frac{i_k^{(\nu,s)}}{g_\nu^{(s)}} \leq d_\nu^{(s)} j_k^{(\nu,s)} \leq h_\nu^{(s)} i_k^{(\nu,s)} \tag{22}$$

for $k = \overline{1, K(\nu, s)}$; $\nu = \overline{1, 2^{s-1}}$; $s \in \mathbb{N}$. Taking into account (11), the definitions of the numbers $h_\nu^{(s)}$ and $g_\nu^{(s)}$ and (22), we obtain that there is a constant $M$ independent of $\nu$ and $s$ such that $p_\nu^{(s)}(t) \leq q_\nu^{(s)}(Mt)$, $q_\nu^{(s)}(t) \leq p_\nu^{(s)}(Mt)$, $t > 0$.Thus, the mapping $\rho : \mathbb{N} \to \mathbb{N}$ is constructed so that the condition $(i)$ is fulfilled.

It remains to check the condition $(ii)$. Take any subsequence $(i_n)$ such that $\alpha(i_n) \to a \in (0,1]$. For every $n$ we find $s = s(n)$, $\nu = \nu(n)$ and $k = k(n)$ such that $i_n = i_{k(n)}^{(\nu(n),s(n))} \in N_{\nu(n)}^{(s(n))}$.Then $\alpha_{\nu(n)-1}^{(s(n))} < \alpha(i_n) \leq \alpha_{\nu(n)}^{(s(n))}$ and $\alpha_{\nu(n)}^{(s(n))} \to a$.By the construction, $\rho(i_n) \in M_{\nu(n)}^{(s(n))}$, therefore $\beta_{\nu(n)-1}^{(s(n))} < \beta(\rho(i_n)) \leq \beta_{\nu(n)}^{(s(n))}$. Hence, by smoothness of $\sigma$, we have

$$\lim_{n\to\infty} \beta(\rho(i_n)) = \sigma(a), \quad \lim_{n\to\infty} d_{\nu(n)}^{(s(n))} = \lim_{n\to\infty} d_{\varsigma(\nu(n))}^{(s(n)-1)} = \sigma'(a). \tag{23}$$

Then, taking into account (19), (21), (23), we conclude that $\lim_{n\to\infty} h_{\nu(n)}^{(s(n))} = \lim_{n\to\infty} g_{\nu(n)}^{(s(n))} = 1$.Combining this with (22), (23), we obtain that $i_{k(n)}^{(\nu(n),s(n))} \sim \sigma'(a)\ j_{k(n)}^{(\nu(n),s(n))}$. Hence the condition (ii) is also proved. The proof is complete. ■

## 4 Proof of Theorem 7

**Lemma 9** *Let $\varphi$, $\widetilde{\varphi}$ be two functions from $\mathbb{N}$ to $(0,1]$ with equidistributed values and $g : (0,1] \to \mathbb{R}_+$ a decreasing continuous function such that $g(\xi) \to +\infty$ as $\xi \to 0$. Then $\Phi^{(n)}(\varphi, g) = F\left(g(\varphi(i)), \left(i^{\frac{1}{n}}\right)\right)$ is quasidiagonally isomorphic to $\Phi^{(n)}(\widetilde{\varphi}, g) = F\left(g(\widetilde{\varphi}(i)), \left(i^{\frac{1}{n}}\right)\right)$.*

**Proof.** Assume that the mapping $\sigma$ in Lemma 8 is the identity. Then the bijection $\rho : \mathbb{N} \to \mathbb{N}$, constructed there, satisfies the condition $i \asymp \rho(i)$ and for any subsequence $i_k$ such that $\varphi(i_k) \to \alpha \neq 0$ the conditions $\widetilde{\varphi}(\rho(i_k)) \to \alpha$ and $i_k \sim \rho(i_k)$ hold. Then, by Proposition 3, the operator $T : \Phi^{(n)}(\varphi, g) \to \Phi^{(n)}(\widetilde{\varphi}, g)$ defined by $Te_i = e_{\rho(i)}$, $i \in \mathbb{N}$, is a required isomorphism. ■

**Proof of Theorem 7.** By Lemma 9, we assume that $\widetilde{\varphi} = \varphi$. Let us introduce the functions $G(\alpha) := \int_0^\alpha \frac{d\lambda}{(g(\lambda))^n}$, $\widetilde{G}(\alpha) := \int_0^\alpha \frac{d\lambda}{(\widetilde{g}(\lambda))^n}$ and choose $q \in (0,1)$ so that $G(q) < \widetilde{G}(1)$. Then $\widetilde{q} := \widetilde{G}^{-1}(G(q)) < 1$ and the function $\sigma := \widetilde{G}^{-1} \circ G : [0,q] \longrightarrow [0,\widetilde{q}]$ is continuous on $[0,q]$, continuously differentiable on $(0,q]$ and satisfies the equation (4) and the condition $\sigma(0) = 0$. We extend the function $\sigma$ to a bijection of the interval $[0,1]$ onto itself preserving continuous differentiability and denote this mapping by the same symbol $\sigma$. The constructed mapping meets the conditions of Lemma 8, hence there is a bijection $\rho:\mathbb{N}\to\mathbb{N}$ satisfying the conditions (i), (ii) of this lemma. Applying Proposition 3, one can easily check that a required isomorphism can be realized as the quasidiagonal operator defined by $Te_i := e_{\rho(i)}$ for $0<\varphi(i)\le q$, and by $Te_i := \left(\exp\left(g\left(\varphi\left(i\right)\right)\ i^{1/n} - \widetilde{g}\left(\varphi\left(\rho\left(i\right)\right)\right)\right)\left(\rho\left(i\right)\right)^{1/n}\right)\ e_{\rho(i)}$ for the rest of $i$'s.

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