A General Framework for Economic Order Quantity Models With Discounts and Transportation Costs

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Abstract: We propose a general framework for economic order quantity type models with unit out-of-pocket holding costs, unit opportunity costs of holding, fixed ordering costs, and general transportation costs. For these models, we analyze the associated optimization problem and derive an easy procedure for determining a bounded interval containing the optimal cycle length. Also for a special class of transportation functions, like the carload discount schedule, we specialize these results and give fast and easy algorithms to calculate the optimal lot size and the corresponding optimal order-up-to-level.

Keywords: Inventory; EOQ-type model; transportation cost function; upper bounds; exact solution.

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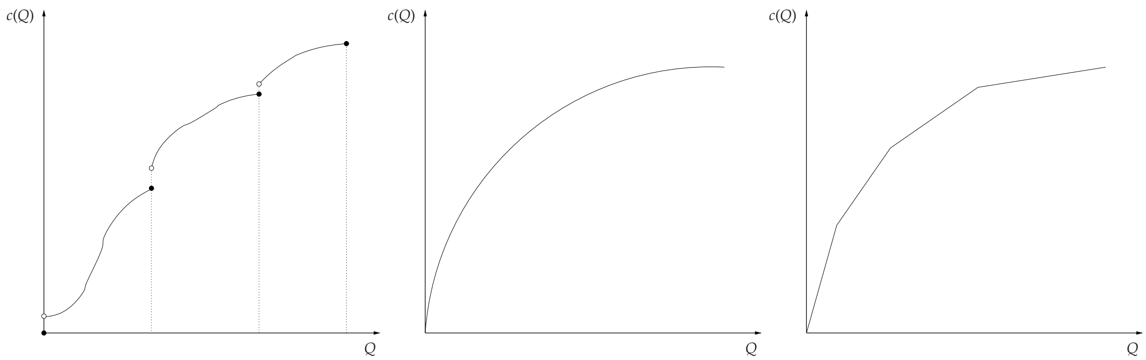
1. Introduction. In inventory control the economic order quantity model (EOQ) is the most fundamental model, which dates back to the pioneering work of Harris (1913). The environment of the model is somewhat restricted. The demand for a single item occurs at a known and constant rate, shortages are not permitted, there is a fixed setup cost and the unit purchasing and holding costs are independent of the size of the replenishment order. In this simplest form, the model describes the trade-off between the fixed setup and the holding costs. Though the model has several simplifying assumptions, it has been effectively used in practice. The standard EOQ model has also been extended to different settings, where shortages, discounts, production environments, and other extensions are considered (Hadley and Whitin, 1963; Nahmias, 1997; Silver et al., 1998; Zipkin, 2000; Muckstadt and Sapra, 2009).

In this paper, we propose a general framework that encompasses a large class of EOQ models studied in the literature. We pay particular attention to transportation and purchase costs, which involve quantity discounts both in purchasing and freight. Moreover, we also allow fixed setup costs of using multiple vehicles (or trucks) to meet an order. As our literature review given in Section 2 shows, there is a sizable list of work on EOQ models that account for the impact of the transportation costs on the lot sizing decision. Less-than-truckload (LTL) or full-truck-load (FTL) shipments, in particular, have been the focal point of many studies. The framework proposed here gives an overall approach to solve most of those problems posed in the literature. In addition, we also introduce several extensions that have not been studied in the literature before and show that these new models can also be handled within the proposed framework.

We start with a generic cost function that incorporates both the transportation and the purchase costs. This form of the transportation-purchase function allows us to analyze several different models including various discounting schemes as well as multiple setup costs. Our approach to these models is to derive, in Section 4, a bounded interval containing the optimal cycle length (reorder interval). We will first construct an upper bound on the optimal solution for a left continuous and increasing transportation-purchase function, $c(\cdot)$ as shown in Figure 1(a), where Q denotes the order quantity. This upper bound is represented by an easy analytical formula for the special case of an increasing polyhedral concave transportation-purchase function.

Such a function is shown in Figure 1(c) and represents a typical economies of scale situation. For the other more general transportation-purchase functions, it is possible to evaluate this upper bound by an algorithm. However, since this might take some computational time, we also derive a weaker analytical upper bound under some reasonable bounding condition on a transportation-purchase function. To improve the trivial zero bound on an optimal solution, we only provide an analytical positive lower bound for an increasing concave transportation-purchase function as illustrated in Figure 1(b).

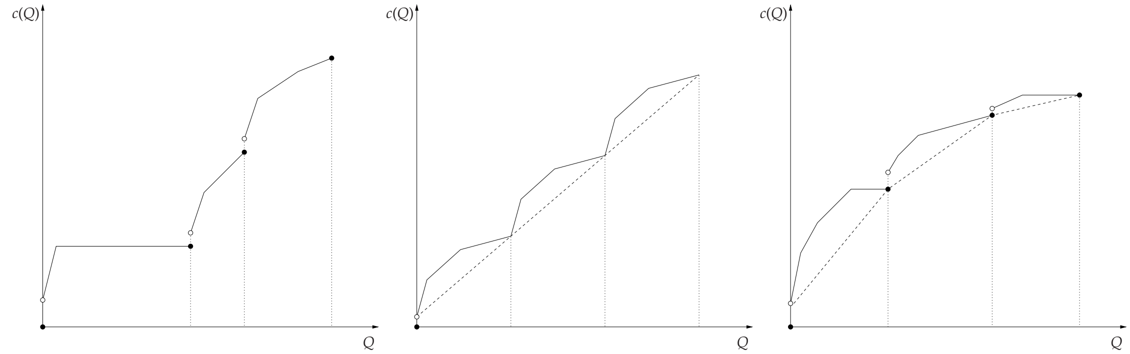


(a) General (b) Concave (c) Polyhedral concave

Figure 1: Some transportation-purchase functions for which the bounds on optimal T are studied.

We shall then show in Section 5 that there exists an important class of functions, for which the optimal solution can be identified by a fast algorithm. Figure 2 shows some important instances that belong to this class. Clearly, the well-known carload discount schedule in combination with linear purchase costs is a representative among these instances. To design these algorithms, we shall first show for an increasing linear transportation-purchase function that the resulting problem is a simple convex optimization problem that can be solved very efficiently. In particular, we shall also derive analytic solutions for two special cases: (i) when there are no shortages, or (ii) when there are shortages but the inventory holding cost rate is zero. Having analyzed an increasing linear transportation-purchase function, we shall then give a fast algorithm to solve the problem when the transportation-purchase function is increasing piecewise polyhedral concave as shown in Figure 2(a). This algorithm is based on solving a series of simple problems that correspond to the increasing linear pieces on the piecewise polyhedral concave function. To further improve the performance of the proposed algorithm, we shall then concentrate on two particular instances as shown in Figures 2(b) and 2(c). The former is a typical carload schedule with identical setups, and the latter represents a general carload schedule with nonincreasing truck setup costs. Both cases admit a lower bounding function, which is linear in the former case and polyhedral concave in the latter case. These lower bounding functions, shown with dashed lines in Figure 2, allow us to concentrate on solving only a few simple problems. Finally, in Section 6 we will give some numerical examples to illustrate our results.

In summary, the primary contributions of this work are (i) presenting a general framework to solve EOQ models with transportation costs and discounts, (ii) integration of LTL and FTL shipment schemes along with decreasing truck setup costs, (iii) analyzing the upper and lower bounds for a very general transportation-purchase cost function, (iv) providing analytical solutions or fast algorithms for several well-known special cases of the transportation-purchase functions.



(a) Piecewise polyhedral concave (b) Typical carload schedule (c) General carload schedule

Figure 2: Some transportation-purchase functions for which fast algorithms are developed.

2. Review of Related Literature. In this section, we shall review the literature on EOQ and lot-sizing models, where the main focus is the incorporation of transportation costs. We refer the reader to (Carter and Ferrin, 1996) for an overview and an informative discussion on the role of transportation costs in inventory control. We shall also occasionally consider quantity and, in particular, freight discounts. Das (1988) gives a general discussion about various discounting schemes.

One of the earliest works discussing the importance of transportation costs on controlling the inventory levels is given by Baumol and Vinod (1970). They try to place the freight decisions within inventory-theoretic models and point out that LTL shipments make the overall problem difficult to solve. Around the same time, Lippman (1969) considers a single-product in a multiple period setting, where charges due to multiple trucks with different sizes are taken into account. These charges create discontinuities (jumps) in the considered objective functions. Lippman obtains the optimal policies for two special cases of the objective function resulting in a monotone cost model and a concave cost model. He also analyzes the stationary, infinite horizon case and discusses the asymptotic properties of the optimal schedules. In a follow-up work, Lippman (1971) considers a similar setup for finding the economic order quantities. In this work, he assumes that the excess truck space cannot be used, and hence, the shipment cost should be incurred in the multiples of the trucks. In both of his works, no discounting scheme is present and shortages are not allowed. Iwaniec (1979) investigates the inventory model of a single product system, where the demand is stochastic and a fixed cost is charged and included in the ordering cost. The conditions under which the full load orders minimize the total expected cost are characterized. The multiple setup cost structure of Lippman (1971) is used also in this work. However, Iwaniec considers full backlogging, and hence, the holding and ordering costs are coupled with backlogging costs but no discounting scheme exists. Aucamp (1982) solves the continuous review case of the multiple setup problem discussed by Lippman (1971) and Iwaniec (1979). The main difference between the standard EOQ model and the Aucamp's model is the addition of vehicle costs to the setup cost. Like others above, no discounting scheme is considered. Lee (1986) discusses an EOQ model with a setup cost term that consists of fixed and freight costs. He also considers the case where the freight cost benefits from a discount scheme. The freight cost depends on the order size and added to the setup cost of placing an order. Noting that the convexity structure does not change within each interval, Lee proposes an algorithm based on finding the interval where the global minimum point resides. This algorithm is an alternate solution approach to that of Aucamp (1982), when the multiple setup cost structure of Lippman (1971) is adopted in the model.

Jucker and Rosenblatt (1985) incorporate the quantity discount schemes into the standard newsboy problem. These discounts play a role in purchasing or transporting units at the beginning of the period. Aside from the well-known all-units and incremental quantity discounts, they also discuss, what they call, carload-lot discounts. The transportation cost function is of the type shown in Figure 2(b). That is, the shipping-cost can be reduced or even exempted when the quantity of purchase is LTL. Knowles and Pantumsinchai (1988) consider an all-units discount schedule with no shortages. The products are sold in containers of various sizes. The seller offers discounts when the products are shipped in larger container sizes. They impose FTL orders by adding a restriction on the order quantity which dictates that the order quantities should be in integral multiples of the container sizes. They give a solution algorithm based on solving a series of knapsack problems. They also develop a more efficient algorithm for a restricted policy, which is based on filling the order starting from the largest container and then carrying on with smaller ones. A different perspective to transportation costs is given by Larson (1988). He introduces several models, where three stages of inventory levels are considered: at the origin, in-transit, and at the destination. Then, the objective becomes minimization of total logistics costs. Hwang et al. (1990) investigate both all-units quantity and freight cost discounts within the standard EOQ context. The economies of scale realized on the freight cost is the same as in (Lee, 1986). Tersine and Barman (1991) combine quantity and freight rate discounts from suppliers and shippers, respectively. They consider all-units and incremental quantity discount schemes both in purchasing and freight cost. However, the truck setup costs and the shortages are omitted. Arcelus and Rowcroft (1991) examine three types of freight-rate structures, where the incremental discount is applied only to purchasing. The objective function of the resulting problem is analyzed over nonoverlapping intervals, and it is shown that the objective function is convex over each interval. Thus, an algorithm, which is based on identifying the local solution within each interval, is proposed to solve the overall problem.

Russell and Krajewski (1991) study the transportation cost structure for LTL shipments. They consider over-declared shipments, which result from an opportunity to reduce the total freight costs by artificially inflating the actual shipping weight to the next breakpoint. In other words, for a freight rate schedule, it may be more economical to ship LTL at a FTL rate. The decision makers then need to transform this nominal freight rate schedule into an effective one, which appropriately represents the best rate schedule for them. This effective schedule consists of intervals over which the transportation cost is determined by a polyhedral concave function consisting of a linear and a constant piece. This is again a special case of what we consider in our work as illustrated by Figure 2(a). Carter et al. (1995a) discuss in-detail the role of anomalous weight breaks in LTL shipping and examine the causes behind this anomaly with its implications in logistics management. These points occur when the discount is so large that the indifference point weight is less than even the lower rate interval. Their observation on anomalous weight breaks has led them to correct the effective freight rate schedule in (Russell and Krajewski, 1991) as they reported in their subsequent work (Carter et al., 1995b). Burwell et al. (1997) consider an EOQ environment under quantity and freight discounts very similar to Tersine and Barman (1991). Unlike Tersine and Barman, their demand is not constant but depends on the price. Therefore, the proposed algorithm to solve the model also determines the selling price besides the optimal lot size. However, they ignore the option of over-declaring the shipments, and they do not consider LTL or FTL freight rates. Swenseth and Godfrey (2002) carry on with a similar discussion about over-declared shipments as in (Russell and Krajewski, 1991). They do not take quantity discounts or shortages into account. Therefore, the resulting transportation cost function can be thought as a special case of the function shown in Figure 2(c). To solve the resulting problem, they propose a heuristic, which is based on evaluating two inverse functions that over- and under-shoot the optimal order quantity. Abad and Aggarwal (2005) extend the model proposed by Burwell et al. (1997) by considering both over-declaring and LTL (or FTL) shipments like Russell and Krajewski (1991) and Swenseth and Godfrey (2002). They propose a solution procedure based on solving a series of nonlinear equations to obtain the optimal order quantity as well as the selling price. In two recent works (Rieskts and Ventura, 2008; Mendoza and Ventura, 2008), the optimal inventory policies with both FTL and LTL transportation modes are examined. Rieskts and Ventura provide focus on both infinite and finite horizon single-stage models with no shortages. Later, Mendoza and Ventura extend the work of Rieskts and Ventura by incorporating all-units and incremental quantity discounts into their models.

3. Mathematical Model. We consider an EOQ-type, infinite planning horizon model with complete backordering, where $\lambda > 0$ is the demand rate and $a > 0$ is the fixed ordering cost. The inventory holding costs consist of a unit out-of-pocket holding cost of $h > 0$ per item per unit of time and a unit opportunity cost of holding with inventory holding cost rate $r \geq 0$. Moreover, the penalty cost of backlogging is $b > 0$ per item per unit of time. To avoid pathological cases we assume that $b > h$. Clearly, when $b = \infty$ there are no shortages in the problem. The function $p : [0, \infty) \to \mathbb{R}$ with $p(0) = 0$ represents the purchase price function, and it is assumed that $p(\cdot)$ is left continuous on $(0, \infty)$. This means that the well-known all-units-discount scheme is also included in, especially the first part of, our analysis. At the same time, the function $t : [0, \infty) \to \mathbb{R}$ with $t(0) = 0$, denotes the transportation cost function and this function is also assumed to be left continuous on $(0, \infty)$. The structure of the function $t(\cdot)$ allows us to model truck costs. Consequently, the total transportation-purchase cost of an order of size Q is given by

$$c(Q) := t(Q) + p(Q), \tag{1}$$

where $c(\cdot)$ denotes the transportation-purchase function. Since the addition of two left continuous functions is again left continuous, the function $c(\cdot)$ is, in general, a left continuous function. This means for every $Q > 0$ that

$$c(Q) = c(Q^-) := \lim_{x \uparrow Q} c(x)$$

and

$$c(Q^+) := \lim_{x \downarrow Q} c(x).$$

Using this left continuous transportation-purchase function $c(\cdot)$ implies that the cost rate function of an EOQ-type model is given by

$$f(T, x) = \begin{cases} u(T)x, & \text{if } x \geq 0; \\ -bx, & \text{if } x < 0, \end{cases} \tag{2}$$

with

$$u(T) := h + r(\lambda T)^{-1} c(\lambda T). \tag{3}$$

For a detailed discussion of this cost rate function within a production environment, the reader is referred to (Bayındır et al., 2006), and a similar derivation for the standard EOQ model is given by, for instance, Muckstadt and Sapra (2009). Since it is easy to see that for a given cycle length $T > 0$, any order-up-to-level $S > \lambda T$ is dominated in cost by $S = \lambda T$, we only derive the average cost expression for (S, T) control rules within the interval $0 \leq S \leq \lambda T$. For such control rules, the average cost $g(S, T)$ has the form

$$g(S, T) = \frac{a + c(\lambda T) + \int_0^T f(T, S - \lambda t)dt}{T}. \tag{4}$$

Hence, to determine the optimal (S,T) rule, we need to solve the optimization problem

$$\min\{g(S,T) : T > 0, 0 \leq S \leq \lambda T\}.$$

By relation (4), this problem reduces to

$$\min\left\{\frac{a + c(\lambda T) + \varphi(T)}{T} : T > 0\right\},$$

where $\varphi : (0,\infty) \to \mathbb{R}$ is given by

$$\varphi(T) = \min\left\{\int_0^T f(T, S - \lambda t)dt : 0 \leq S \leq \lambda T\right\}. \tag{5}$$

Since by relation (2) it is easy to verify for $0 \leq S \leq \lambda T$ that

$$\int_0^T f(T, S - \lambda t)dt = \frac{\lambda^{-1}u(T)S^2}{2} + \frac{\lambda^{-1}b(S - \lambda T)^2}{2}, \tag{6}$$

and the derivative of this function for T fixed is equal to $\lambda^{-1}u(T)S + \lambda^{-1}b(S - \lambda T)$, the optimal value $S(T)$ of the optimization problem listed in relation (5) is given by

$$S(T) = \begin{cases} \frac{b\lambda T}{b+u(T)}, & \text{for } 0 < b < \infty; \\ \lambda T, & \text{for } b = \infty. \end{cases}$$

Hence, we obtain by relation (6) that

$$\varphi(T) = \begin{cases} \frac{\lambda b u(T) T^2}{2(b+u(T))}, & \text{for } 0 < b < \infty; \\ \frac{\lambda u(T) T^2}{2}, & \text{for } b = \infty. \end{cases}$$

This shows by relation (3) for $b < \infty$ (shortages are allowed) that we need to solve the optimization problem

$$\min\{\Phi_b(T) : T > 0\},$$

where

$$\Phi_b(T) := \frac{a + c(\lambda T)}{T} + \frac{b\lambda T}{2} - \frac{(\lambda b T)^2}{2\lambda(b+h)T + 2rc(\lambda T)}. \tag{7}$$

Similarly for $b = \infty$ (no shortages allowed), we obtain the optimization problem

$$\min\{\Phi_\infty(T) : T > 0\},$$

where

$$\Phi_\infty(T) := \frac{a + c(\lambda T)}{T} + \frac{h\lambda T + rc(\lambda T)}{2}. \tag{8}$$

By the additivity of the costs, it is obvious that including the left continuous transportation-purchase function $c(\cdot)$ as a separate cost component into the EOQ-type models does not change the structural form of the objective function. However, since $c(\cdot)$ is left continuous, we can only conclude that the objective functions in relations (7) and (8) are also left continuous, and hence, they may contain points of discontinuity. In general, these functions (as a function of the length of the replenishment cycle T) are not unimodal anymore as in the classical EOQ models. Hence, they may contain several local minima and so, it might be difficult to guarantee that a given solution is indeed optimal.

4. Bounding The Optimal Cycle Length. In this section we show that one can identify an upper bound on the optimal cycle length of the previous EOQ-type models for left continuous increasing transportation-purchase functions $c(\cdot)$ as shown in Figure 1(a). For very general functions $c(\cdot)$, it might be difficult to compute this upper bound by means of an easy algorithm. Therefore, we show that under an affine bounding condition on the function $c(\cdot)$, this upper bound can be replaced by a weaker upper bound having an elementary formula. To derive these results, we first identify the general structure of the considered EOQ-type models.

Let $F : [0,\infty) \times (0,\infty) \to \mathbb{R}$ be given by

$$F(x,T) := \frac{a+x}{T} + \frac{b\lambda T}{2} - \frac{(\lambda b T)^2}{2\lambda(h+b)T + 2rx}, \tag{9}$$

then it follows from relation (7) that the EOQ-type optimization problem with shortages is given by

$$\min\{F(c(\lambda T), T) : T > 0\}. \tag{P_b}$$

Similarly, we also introduce the function $G : [0,\infty) \times (0,\infty) \to \mathbb{R}$ given by

$$G(x,T) := \frac{a+x}{T} + \frac{h\lambda T + rx}{2}. \tag{10}$$

Then, it is clear from relation (8) that the EOQ-type model with no shortages allowed ($b = \infty$) transforms to

$$\min\{G(c(\lambda T), T) : T > 0\}. \tag{P_∞}$$

By relations (9) and (10), it is obvious that the functions $F(\cdot,\cdot)$ and $G(\cdot,\cdot)$ belong to the following class of functions.

Definition 4.1 *A function $H : [0,\infty)\times(0,\infty) \to \mathbb{R}$ belongs to the set $\mathcal{H}$ if the function $H(\cdot,\cdot)$ is continuous, $x \mapsto H(x,T)$ is increasing on $[0,\infty)$ for every $T > 0$, and $\lim_{T\downarrow 0} H(x,T) = \lim_{T\uparrow\infty} H(x,T) = \infty$ for every $x \geq 0$.*

Hence, both EOQ-type optimization models are particular instances of the optimization problem

$$\min\{H(c(\lambda T), T) : T > 0\}, \tag{P}$$

where $H(\cdot,\cdot)$ belongs to the set $\mathcal{H}$ and $c(\cdot)$ is an increasing left continuous function on $[0,\infty)$. We show in Lemma A.1 of Appendix A that the function $T \mapsto H(c(\lambda T), T)$ is lower semi-continuous for any $H(\cdot,\cdot)$ belonging to $\mathcal{H}$. Consequently, an optimal solution for problem (P) indeed exists, and hence, our search for a bounded interval containing an optimal solution is justified.

4.1 Dominance Results. In this section, we shall give two simple dominance results that will be instrumental for finding a bounded interval for different EOQ-type models. We start with the following lemma, which has a straightforward proof. The functions $c(\cdot)$ and $c_1(\cdot)$ satisfying the conditions of the lemma are exemplified in Figure 3.

Lemma 4.1 *Let the functions $c_1(\cdot)$, $c(\cdot)$ be left continuous on $[0,\infty)$ with $c_1(\cdot)$ increasing and $H(\cdot,\cdot)$ belong to $\mathcal{H}$.*

(*i*) *If $c(Q) \geq c_1(Q)$ for every $Q > \lambda d$ and $c(\lambda d) = c_1(\lambda d)$ and $T \mapsto H(c_1(\lambda T), T)$ is increasing on (d,∞), then $H(c(\lambda T), T) \geq H(c(\lambda d), d)$ for every $T > d$.*

(*ii*) *If $c(Q) \geq c_1(Q)$ for every $Q \leq \lambda d$ and $c(\lambda d) = c_1(\lambda d)$ and $T \mapsto H(c_1(\lambda T), T)$ is decreasing on $(0,d)$, then $H(c(\lambda T), T) \geq H(c(\lambda d), d)$ for every $T < d$.*

Proof. Since the function $H(\cdot,\cdot)$ belongs to $\mathcal{H}$ and the function $c_1(\cdot)$ is left continuous and increasing, it follows that

$$\lim_{T\downarrow d} H(c_1(\lambda T), T) = H(c_1((\lambda d)^+), d) \geq H(c_1(\lambda d), d) = H(c(\lambda d), d).$$

Using again $H(\cdot,\cdot) \in \mathcal{H}$, $c(\lambda T) \geq c_1(\lambda T)$ for every $T > d$, and $T \mapsto H(c_1(\lambda T), T)$ is increasing on (d,∞), we have for every $T > d$ that

$$H(c(\lambda T), T) \geq H(c_1(\lambda T), T) \geq \lim_{T\downarrow d} H(c_1(\lambda T), T) \geq H(c(\lambda d), d).$$

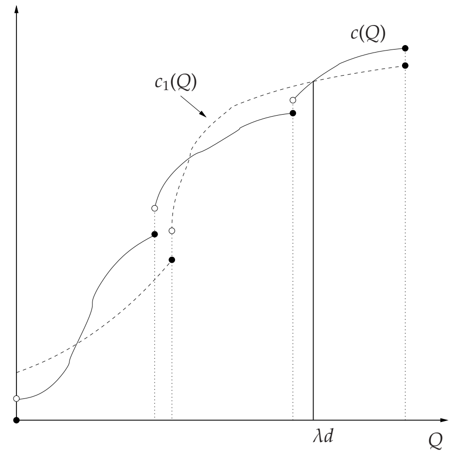


Figure 3: The increasing left-continuous functions used in Lemma 4.1

By a similar proof the second part can also be shown. □

An easy implication of Lemma 4.1 is given by the following result.

Lemma 4.2 *Let the functions $c_1(\cdot)$, $c(\cdot)$ be left continuous on $[0,\infty)$ with $c_1(\cdot)$ increasing and $H(\cdot,\cdot)$ belong to $\mathcal{H}$. If*

(*i*) *$c(Q) \geq c_1(Q)$ for every $Q \geq 0$ and $c(\lambda d_n) = c_1(\lambda d_n)$ for some strictly increasing sequence $d_n \uparrow \infty$ with $d_0 := 0$, and*

(*ii*) *there exists some $y_1 \geq y_0 > 0$ such that the function $T \mapsto H(c_1(\lambda T), T)$ is decreasing on $(0, y_0)$ and increasing on $[y_1, \infty)$,*

then for $n_* := \max\{n \in \mathbb{Z}_+ : d_n < y_0\}$ and $n^* := \min\{n \in \mathbb{Z}_+ : d_n \geq y_1\}$, the interval $[d_{n_*}, d_{n^*}]$ contains an optimal solution of the optimization problem (P)*.*

Proof. Since the function $T \mapsto H(c_1(\lambda T), T)$ is decreasing on $(0, d_{n_*})$ and increasing on (d_{n^*}, ∞), and $c(\lambda d_{n_*}) = c_1(\lambda d_{n_*})$ and $c(\lambda d_{n^*}) = c_1(\lambda d_{n^*})$, we can apply Lemma 4.1 to show the desired result. □

Clearly, if $T \mapsto H(c_1(\lambda T), T)$ is unimodal, then we obtain that $y_1 = y_0$ and hence $n^* = n_* + 1$. In the next subsection we will apply the above localization results to the EOQ-type models.

4.2 Applications of The Dominance Results to The EOQ-Type Models. In this section we will show some applications of Lemma 4.1 and Lemma 4.2 on different EOQ-type models. We first examine the simple EOQ-type model with no shortages. To obtain an easily computable upper bound on an optimal solution, we impose on the function $c(\cdot)$ the following bounding condition.

Assumption 4.1 *The transportation-purchase function $c(\cdot)$ satisfies*

$$c(Q) \leq \alpha Q + \beta \tag{11}$$

for some $\alpha, \beta > 0$.

By definition of a transportation-purchase function, Assumption 4.1 seems to be a reasonable condition. Moreover, in the subsequent discussion we shall additionally assume that the transportation purchase function

$c(\cdot)$ is increasing. Notice that the analysis up to this point applies to *any* type of EOQ-type model, but with this monotonicity assumption on $c(\cdot)$ we exclude the all-units discount model.

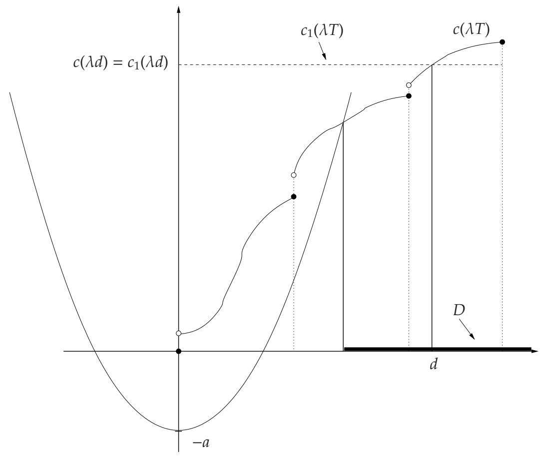


Figure 4: The construction used in Example 4.1 and Example 4.2

Example 4.1 (Upper Bound for Increasing $c(\cdot)$ With No Shortages) *If the transportation-purchase function $c(\cdot)$ is increasing and left continuous, consider the set*

$$D := \left\{ d \geq 0 : c(\lambda d) \leq \frac{h\lambda d^2}{2} - a \right\}, \tag{12}$$

and assume D is nonempty (see Figure 4). We will next show for any $d \in D$ that an optimal solution of this problem can be found within the interval $[0, d]$. To verify this claim, consider some $d \in D$ and introduce the constant function $c_1 : (0, \infty) \to \mathbb{R}$ given by

$$c_1(Q) := c(\lambda d). \tag{13}$$

Since $c(\cdot)$ is increasing, clearly $c(Q) \geq c_1(Q)$ for every $Q > \lambda d$ and $c(\lambda d) = c_1(\lambda d)$. Moreover, if $c_1(\cdot)$ is the considered transportation-purchase function and no shortages are allowed, then the objective function $\Psi_d : (0, \infty) \to \mathbb{R}$ has the form

$$\Psi_d(T) = G(c_1(\lambda T), T) = G(c(\lambda d), T),$$

where $G(\cdot, \cdot)$ is given in relation (10). By elementary calculus, it is easy to verify that the optimal solution $T_{opt}(d)$ of the optimization problem $\min\{\Psi_d(T) : T > 0\}$ is given by

$$T_{opt}(d) = \sqrt{\frac{2(a + c(\lambda d))}{h\lambda}}. \tag{14}$$

Moreover, since $\Psi_d(\cdot)$ is a strictly convex function, it is strictly decreasing on $(0, T_{opt}(d))$ and strictly increasing on $(T_{opt}(d), \infty)$. Since d belongs to D, this implies by relation (14) that $T_{opt}(d) \leq d$. Consequently, we may conclude that the function $\Psi_d(\cdot)$ is increasing on (d, ∞). By applying now the first part of Lemma 4.1, it follows that an optimal solution of an EOQ-type model with no shortages is contained in $[0, d]$. To find the best possible upper bound, we introduce

$$d_{\min} := \inf\{d \geq 0 : d \in D\}. \tag{15}$$

Since $c(\cdot)$ is increasing and left continuous, it follows that $d_{\min}$ also belongs to D, and so, an optimal solution is contained in $[0, d_{\min}]$. However, due to the general form of the transportation-purchase function $c(\cdot)$, it might be difficult to give a fast procedure to compute the value of $d_{\min}$. To replace $d_{\min}$ by an easy computable bound, we now use Assumption 4.1 as $c(\lambda d) \leq \alpha\lambda d + \beta$. Observe this bounding condition guarantees that the set D is nonempty and

$$\left\{ d \geq 0 : \alpha\lambda d + \beta \leq \frac{h\lambda d^2}{2} - a \right\} \subseteq D. \tag{16}$$

Since it is easy to see that $\{d \geq 0 : \alpha\lambda d + \beta \leq \frac{h\lambda d^2}{2} - a\} = [v_{\alpha,\beta}, \infty)$ *with*

$$v_{\alpha,\beta} := \alpha h^{-1} + \sqrt{\alpha^2 h^{-2} + 2h^{-1}\lambda^{-1}(a+\beta)}, \tag{17}$$

we obtain by relation (16) *that*

$$v_{\alpha,\beta} \geq d_{\min}. \tag{18}$$

Therefore, an optimal solution is contained in $(0, v_{\alpha,\beta}]$.

Due to the specific form of the function $c_1(\cdot)$, it follows by relation (10) that for the EOQ-type model with no shortages and transportation-purchase function $c_1(\cdot)$, the inventory holding cost rate is a fixed cost independent of the decision variable T. Hence, the optimal $T_{opt}(d)$ given by relation (14) does not contain the value of r. This means for our procedure discussed in Example 4.1 that the constructed upper bound on an optimal solution does not contain this parameter r and holds uniformly for every $r \geq 0$. Hence, it seems likely that this upper bound might be far away from an optimal solution of an EOQ-type model with function $c(\cdot)$ and a given inventory holding cost rate. We explore this issue by our computational study in Section 6. In case we do not have any structure on $c(\cdot)$ –the structured case will be considered in the next section– we might now use some discretization method over $(0, v_{\alpha,\beta}]$ to approximate the optimal solution for the no shortages case.

We shall consider next the general EOQ-type model with shortages. Before discussing the construction of an upper bound for this model, we first need the following result.

Lemma 4.3 *If* $T^{(r)}_{opt}(d)$ *denotes the optimal solution of the EOQ model with shortages allowed, inventory holding cost rate* $r \geq 0$ *and the constant transportation-purchase function* $c_1(\cdot)$ *listed in relation* (13)*, then for all* $r \geq 0$*, we have*

$$T^{(r)}_{opt}(d) \leq T^{(0)}_{opt}(d) = \sqrt{\frac{2(a + c(\lambda d))}{h\lambda}\frac{h+b}{b}}.$$

Proof. The objective function of the considered EOQ-model with inventory holding cost rate $r > 0$ is given by $T \mapsto F(c_1(\lambda T), T)$ with $F(\cdot,\cdot)$ listed in relation (9). Since it is easy to check for every $x \geq 0$ that

$$\frac{(\lambda b T)^2}{2\lambda(h+b)T + 2rx} = \frac{\lambda b^2}{2(h+b)}\left(T - \frac{rxT}{\lambda(h+b)T + rx}\right),$$

we have

$$F(c_1(\lambda T), T) = \frac{a + c(\lambda d)}{T} + \frac{b}{h+b}\frac{\lambda h T}{2} + \frac{\lambda b^2 r}{2(h+b)}\left(\frac{c(\lambda d)T}{\lambda(h+b)T + rc(\lambda d)}\right). \tag{19}$$

Introducing now the convex function $T \mapsto F_0(c_1(\lambda T), T)$ with

$$F_0(x, T) := \frac{a+x}{T} + \frac{b}{h+b}\frac{\lambda h T}{2}$$

and the increasing function $K : (0, \infty) \to \mathbb{R}$ given by

$$K(T) := \frac{\lambda b^2 r}{2(h+b)}\left(\frac{c(\lambda d)T}{\lambda(h+b)T + rc(\lambda d)}\right),$$

we obtain by relation (19) that

$$F(c_1(\lambda T), T) = F_0(c_1(\lambda T), T) + K(T). \tag{20}$$

By looking at relation (20), we observe that the function

$$T \mapsto F_0(c_1(\lambda T), T)$$

is the objective function of an EOQ-model with shortages allowed, $r = 0$, and the transportation-purchase function $c_1(\cdot)$. Also, it is easy to check in relation (20) that the remainder function K is increasing with a positive derivative. This shows that the derivative of the function

$$T \mapsto F(c_1(\lambda T), T)$$

evaluated at the optimal solution $T^{(0)}_{opt}(d)$ of an EOQ-type model with shortages allowed and $r = 0$ is positive. Using now relation (9) with $r = 0$, it is easy to check that

$$T^{(0)}_{opt}(d) = \sqrt{\frac{2(a + c(\lambda d))}{h\lambda}\frac{h+b}{b}}.$$

Since by the definition of $T^{(r)}_{opt}(d)$ the derivative of the function $T \to F(c_1(\lambda T), T)$ evaluated at this point equals 0, the inequality

$$T^{(r)}_{opt}(d) \leq T^{(0)}_{opt}(d)$$

holds once we have verified that the function $T \mapsto F(c_1(\lambda T), T)$ is unimodal. To show this property, we first observe that the function $K_1 : (0, \infty) \to \mathbb{R}$ given by

$$K_1(T) := TK(T)$$

being the ratio of a squared convex function and an affine function is convex (Bector, 1968). This implies that the function $T \mapsto TK_1(T^{-1}) = K(T^{-1})$ is convex (Hiriart-Urruty. and Lemarechal, 1993). Moreover, it is easy to verify by its definition that the function $T \mapsto F_0(c_1(\lambda T^{-1}), T^{-1})$ is convex, and this shows by relation (20) that the function $T \mapsto F(c_1(\lambda T^{-1}), T^{-1})$ is convex implying $T \mapsto F(c_1(\lambda T), T)$ is unimodal. □

Lemma 4.3 shows that the optimal solution of an EOQ-type model with the constant transportation-purchase function $c_1(\cdot)$ and nonzero inventory holding cost rate is bounded from above by the optimal solution of an EOQ-type model with the transportation-purchase function $c_1(\cdot)$ and zero inventory holding cost rate. Using this result we will construct in the next example an upper bound on the optimal solution of an EOQ-type model with shortages allowed, inventory holding cost rate $r \geq 0$ and left-continuous increasing transportation-purchase function $c(\cdot)$.

Example 4.2 (Upper Bound for Increasing $c(\cdot)$ With Shortages) *If the transportation-purchase function $c(\cdot)$ is increasing and left continuous, consider the set*

$$D := \left\{ d \geq 0 : c(\lambda d) \leq \frac{h\lambda d^2}{2}\frac{b}{h+b} - a \right\}, \tag{21}$$

and assume that D is nonempty (see also Figure 4). Let $d \in D$ and consider the constant function $c_1 : (0, \infty) \to \mathbb{R}$ given by

$$c_1(Q) := c(\lambda d).$$

Since c is increasing clearly $c(Q) \geq c_1(Q)$ for every $Q > \lambda d$ and $c(\lambda d) = c_1(\lambda d)$. Moreover, if shortages are allowed, then the objective function $\Psi_d : (0, \infty) \to \mathbb{R}$ has the form

$$\Psi_d(T) = F(c_1(\lambda T), T) = F(c(\lambda d), T),$$

where $F(\cdot, \cdot)$ is given in relation (9). In the proof of Lemma 4.3, it is shown that the objective function Ψ_d is unimodal, and for any $r \geq 0$ we have

$$T^{(r)}_{opt}(d) \leq T^{(0)}_{opt}(d) = \sqrt{\frac{2(a + c(\lambda d))}{\lambda h}\frac{h+b}{b}}. \tag{22}$$

Applying now the unimodality of the function $\Psi_d(\cdot)$ this yields that $\Psi_d(\cdot)$ is increasing on the interval $(T^{(r)}_{opt}(d), \infty)$, and since d belongs to D, we also obtain by relation (22) that

$$T^{(r)}_{opt}(d) \leq T^{(0)}_{opt}(d) \leq d.$$

This shows that the function $\Psi_d(\cdot)$ is increasing on (d, ∞), and by applying part (i) of Lemma 4.1, we conclude that an optimal solution of the EOQ-type model with the general transportation-purchase function $c(\cdot)$ is contained in the interval $[0, d]$. As in Example 4.1 the best possible upper bound is now given by

$$d_{\min} := \inf\{d \geq 0 : d \in D\}. \tag{23}$$

Again due to the particular instance of $c(\cdot)$ it might be difficult to compute $d_{\min}$. To replace d_{min} by an easily computable upper bound, we again use the bounding condition given in Assumption 4.1 and obtain $c(\lambda d) \leq \alpha\lambda d + \beta$. This implies that D is nonempty and it follows as in Example 4.1 that $d_{\min} \leq w_{\alpha,\beta}$ with

$$w_{\alpha,\beta} := \alpha h^{-1}(h+b)b^{-1} + \sqrt{\alpha^2 h^{-2}((h+b)b^{-1})^2 + 2h^{-1}\lambda^{-1}(a+\beta)(h+b)b^{-1}}. \tag{24}$$

Therefore, under the bounding condition, $w_{\alpha,\beta}$ serves as an upper bound on an optimal solution of the original problem.

Remark 4.1 *By relations (12) and (21), it is easy to see that an upper bound on an optimal solution for an EOQ-type model with no shortages (Example 4.1) is always smaller than an upper bound on an optimal solution of an EOQ-type model with shortages (Example 4.2). Similarly, we obtain by relations (17) and (24) that this also holds for the easily computable upper bounds under the bounding condition.*

In case we additionally know that the function $c(\cdot)$ is concave, which corresponds to some incremental discount scheme for either the purchase function or the transportation cost function, it is also possible to compute a (nontrivial) lower bound on the optimal solutions of the EOQ-type models considered in the previous two examples. The next example discusses this lower bound explicitly for the no shortages case.

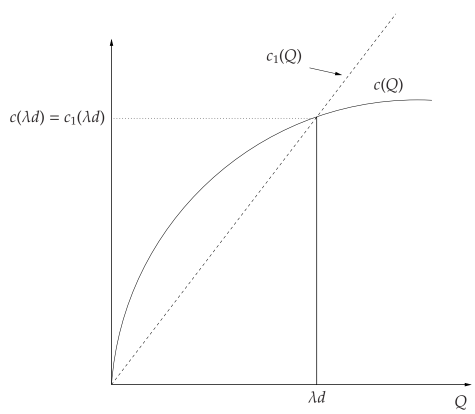


Figure 5: The construction used in Example 4.3

Example 4.3 (Lower Bound for Increasing Concave $c(\cdot)$ With No Shortages) *If we know additionally that the transportation-purchase function $c(\cdot)$ is concave, and hence continuous, it is also possible to give a lower bound on the optimal solution. Observe in this case that Assumption 4.1 is trivially satisfied (see Figure 5). Take for simplicity, $Q \mapsto \frac{c(\lambda d)}{\lambda d}Q + c(\lambda d)$, which clearly satisfies Assumption 4.1 with $\alpha = \frac{c(\lambda d)}{\lambda d}$ and $\beta = c(\lambda d)$. Consider now for $d > 0$, the function $c_1 : (0, \infty) \to \mathbb{R}$ given by*

$$c_1(Q) = \frac{c(\lambda d)}{\lambda d}Q.$$

By the concavity of $c(\cdot)$ *and* $c(0) = 0$, *we obtain for every* $Q < \lambda d$ *that*

$$c(Q) = c(Q\lambda^{-1}d^{-1}\lambda d) \geq Q\lambda^{-1}d^{-1}c(\lambda d)$$

and this shows $c(Q) \geq c_1(Q)$ *for every* $Q < \lambda d$ *and* $c(\lambda d) = c_1(\lambda d)$*. As in Example 4.1 the objective function has the form*

$$\Psi_d(T) = G(c_1(\lambda T), T)$$

where $G(\cdot, \cdot)$ *is given in relation (10). By elementary calculus, it is easy to verify that the optimal solution* $T_{opt}(d)$ *of the optimization problem* $\min\{\Psi_d(T) : T > 0\}$ *is given by*

$$T_{opt}(d) = \sqrt{\frac{2a}{h\lambda + rc(\lambda d)d^{-1}}}. \tag{25}$$

Since the function $x \mapsto c(\lambda x)x^{-1}$ *is decreasing and continuous with* $\lim_{x\downarrow 0} c(\lambda x)x^{-1} \leq \infty$, *it follows by relation (25) that the function* $T_{opt} : (0, \infty) \to \mathbb{R}$ *is increasing and continuous. Also, by the strict convexity of the function* Ψ_d *this function is strictly decreasing on* $(0, T_{opt}(d))$ *and strictly increasing on* $(T_{opt}(d), \infty)$*. This implies*

$$\Psi_d \text{ decreasing on } (0, d) \Leftrightarrow T_{opt}(d) \geq d.$$

Since the set $\{d \geq 0 : T_{opt}(d) \geq d\}$ *contains* 0 *it follows by the second part of Lemma 4.1 that an optimal solution of the EOQ-type model with no shortages allowed and a concave transportation-purchase function* $c(\cdot)$ *is contained in* $[d_{max}, \infty)$, *where*

$$d_{\max} := \sup\{d \geq 0 : T_{opt}(d) \geq d\} = \sup\{d \geq 0 : h\lambda d^2 + rdc(\lambda d) \leq 2a\}. \tag{26}$$

Since the function $d \mapsto h\lambda d^2 + rdc(\lambda d)$ *is strictly increasing and continuous on* $[0, \infty)$, *we obtain that* d_{max} *is the unique solution of the system*

$$h\lambda x^2 + rxc(\lambda x) = 2a.$$

Also, by the nonnegativity of c *we obtain that*

$$d_{max} \in [0, \sqrt{2a\lambda^{-1}h^{-1}}].$$

Thus, one can apply a computationally fast derivative free one-dimensional search algorithm over the interval of uncertainty $[0, \sqrt{2a\lambda^{-1}h^{-1}}]$ *to compute the lower bound* d_{max} *(Bazaraa et al., 1993).*

Since the derivation is very similar, we omit the lower bound for the shortages case.

As shown in the above examples, under the affine bounding condition stated in Assumption 4.1, it is possible to identify by means of an elementary formula a bounded interval I containing an optimal solution of the EOQ-type model with increasing transportation-purchase function $c(\cdot)$. Hence, we obtain for the two different cases represented by the optimization problems (P_b) and (P_∞) that

$$\min_{T>0} H(c(\lambda T), T) = \min_{T\in I} H(c(\lambda T), T). \tag{27}$$

However, for the general increasing left continuous transportation-purchase functions, the function $T \mapsto H(c(\lambda T), T)$ does not have the desirable unimodal structure. Since we are interested in finding an optimal solution, the only thing we could do is to discretize the interval I and select among the evaluated function values on this grid the one with a minimal value. In case the objective function has a finite number of discontinuities and it is Lipschitz continuous between any two consecutive discontinuities with known (maybe different) Lipschitz constants, it is possible by using an appropriate chosen grid to give an error on the deviation of the objective value of this chosen solution from the optimal objective value. We leave the details of this construction to the reader and refer to the literature on one-dimensional Lipschitz optimization algorithms (Horst et al., 1995).

However, for some left continuous increasing transportation-purchase functions $c(\cdot)$, it is possible to compute explicitly the value of d_{min} listed in relations (15) and (23) by means of an easy algorithm. This means that for these functions we do not need the easily computable upper bound and so in this case the upper bound on an optimal solution can be improved. An example of such a class of transportation-purchase functions is given in the next definition.

Definition 4.2 (Rockafellar (1972)) *A function $c : (0,\infty) \to \mathbb{R}$ is called a polyhedral concave function on $(0,\infty)$, if $c(\cdot)$ can be represented as the minimum of a finite number of affine functions on $(0,\infty)$. It is called polyhedral concave on an interval I, if $c(\cdot)$ is the minimum of a finite number of affine functions on I.*

We will now give an easy algorithm to identify the value d_{min} in case $c(\cdot)$ is an increasing polyhedral concave function. Observe it is easy to verify that polyhedral concave functions defined on the same interval are closed under addition. Within the inventory theory, polyhedral concavity on $[0,\infty)$ of the transportation-purchase function $c(\cdot)$ describes incremental discounting either with respect to the purchase costs or the transportation costs or both.

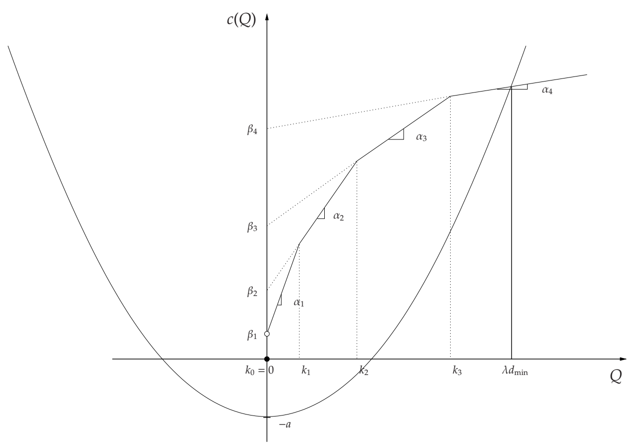


Figure 6: A polyhedral concave transportation-purchase function.

Clearly, a polyhedral concave function on $(0,\infty)$ can be represented for every $Q > 0$ as

$$c(Q) = \min_{1\leq n\leq N}\{\alpha_n Q + \beta_n\}, \tag{28}$$

where N denotes the total number of affine functions, $\alpha_1 > .... > \alpha_N \geq 0$, and $0 \leq \beta_1 < \beta_2 < ... < \beta_N$. An example of a polyhedral concave function $c(\cdot)$ is given in Figure 6. Between k_{n-1} and k_n the minimum in relation (28) is attained by the affine function $Q \mapsto \alpha_n Q + \beta_n$. To compute the values α_n and β_n in terms of our original data given by the finite set of breaking points $0 = k_0 < k_1 < ... < k_{N-1} < k_N = \infty$, and function values $c(k_n)$, $n = 1, ...N-1$ we observe that

$$\alpha_n = \frac{c(k_n) - c(k_{n-1})}{k_n - k_{n-1}} \tag{29}$$

for $n = 1, ..., N-1$ and

$$\alpha_N = c(k_{N-1} + 1) - c(k_{N-1}). \tag{30}$$

Also, by the same figure we obtain for $k_{n-1} < Q \leq k_n$, $n = 1, .., N$ that

$$c(Q) = c(k_{n-1}) + \alpha_n(Q - k_{n-1}) = \alpha_n Q + \beta_n$$

and this implies

$$\beta_n = c(k_{n-1}) - \alpha_n k_{n-1} \tag{31}$$

for $n = 1, .., N$.

We will now give an easy algorithm to identify the value d_{min}, if $c(\cdot)$ is a polyhedral concave function with the representation given in relation (28). Using now relations (15) and (23), we have

$$d_{\min} = \min\{d > 0 : c(\lambda d) \leq \frac{h\lambda d^2\zeta}{2} - a\}, \tag{32}$$

where $\zeta = 1$ for the no shortages case and $\zeta = \frac{b}{h+b}$ for the shortages case. Since $c(\cdot)$ is concave and increasing, and the function $d \mapsto \frac{h\lambda d^2\zeta}{2} - a$ is strictly convex and increasing on $[0, \infty)$ (see Figure 6), each region D, given by relation (12) or relation (21), is an interval $[d_{min}, \infty)$. The next algorithm clearly yields d_{min} as an output .

Algorithm 1: Finding $d_{\min}$ for polyhedral $c(\cdot)$

1: $n_* := \max\{0 \leq n \leq N-1 : c(k_n) > \frac{hk_n^2\zeta}{2\lambda} - a\}$

2: Determine in $[k_{n_*}, k_{n_*+1}]$ or in $[k_{n_*}, \infty)$ the unique analytical solution d_* of the equation

$$\alpha_{n_*+1}\lambda d + \beta_{n_*+1} = \frac{h\lambda d^2\zeta}{2} - a$$

given by

$$d_* = \frac{\alpha_{n_*+1}\lambda + \sqrt{(\alpha_{n_*+1}\lambda)^2 + 2h\lambda\zeta(a + \beta_{n_*+1})}}{h\lambda\zeta}$$

3: $d_{\min} \leftarrow d_*$

In the next section we shall identify a subclass of the increasing left continuous transportation-purchase functions, for which it is easy to identify an optimal solution instead of only a bounded interval containing an optimal solution.

5. Fast Algorithms for Solving Some Important Cases. Unless we impose some additional structure on $c(\cdot)$, it could be difficult to find a fast algorithm to solve optimization problem (P) due to the existence of many local minima. Clearly, if $c(\cdot)$ is an affine function given by

$$c(Q) = \alpha Q + \beta$$

with $\alpha > 0$, $\beta \geq 0$, it is already shown by Bayındır et al. (2006) that the objective functions of both EOQ-type models given by (P_b) and (P_∞) are unimodal functions. Also for the no shortages model (P_∞), it is easy to check by relation (10) that the optimal solution T_{opt} is given by

$$T_{opt} = \sqrt{\frac{2(a + \beta)}{\lambda(h + r\alpha)}}, \tag{33}$$

while for the shortages model (P_b) with zero inventory holding cost rate ($r = 0$), it follows by relation (9) that the optimal solution T_{opt} has the form

$$T_{opt} = \sqrt{\frac{2(a + \beta)}{\lambda h}\frac{h + b}{b}}. \tag{34}$$

Finally, for the most general model with shortages allowed and nonzero inventory holding cost rate, it follows that the function

$$T \mapsto F(c(\lambda T^{-1}, T^{-1})$$

is a convex function on $[0, \infty)$; see, (Bayındır et al., 2006, Lemma 3.2). Hence, solving problem (P_b) using the decision variable T^{-1} is an easy one-dimensional convex optimization problem, and so, we can find T_{opt} rather quickly. Consequently, this observation helps us to come up with fast algorithms when $c(\cdot)$ consists of linear pieces. Among such functions, the most frequently used ones are the polyhedral concave functions given in (28). Using this representation and $H(\cdot, \cdot) \in \mathcal{H}$, the overall objective function for both EOQ-type models becomes

$$H(c(\lambda T), T) = \min_{1 \leq n \leq N} H(\alpha_n \lambda T + \beta_n, T). \tag{35}$$

This shows by our previous observations that the function $T \mapsto H(c(\lambda T^{-1}), T^{-1})$ is simply the minimum of N different convex functions. In general this function is not convex anymore and even not unimodal. However, due to relation (35) it follows that

$$\min_{T>0} H(c(\lambda T^{-1}), T^{-1}) = \min_{1 \leq n \leq N} \min_{T>0} H(\alpha_n \lambda T^{-1} + \beta_n, T^{-1}), \tag{36}$$

and by relation (36), we need to solve N one-dimensional unconstrained convex optimization problems to determine an optimal solution. Notice by relation (35) that each of these N problems involve an affine function. This implies that if we consider the no shortages model (P_b) or the shortages model (P_∞) with $r = 0$, then we have the analytic solutions (33) and (34), respectively. Therefore, solving (36) boils down to selecting the minimum among N different values in these cases.

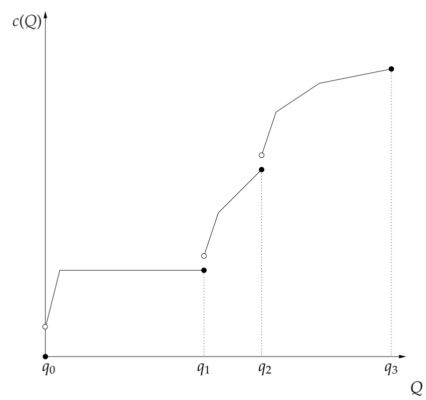


Figure 7: A piecewise polyhedral concave transportation-purchase function.

We next introduce a more general class containing as a subclass the polyhedral concave functions on $[0, \infty)$. An illustration of a function in this class is given in Figure 7.

Definition 5.1 *A finite valued function $c : (0, \infty) \to \mathbb{R}$ is called a piecewise polyhedral concave function if there exists a strictly increasing sequence q_n, $n \in \mathbb{Z}_+$ with $q_0 := 0$ and $q_n \uparrow \infty$ such that the function $c(\cdot)$ is polyhedral concave on $(q_n, q_{n+1}]$, $n \in \mathbb{N}$.*

A piecewise concave polyhedral function might be discontinuous at the points q_n, $n \in \mathbb{Z}_+$. If the function $c(\cdot)$ is a piecewise polyhedral concave function, then it follows by relation (28) that

$$c(Q) = \min_{1 \leq n \leq N_k} \{\alpha_{nk} Q + \beta_{nk}\} \tag{37}$$

for $q_{k-1} < Q \leq q_k$ and finite N_k. If, additionally, the function $c(\cdot)$ satisfies Assumption 4.1, then we have shown in Subsection 4.2 that an easily computable upper bound exists on the optimal solution. We denote this upper bound by U. For problem (P_∞), U is given by relation (17), while for problem (P_b) it is given by relation (24). Since $q_n \uparrow \infty$ and U is a finite upper bound on an optimal solution it follows that

$$m^* := \min\{n \in \mathbb{N} : q_n > \lambda U\} < \infty \tag{38}$$

and an optimal solution is contained in the bounded interval $[0, \lambda^{-1}q_{m^*})$. Since $c(\cdot)$ is increasing this implies that

$$\begin{aligned} \min\nolimits_{T>0} H(c(\lambda T), T) &= \min\nolimits_{0<T\leq \lambda^{-1}q_{m^*}} H(c(\lambda T), T) \\ &= \min\nolimits_{1\leq k\leq m^*} \min\nolimits_{\lambda^{-1}q_{k-1}\leq T\leq \lambda^{-1}q_k} H(c(\lambda T), T). \end{aligned}$$

By relation (37), it follows now that

$$\min_{\lambda^{-1}q_{k-1}\leq T\leq \lambda^{-1}q_k} H(c(\lambda T), T) = \min_{1\leq n\leq N_k} \min_{\lambda^{-1}q_{k-1}\leq T\leq \lambda^{-1}q_k} H(\alpha_{nk}\lambda T + \beta_{nk}, T),$$

and so, we have to solve for $1 \leq k \leq m^*$ and $n \leq N_k$, the constrained convex one-dimensional optimization problems

$$\min_{\lambda^{-1}q_{k-1}\leq T\leq \lambda^{-1}q_k} H(\alpha_{nk}\lambda T^{-1} + \beta_{nk}, T^{-1}).$$

Solving these subproblems can be done relatively fast, but since we have to solve $\sum_{k=1}^{m^*} N_k$ of those subproblems this might take a long computation time for the most general case. Observe once again, if we only consider the no shortages model or the shortages model with zero inventory holding cost rate, the subproblems $\min_{T>0} H(\alpha_{nk}\lambda T + \beta_{nk}, T)$ have analytical solutions given by relations (33) and (34), respectively. Hence, using the unimodality of the considered objective functions, the optimal solution can be determined simply by checking whether the optimal solution of the unconstrained problem lies within $[\lambda^{-1}q_{k-1}, \lambda^{-1}q_k]$. Hence, for the piecewise polyhedral transportation-purchase function, we have the steps outlined in Algorithm 2.

Algorithm 2: Finding T_{opt} for piecewise polyhedral $c(\cdot)$

1: Determine U and determine m^* by relation (38)

2: Solve for $k = 1, ..., m^*$ the optimization problems

$$\varphi_k := \min_{\lambda^{-1}q_{k-1}\leq T\leq \lambda^{-1}q_k} H(c(\lambda T), T)$$

3: $n_{opt} := \arg\min\{\varphi_k : 1 \leq k \leq m^*\}$

4: $T_{opt} \leftarrow \arg\min_{\lambda^{-1}q_{n_{opt}-1}\leq T\leq \lambda^{-1}q_{n_{opt}}} H(c(\lambda T), T)$

In Algorithm 2 we need to solve in Step 2 many relatively simple optimization problems. However, for m^* large this still might take some computation time. In the next example, we consider a subclass of the set of piecewise polyhedral concave functions with some additional structure for which it is possible to give a faster algorithm. For this class, we have to solve only one subproblem in Step 2. The well-known carload discount schedule transportation function with identical trucks belongs to this class (Nahmias, 1997).

Example 5.1 (Carload Discount Schedule With Identical Trucks) *Let $C > 0$ be the truck capacity, $g : (0, C] \to \mathbb{R}$ be an increasing polyhedral concave function satisfying $g(0) = 0$ and $s \geq 0$ be the setup cost of using one truck. Here,*

$g(Q)$ corresponds to the transportation cost for transporting an order of size Q with $0 < Q \leq C$. If no discount is given on the number of used (identical) trucks, then the total transportation cost function $t : [0, \infty) \to \mathbb{R}$ has the form

$$t(Q) = \begin{cases} 0, & \text{if } Q = 0; \\ g(Q) + s, & \text{if } 0 < Q \leq C, \end{cases}$$

and

$$t(Q) = ng(C) + g(Q - nC) + (n + 1)s$$

for $nC < Q \leq (n+1)C$ with integer $n \geq 1$. Clearly, the above transportation function $t(\cdot)$ belongs to the class of piecewise polyhedral concave functions with $q_n = nC$. When we use the above transportation function $t(\cdot)$ with a linear purchase function $p(\cdot)$, then we obtain a transportation-function $c(\cdot)$ similar to the one shown in Figure 8.

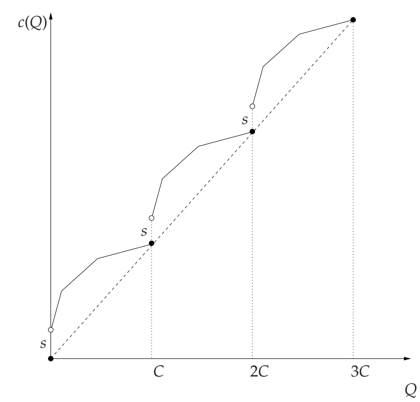


Figure 8: A transportation-purchase function for carload discount schedule with identical trucks.

For this class of functions it follows $t(Q) \geq t_1(Q)$ for every $Q \geq 0$ with

$$t_1(Q) := \frac{g(C) + s}{C} Q$$

and for $d_n := \lambda^{-1} nC$ the equality

$$t(\lambda d_n) = t_1(\lambda d_n)$$

holds for every $n \in \mathbb{Z}_+$. If the price of each ordered item equals $\pi > 0$ (no quantity discount), and hence the purchase function $p : [0, \infty) \to \mathbb{R}$ is given by $p(Q) = \pi Q$, it follows that the lower bounding function $c_1(\cdot)$ of the transportation-purchase function $c(Q) = t(Q) + p(Q)$ is given by

$$c_1(Q) = t_1(Q) + p(Q) = \left(\frac{g(C) + s}{C} + \pi \right) Q$$

and

$$c(\lambda d_n) = c_1(\lambda d_n)$$

for every $n \in \mathbb{Z}_+$. Adding a linear function $p(\cdot)$ to the piecewise polyhedral concave function $t(\cdot)$ yields that $c(\cdot)$ is a piecewise polyhedral concave function (see also Figure 8). Since for the EOQ-type model with linear function $c_1(\cdot)$ both the no shortages objective function $T \mapsto G(c_1(\lambda T), T)$ and the shortages objective function $T \mapsto F(c_1(\lambda T), T)$ are unimodal, it follows by Lemma 4.2 that an optimal solution of the EOQ-type model with transportation-purchase function $c(\cdot)$ is contained within the interval $[d_{n_*}, d_{n_*+1}]$ with

$$n_* := \max\{n \in \mathbb{Z}_+ : d_n \leq T_{opt}\}, \tag{39}$$

where T_{opt} is the optimal solution of the EOQ-type model with linear transportation-purchase function $c_1(\cdot)$. Since $d_n = \lambda^{-1}nC$, this implies

$$n_* = \lfloor \lambda T_{opt} C^{-1} \rfloor, \tag{40}$$

where $\lfloor \cdot \rfloor$ denotes the floor function. In particular, if we consider the no shortages case ($b = \infty$), then we obtain using relation (33) that the optimal solution T_{opt} of the EOQ-type model with function $c_1(\cdot)$ has an easy analytical form given by

$$T_{opt} = \sqrt{\frac{2a}{\lambda(h + rp + r(g(C) + s)C^{-1})}}. \tag{41}$$

Likewise, for the EOQ-model with shortages ($b < \infty$) and no inventory holding cost rate ($r = 0$), we obtain using relation (34) that

$$T_{opt} = \sqrt{\frac{2a(h+b)}{\lambda hb}}. \tag{42}$$

Finally, for the most general EOQ-type model with shortages allowed and positive inventory holding cost rate r, there exists a fast algorithm to compute its optimal solution T_{opt}. If T_{opt} equals d_{n_*} or equivalently T_{opt} is an integer multiple of $\lambda^{-1}C$ the optimal solution of the EOQ model with function $c(\cdot)$ also equals T_{opt}. Otherwise, as already observed, the optimal solution of this EOQ model with function $c(\cdot)$ can be found in the interval $(d_{n_*}, d_{n_*+1}]$, and so, we have to solve in the second step the optimization problem

$$\min_{d_{n_*} < T \leq d_{n_*+1}} H(c(\lambda T), T).$$

Algorithm 3 gives the details of solving the carload discount schedule with identical trucks.

Algorithm 3: Finding T_{opt} for carload discount schedule with identical trucks

1: $T^* = \arg\min_{T>0} H(c_1(\lambda T), T)$
2: **if** *T^* is not an integer multiple of $\lambda^{-1}C$* **then**
3: $\quad n_* = \lfloor \lambda T_{opt} C^{-1} \rfloor$
4: $\quad T^* = \arg\min_{d_{n_*} < T \leq d_{n_*}+1} H(c(\lambda T), T)$
5: $T_{opt} \leftarrow T^*$

When we generalize Example 5.1 to nonidentical trucks, we can use our results given for arbitrary piecewise polyhedral concave functions. If we further concentrate on the carload discount schedule with nonincreasing truck setup costs as shown in Figure 9, then the lower bounding function $c_1(\cdot)$ becomes polyhedral concave. In this case, we can develop a faster algorithm. To obtain a polyhedral concave $c_1(\cdot)$, we assume for $n \geq 1$ that the sequence

$$\delta_n := \frac{c(q_n) - c(q_{n-1})}{q_n - q_{n-1}}$$

is decreasing. Then, the function $c_1 : [0, \infty) \to \mathbb{R}$ becomes

$$c_1(Q) = c(q_{n-1}) + \delta_n(Q - q_{n-1}) = \delta_n Q + \gamma_n \tag{43}$$

for $q_{n-1} \leq Q \leq q_n$, $n \geq 1$ with $\gamma_n = c(q_{n-1}) - \delta_n q_{n-1}$. As shown in Figure 9, $c(q_n) = c_1(q_n)$, $n \in \mathbb{N}$, and $c(Q) \geq c_1(Q)$ for every $Q \geq 0$.

Since by construction $c(Q) \geq c_1(Q)$ it follows that

$$H(c(\lambda T), T) \geq H(c_1(\lambda T), T).$$

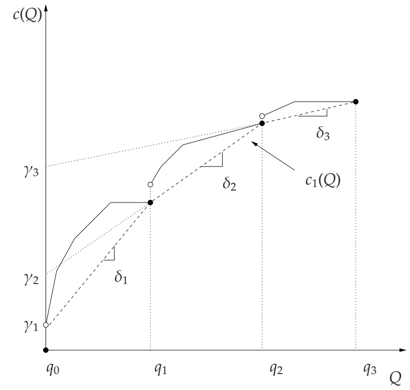


Figure 9: A transportation-purchase function for carload discount schedule with nonincreasing truck setup costs.

We will now show by means of the concavity of the lower bounding function $c_1(\cdot)$ that one can determine a better upper bound than (38). We know for any d belonging to the set

$$D_\zeta = \{d > 0 : c_1(\lambda d) \leq \frac{h\lambda d^2\zeta}{2} - a\}$$

that

$$H(c_1(\lambda T), T) \geq H(c_1(\lambda d), d) \tag{44}$$

for any $T \geq d$. By the concavity of $c_1(\cdot)$, this implies for

$$n_* := \max\{n \in \mathbb{N} : c_1(q_n) > \frac{h{q_n}^2\zeta}{2\lambda} - a\}$$

that

$$H(c_1(\lambda T), T) \geq H(c_1(q_{n_*+1}), \lambda^{-1}q_{n_*+1}) \tag{45}$$

for every $T \geq \lambda^{-1}q_{n_*+1}$. This implies by relation (44) and $c(q_{n_*+1}) = c_1(q_{n_*+1})$ that

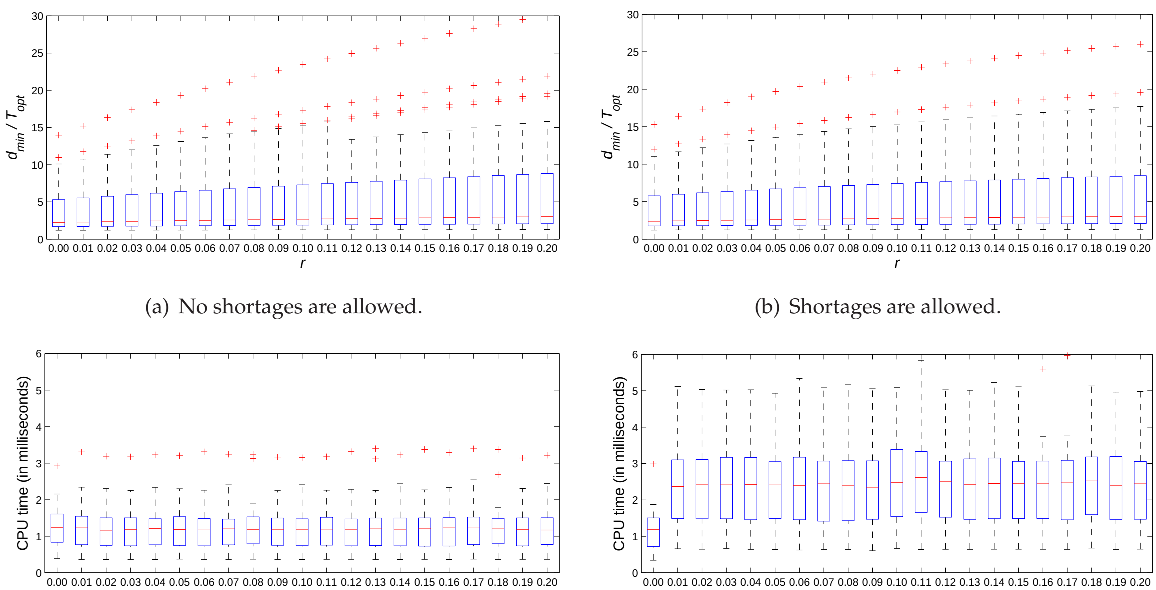
$$H(c(\lambda T), T) \geq H(c(q_{n_*+1}), \lambda^{-1}q_{n_*+1})$$

for every $T \geq \lambda^{-1}q_{n_*+1}$. Hence we have shown that any optimal solution of the original EOQ model with transportation-purchase function $c(\cdot)$ is contained in $[0, \lambda^{-1}q_{n_*+1}]$. By the discussion at the end of Subsection 4.2 and relation (38), it follows that $n_* \leq m^*$ and this shows that the newly constructed upper bound is at least as good as the constructed bound for an arbitrary piecewise polyhedral concave function. Therefore, the number of subproblems to be solved could be far less than m^*. We investigate this issue in the next section.

6. Computational Study. We designed our numerical experiments with two basic goals in mind. First, we would like to demonstrate that the EOQ model is amenable to fast solution methods in the presence of a general class of transportation functions introduced in this paper. Second, we aim to shed some light into the dynamics of the EOQ model under the carload discount schedule which seems to be the most well-known transportation function in the literature. Recall that in our analysis we assumed that there exists an affine upper bound on the transportation-purchase function (Assumption 4.1). Though straightforward, for completeness we explicitly give in Appendix B the steps to compute these affine bounds for the functions that are used in our computational experiments.

The algorithms we developed were implemented in Matlab R2008a, and the numerical experiments were performed on a Lenovo T400 portable computer with an Intel Centrino 2 T9400 processor and 4GB of memory.

6.1 Tightness of The Upper Bounds on T_{opt} for Polyhedral Concave and Piecewise Polyhedral Concave $c(\cdot)$. In the final paragraph of Example 4.1, we reckoned that the constructed upper bound $v_{\alpha,\beta}$ on d_{min} given in (17) for the no shortages case may be weak for problems with strictly positive inventory holding cost rate r because $v_{\alpha,\beta}$ does not contain the value of r. The same is true for the upper bound $w_{\alpha,\beta}$ on d_{min} defined in (24) if shortages are allowed. Thus, in the first part of our computational study we explore the strength of the upper bounds on T_{opt} as r changes. To this end, 100 instances are created and solved for varying values of r for both polyhedral concave and piecewise polyhedral concave transportation-purchase functions. For all of these instances, we set $\lambda = 1500$, $a = 200$, $h = 0.05$. Piecewise polyhedral concave functions consist of 20 intervals over which the transportation-purchase function $c(\cdot)$ is polyhedral concave. In this case, each polyhedral concave function is constructed by the minimum of a number of affine functions where this number is chosen randomly from the range $[2, 5]$. If $c(\cdot)$ is polyhedral concave on $[0, \infty)$, then the number of linear pieces on $c(\cdot)$ is selected randomly from the range $[2, 20]$. For both piecewise polyhedral concave and polyhedral concave $c(\cdot)$, the slope of the first affine function on each polyhedral concave function is distributed as $U[0.50, 1.00]$. The following slopes are calculated by multiplying the immediately preceding slope by a random number in the range $[0.80, 1.00]$. All (truck) setup costs are identical to 50, and the distance between two breakpoints on $c(\cdot)$ is generated randomly from the range $[0.05\lambda, 0.20\lambda]$. If shortages are allowed, b takes a value of 0.25, otherwise $b = \infty$. The inventory holding cost rate r is varied in the interval $[0, 0.20]$ at increments of 0.01. The results of these experiments are summarized in figures 10 - 11 .



(a) No shortages are allowed.

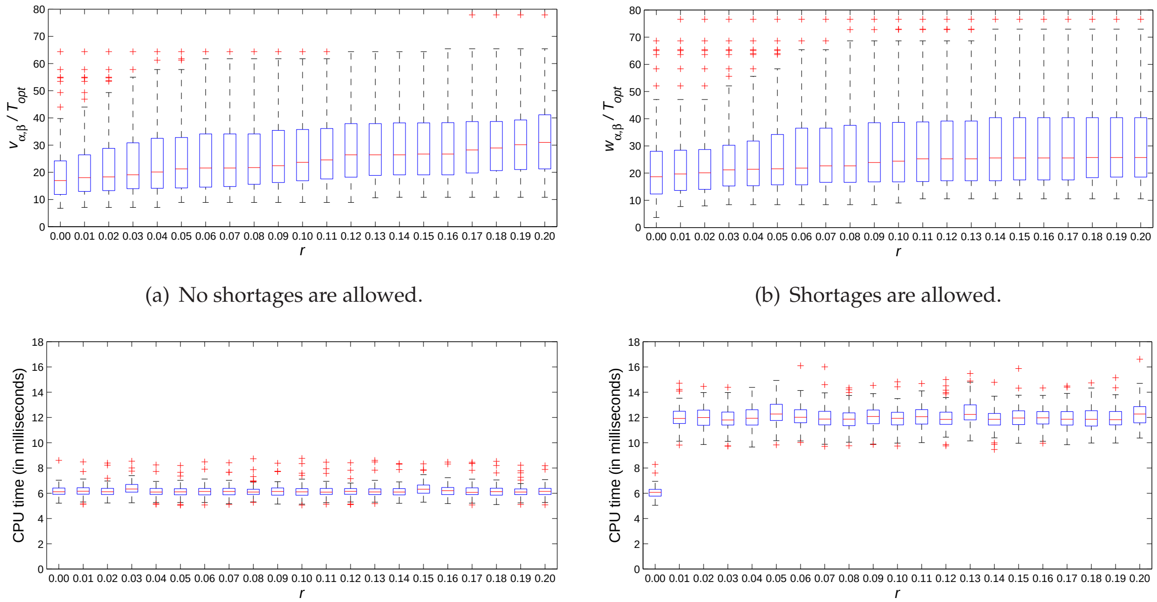
(b) Shortages are allowed.

(c) No shortages are allowed.

(d) Shortages are allowed.

Figure 10: Quality of the upper bound on T_{opt} for polyhedral concave functions with respect to r and associated solution times.

For polyhedral concave functions, the upper bound d_{min} on T_{opt} is generally quite tight both for problems with and without shortages. See figures 10(a)-10(b). Unfortunately, we cannot compute d_{min} exactly for piecewise polyhedral concave functions, and we can only determine the upper bounds $v_{\alpha,\beta}$ and $w_{\alpha,\beta}$ on d_{min} for problems with no shortages and with shortages, respectively. (See Examples 4.1-4.2.) Both $v_{\alpha,\beta}$ and $w_{\alpha,\beta}$ rely on the existence of an affine upper bound on $c(\cdot)$ and are not particularly tight as depicted in figures 11(a)-11(b). Thus,



(a) No shortages are allowed. (b) Shortages are allowed.

(c) No shortages are allowed. (d) Shortages are allowed.

Figure 11: Quality of the upper bound on T_{opt} for piecewise polyhedral concave functions with respect to r and associated solution times.

in the future we may formulate the problem of determining the best affine upper bound as an optimization problem which would replace the approach described in Section B.2.

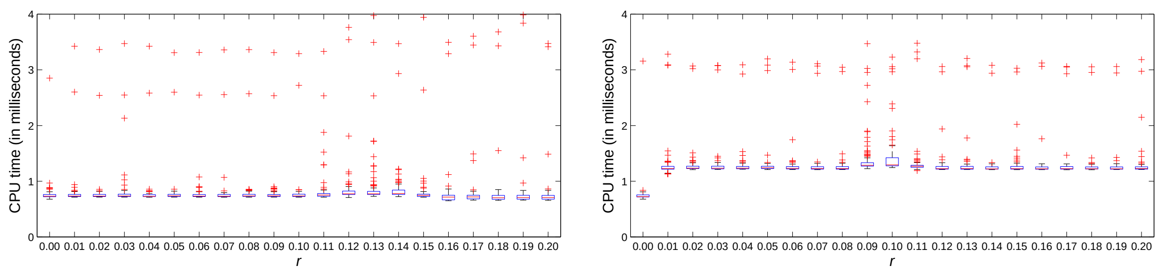
The values of the upper bounds d_{min}, $v_{\alpha,\beta}$, and $w_{\alpha,\beta}$ on T_{opt} are invariant to the inventory holding cost rate r; however, we observe that the ratios d_{min}/T_{opt}, $v_{\alpha,\beta}/T_{opt}$, and $w_{\alpha,\beta}/T_{opt}$ are not significantly affected by increasing values of r in figures 10(a)-10(b) and 11(a)-11(b). These graphs exhibit only slightly increasing trends as r increases from zero to 0.20.

Overall, figures 10(c)-10(d) and 11(c)-11(d) demonstrate clearly that we can solve for the economic order quantity very quickly even when a general class of transportation costs as described in this paper are incorporated into the model. This is important in its own right and also suggests that decomposition approaches may be a promising direction for future research for more complex lot sizing problems with transportation costs. The algorithms proposed in this paper or their extensions may prove useful to solve the subproblems in such methods very effectively.

Two major factors determine the CPU times. First, our algorithms are built on solving many EOQ problems with linear transportation-purchase functions. These subproblems possess analytical solutions if no shortages are allowed or $r = 0$ when shortages are allowed. Otherwise, a line search must be employed to solve these subproblems which is computationally more costly. This fact is clearly displayed in figures 10(c)-10(d) and 11(c)-11(d). Second, the solution times depend on the number of subproblems to be solved which explains the longer solution times for piecewise polyhedral concave $c(\cdot)$ compared to those for polyhedral concave $c(\cdot)$. We will take up on this issue later again in this section.

6.2 Carload Discount Schedule. In the remainder of our computational study we focus our attention on the carload discount schedule which is widely used in the literature (Nahmias, 1997). We first start by providing a negative answer to Nahmias' claim that solving the EOQ model under the carload discount schedule with two linear pieces may be very hard, and then propose some managerial insights into the nature of the optimal order policy under this transportation cost structure. Finally, we conclude by analyzing the impact of the number of linear pieces on $c(\cdot)$ and the improved upper bound on T_{opt} given in relation (45) on the solution times for the carload discount schedule with nonincreasing setup costs; see, Example 5.1.

One hundred instances with transportation-purchase functions based on the carload discount schedule with two linear pieces are generated very similarly to those with piecewise polyhedral $c(\cdot)$ described previously. We only point out the differences in the data generation scheme. The transportation-purchase function $c(\cdot)$ is polyhedral concave over each interval $((k-1)C, kC]$, $k = 1, 2, \ldots$, where $C = 250$ is the truck capacity. All truck setup costs are set to zero. The slope of the first piece of the carload discount schedule is distributed as $U[0.50, 1.00]$, and the cost of a truck increases linearly until the full truck load cost is incurred at a point chosen randomly in the interval $[0.25C, 0.75C]$. Any additional items do not contribute to the cost of a truck. These 100 instances are solved for varying values of r both with and without shortages. The CPU times for solving these instances are plotted in Figure 12. The median CPU time is below 1.5 milliseconds in all cases, and the maximum CPU time is about 4 milliseconds. Clearly, the economic order quantity may be identified very effectively under the classical carload discount schedule.



(a) No shortages are allowed.

(b) Shortages are allowed.

Figure 12: Solution times for the classical carload discount schedule.

In the next set of experiments, our main goal is to illustrate the dynamics of the model if the transportation costs are dictated by the classical carload discount schedule. In particular, we focus on the interplay between the inventory holding costs and the structure of the classical carload discount schedule. We create ten instances for each combination of $h \in \{0.50, 1.00, 1.50, 2.00, 2.50\}$ and $b \in \{\infty, 5h\}$. For all of these instances, we set $\lambda = 1500$, $a = 100$, $r = 0$, and $C = 250$. Then, for each instance we keep the cost of a full truck load fixed at 100 but consider different slopes for the carload discount schedule as depicted in Figure 13. The main insight conveyed by the results in Figure 14 is that the optimal schedule strives to use a truck at full capacity unless holding inventory is expensive. For instance, in Figure 14(a) the optimal order quantity is always 3 full truck loads for $h = 0.50$ until the carload schedule turns into an (ordinary) linear transportation cost function. On the other hand, for $h = 2.50$ the optimal order quantity diverts from a full truck load if the full cost of a truck is incurred at $0.70C$ or higher.

Finally, we explore how the solution times scale as a function of the number of subproblems to be solved. Recall that earlier in this section we argued that the solution times depend heavily on the number of linear pieces on the transportation-purchase function $c(\cdot)$. We illustrate that this relationship is basically linear - as expected - by solving the EOQ model under a general carload discount schedule. That is, the truck setup costs

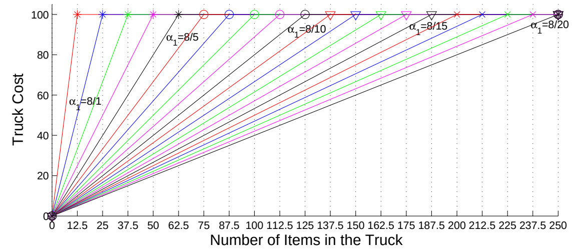
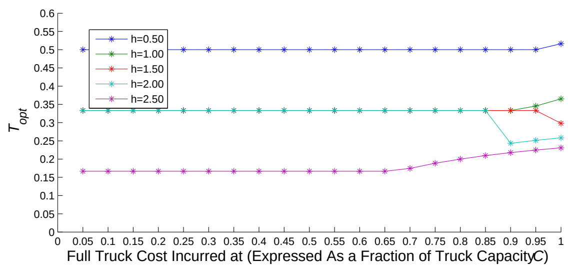
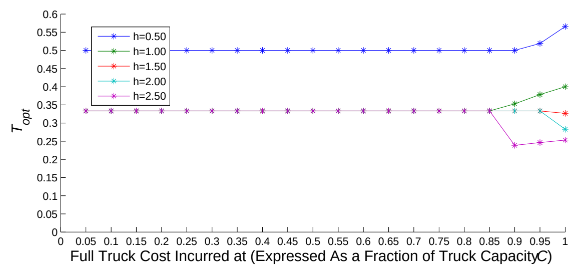


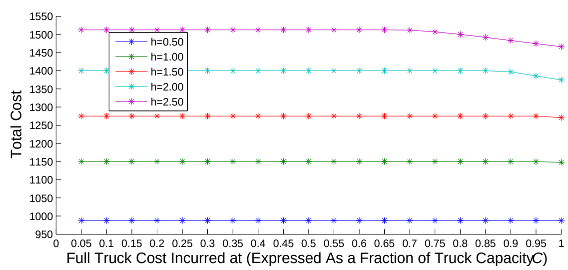
Figure 13: Alternate carload discount schedules for the same capacity and full truck load cost.



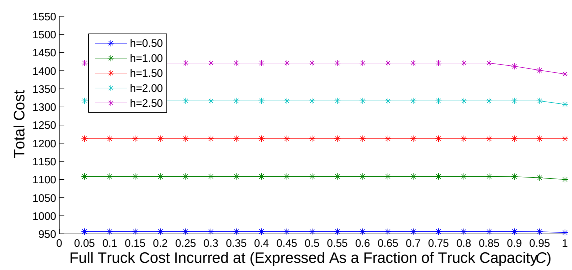
(a) No shortages are allowed.



(b) Shortages are allowed.



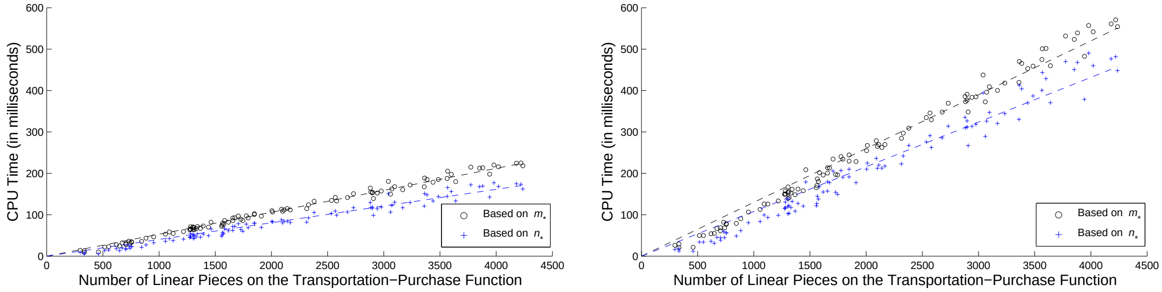
(c) No shortages are allowed.



(d) Shortages are allowed.

Figure 14: Optimal cycle length and cost for alternate carload discount schedules and different h values.

are decreasing although the trucks are identical, and there may be multiple breakpoints on the transportation-purchase function. (See Figure 9). We generate 100 instances where we set $\lambda = 1500, a = 200, h = 0.05, r = 0.10$, and $b = 0.25$ if shortages are allowed, and $b = \infty$ otherwise. As before, the truck capacity is $C = 250$, and the transportation-purchase function $c(\cdot)$ is polyhedral concave over each interval $((k-1)C, kC]$, $k = 1, 2, \ldots$. The setup cost of the first truck is distributed as $U[50, 100]$, and for each following truck the setup cost is computed by multiplying that of the previous truck with a random number in the range $[0.50, 1.00]$. For each truck, the number of breakpoints on the discount schedule is created randomly in the range $[2, 20]$, and the distance between two successive breakpoints is calculated by multiplying the remaining capacity of the truck by a random number in $[0.05, 0.20]$. The slope of the first linear piece is distributed as $U[0.50, 1.00]$ and subsequent slopes are obtained by multiplying the slopes of the immediately preceding pieces by a random number in the range $[0.80, 1.00]$. The final slope is always zero. In Figure 15, we plot the solution times against the number of subproblems solved and conclude that the relationship between these two quantities is linear. The dotted lines in the figure are fitted by simple linear regression through the origin. We also observe that the relatively tighter upper bound on T_{opt} given in relation (45) for carload discount schedules with nonincreasing setup costs provides computational savings of 22% and 28% on average for instances with and without shortages, respectively.



(a) No shortages are allowed. (b) Shortages are allowed.

Figure 15: Solution times for the carload discount schedule with nonincreasing setup costs and multiple linear pieces.

7. Conclusion and Future Research. In this work, we have analyzed the impact of the transportation cost along with discounts in EOQ-type models. We investigated the structures of the resulting problems and derived bounds on their optimal cycle lengths. Observing that the carload discount schedule is frequently used in the real practice, we have identified a subclass of problems that also includes the well-known carload discount schedule. Due to their special structure, we have shown that the problems within this class are relatively easy to solve. Using our analysis, we have also laid down the steps of several fast algorithms. To support our analysis and results, we have setup a thorough computational study and discussed our observations from different angles. Overall, we have concluded that a large group of EOQ-type problems with transportation costs and discounts can be considered as simple problems and they can be solved very efficiently in almost no time.

In the future, we intend to study the extension of the EOQ-type problems to stochastic single item inventory models with arbitrary transportation costs. There exist models in the literature, where the optimal price is determined along with the optimal order quantity. If the demand-price relationship is one-to-one (as it is the case in most of pricing studies within the realm of EOQ), then we may be able to obtain similar results at the expense of complicating the analysis. Lastly, a natural follow-up work could be incorporating the transportation costs and discounts into multi-item lot-sizing. We then need to think about consolidation of many items into a single shipment, which may yield significant savings in transportation costs without comparable increases in inventory holding costs.

Appendix A. Existence Result. In this appendix we show that the optimization problem (P) with $H(\cdot,\cdot)$ belonging to $\mathcal{H}$ and $c(\cdot)$ an increasing left continuous function has an optimal solution.

Definition A.1 *A function $f : [0,\infty) \to \mathbb{R}$ is called lower semi-continuous at $x \geq 0$ if*

$$\liminf_{k\uparrow\infty} f(x_k) \geq f(x)$$

for every sequence x_k satisfying $\lim_{k\uparrow\infty} x_k = x$. The function is called lower semi-continuous if it is lower semi-continuous at every $x \geq 0$.

It is well known (see, for example, Rockafellar (1972) or Frenk and Kassay (2005)) that the function $f : [0,\infty) \to \mathbb{R}$ is lower semi-continuous if and only if for every $\alpha \in \mathbb{R}$ the lower level set

$$L(\alpha) = \{x \in [0,\infty) : f(x) \leq \alpha\}$$

is closed. It is now possible to show the next result. Observe we extend the EOQ-type function $T \mapsto H(c(T), T)$ defined on $(0, \infty)$ to $[0, \infty)$ by defining $H(c(0), 0) = \infty$.

Lemma A.1 *If $c(\cdot)$ is an increasing left continuous function and H belongs to $\mathcal{H}(\cdot,\cdot)$, then the function $T \mapsto H(c(T), T)$ is lower semi-continuous on $[0, \infty)$.*

Proof. By the previous remark we have to show that the lower level set $L(\alpha) := \{T \in [0, \infty) : H(c(T), T) \leq \alpha\}$ is closed for every $\alpha \in \mathbb{R}$. Let $\alpha \in \mathbb{R}$ be given and consider some sequence $(T_n)_{n \in \mathbb{N}} \subseteq L(\alpha)$ satisfying $\lim_{k \uparrow \infty} T_k = T$. Consider now the following two mutually exclusive cases. If there exists an infinite set $\mathcal{N}_0 \subseteq \mathbb{N}$ satisfying $T \leq T_n$ for every $n \in \mathcal{N}_0$, then by the monotonicity of c it follows $c(T) \leq c(T_n)$ for every $n \in \mathcal{N}_0$. This implies by the monotonicity of the function $x \mapsto H(x, T)$ for every $T > 0$ that

$$H(c(T), T_n) \leq H(c(T_n), T_n) \leq \alpha$$

for every $n \in \mathcal{N}_0$. Since $\mathcal{N}_0$ is an infinite set and $\lim_{n \in \mathcal{N}_0 \uparrow \infty} T_n = T$ we obtain by the continuity of $x \mapsto H(c(T), x)$ that

$$H(c(T), T) = \lim_{n \in \mathcal{N}_0 \uparrow \infty} H(c(T), T_n) \leq \alpha.$$

If there does not exist an infinite set $\mathcal{N}_0 \subseteq \mathbb{N}$ satisfying $T \leq T_n$ for every $n \in \mathcal{N}_0$, then clearly one can find a strictly increasing sequence $(T_n)_{n \in \mathcal{N}_1}$ satisfying $\lim_{n \in \mathcal{N}_1} T_n \uparrow T$. This implies by the left continuity of c that $\lim_{n \in \mathcal{N}_1} c(T_n) = c(T)$ and applying now the continuity of H it follows

$$\alpha \geq \lim_{n \in \mathcal{N}_1} H(c(T_n), T_n) = H(c(T), T)$$

Hence for both cases we have shown that $H(c(T), T) \leq \alpha$ and so $L(\alpha)$ is closed. □

By Lemma A.1 and $H(\cdot,\cdot)$ belonging to $\mathcal{H}$ implying

$$\lim_{T \downarrow 0} H(x, T) = \lim_{T \uparrow \infty} H(x, T) = \infty$$

for every $x \geq 0$ we obtain by the Weierstrass-Lebesgue lemma that the optimization problem (P) has an optimal solution (Aubin, 1993).

Appendix B. Computing The Affine Upper Bounds. In this appendix, we demonstrate how an affine function may be computed that satisfies (11) for both the carload discount schedule and the piecewise polyhedral concave transportation-purchase functions.

B.1 The Carload Schedule. Without loss of generality, we only consider carload discount schedules with nonincreasing truck setup costs which also includes trucks with identical setup costs as a special case. Similar to the construction in Example 5.1, we let $g : (0, C] \to \mathbb{R}$ be an increasing polyhedral concave function satisfying $g(0) = 0$ and s_i with $s_i \geq s_{i-1} \geq 0$, $i \geq 1$ be the setup cost of the ith truck. We then define

$$c(Q) = \begin{cases} 0, & \text{if } Q = 0; \\ g(Q) + s_1, & \text{if } 0 < Q \leq C, \end{cases}$$

where

$$g(Q) = \min_{1 \leq k \leq N} \{\alpha_k Q + \beta_k\} \tag{46}$$

with $\alpha_1 > \alpha_2 > \cdots > \alpha_N \geq 0$ and $0 = \beta_1 < \beta_2 < \cdots < \beta_N$, and

$$c(Q) = \sum_{i=1}^{n+1} s_i + ng(C) + g(Q - nC)$$

for $nC < Q \leq (n+1)C$ with integer $n \geq 1$ (see Figure 16).

Lemma B.1 *For a discount carload schedule with nonincreasing setup costs $s_i \geq 0$, $i \geq 1$ it follows that*

$$c(Q) \leq \alpha Q + \beta,$$

where $\alpha = \max(\alpha_1, c(C)C^{-1})$ *and* $\beta = s_1$.

Proof. Since $s_1 \geq 0$, we have $c(0) = 0 \leq s_1 = \beta$. For $0 < Q \leq C$, it follows by relation (46) that

$$c(Q) = \min_{1 \leq k \leq N} \{\alpha_k Q + \beta_k\} + s_1 \leq \alpha_1 Q + s_1 \leq \max(\alpha_1, c(C)C^{-1})Q + s_1 = \alpha Q + \beta.$$

For $nC < Q \leq (n+1)C$ with integer $n \geq 1$, we have

$$\begin{aligned} c(Q) = \textstyle\sum_{i=1}^{n+1} s_i + ng(C) + g(Q - nC) \quad &\leq (n+1)s_1 + ng(C) + g(Q - nC) \\ &= n(s_1 + g(C)) + g(Q - nC) + s_1 \\ &= nc(C) + g(Q - nC) + s_1 \\ &\leq \max(\alpha_1, c(C)C^{-1})nC + \min_{1 \leq k \leq N}\{\alpha_k(Q - nC) + \beta_k\} + s_1 \\ &\leq \max(\alpha_1, c(C)C^{-1})nC + \alpha_1(Q - nC) + s_1 \\ &\leq \max(\alpha_1, c(C)C^{-1})Q + s_1 \\ &= \alpha Q + \beta. \end{aligned}$$

□

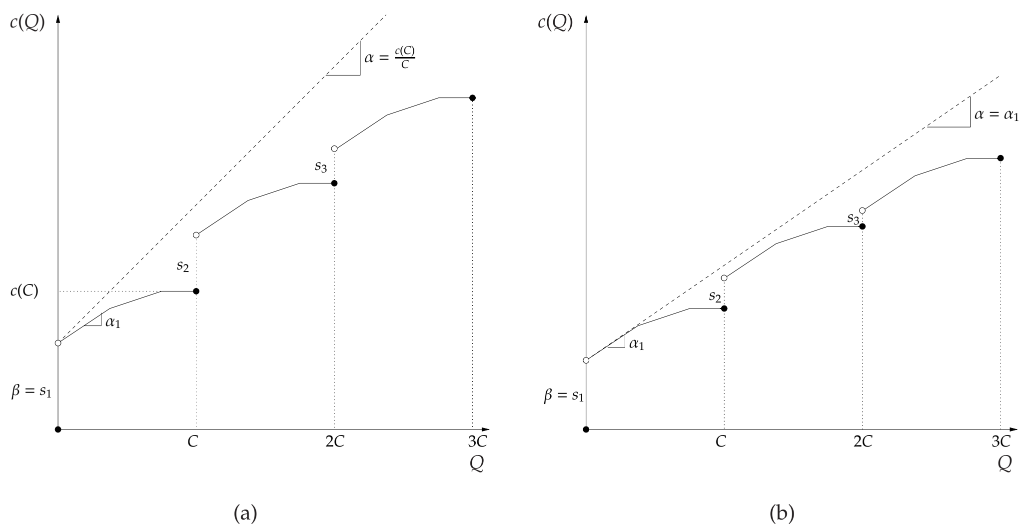


Figure 16: Construction of an upper bound for the carload discount schedule.

B.2 Piecewise Polyhedral Concave Functions. We next compute an affine bound for a piecewise polyhedral concave function over the predefined interval $[0, q_K]$, where K corresponds to the number of trucks under consideration. Let $g_k : (q_{k-1}, q_k] \to \mathbb{R}$ be an increasing polyhedral concave function satisfying $g_k(0) = 0$ and $s_i \geq 0$ be the setup cost of the ith truck. We then define

$$c(Q) = \begin{cases} 0, & \text{if } Q = 0; \\ g_1(Q) + s_1, & \text{if } 0 < Q \leq q_1; \\ \sum_{l=1}^{k-1} (g_l(q_l) + s_l) + g_k(Q - q_{k-1}) + s_k, & \text{if } q_{k-1} < Q \leq q_k, \end{cases}$$

where $2 \leq k \leq K$ and

$$g_k(Q) = \min_{1 \leq n \leq N_k} \{\alpha_{nk} Q + \beta_{nk}\}$$

with $\alpha_{1k} > \alpha_{2k} \cdots > \alpha_{Nk} \geq 0$ and $0 = \beta_{1k} < \beta_{2k} < \cdots < \beta_{Nk}$ (see Figure 17).

LEMMA B.2 *Let $u : [0, q_K] \to \mathbb{R}$ be the piecewise linear convex function given by*

$$u(Q) = \max\left\{\alpha_{N_1 1}Q + \beta_{N_1 1} + s_1, \max_{2\le k\le K}\left\{\sum_{l=1}^{k-1}(g_l(q_l) + s_l) + \alpha_{N_k k}(Q - q_{k-1}) + \beta_{N_k k} + s_k\right\}\right\}.$$

Then, it follows for $0 \le Q \le q_K$ that

$$c(Q) \le \alpha Q + \beta,$$

where $\alpha = \frac{u(q_K)-u(0)}{q_K}$ and $\beta = u(0) \ge 0$.

PROOF. Since $u(\cdot)$ is convex, it follows for $0 \le Q \le q_K$ that

$$u(Q) \le \frac{u(q_K) - u(0)}{q_K}Q + u(0) = \alpha Q + \beta. \tag{47}$$

Clearly, $c(0) = 0 \le u(0) = \beta$. For $0 < Q \le q_1$, we have

$$c(Q) = \min_{1\le n\le N_1}\{\alpha_{n1}Q + \beta_{n1}\} + s_1 \le \alpha_{N_1 1}Q + \beta_{N_1 1} + s_1 \le u(Q).$$

Similarly, for $q_{k-1} < Q \le q_k$ with $2 \le k \le K$, we have

$$c(Q) = \sum_{l=1}^{k-1}(g_l(q_l) + s_l) + \min_{1\le n\le N_k}\{\alpha_{nk}(Q - q_{k-1}) + \beta_{nk}\} + s_k \le \sum_{l=1}^{k-1}(g_l(q_l) + s_l) + \alpha_{N_k k}(Q - q_{k-1}) + \beta_{N_k k} + s_k \le u(Q).$$

The result then follows by using relation (47). □

This construction is illustrated in Figure 17 where $K = 3$.

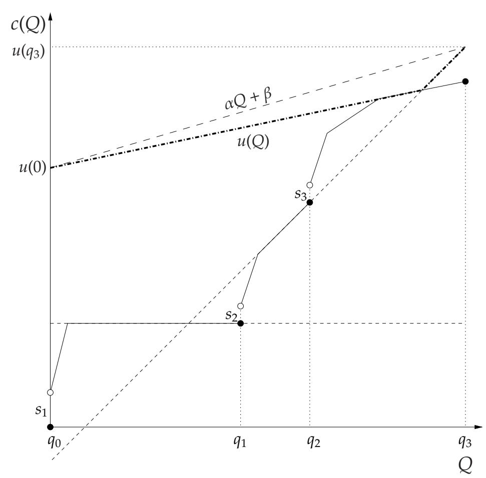


Figure 17: Construction of an upper bound for a piecewise polyhedral concave transportation-purchase function ($K = 3$).

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