Optimality of Linearity with Collusion and Renegotiation*

Mehmet Barlo† — Sabancı University

Ayça Özdoğan‡ — TOBB-ETU

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Abstract

This study analyzes a continuous-time N-agent Brownian hidden-action model with exponential utilities, in which agents' actions jointly determine the mean and the variance of the outcome process. In order to give a theoretical justification for the use of linear contracts, as in Holmstrom and Milgrom (1987), we consider a variant of its generalization given by Sung (1995), into which collusion and renegotiation possibilities among agents are incorporated. In this model, we prove that there exists a linear and stationary optimal compensation scheme which is also immune to collusion and renegotiation.

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†FASS, Sabancı University, Tuzla, Istanbul, 34956, Turkey; +90 216 483 9284; barlo@sabanciuniv.edu

‡**Corresponding Author**: TOBB University of Economics and Technology, Department of Economics, Sögütözü Cad. No:43 Sögütözü, Ankara, 06560, Turkey; +90 312 292 4543; aozdogan@etu.edu.tr

1 Introduction

This study analyzes a continuous-time N-agent Brownian hidden-action model where agents jointly determine the mean and the variance of the outcome process at every instant.[1] The main purpose of this paper is to establish a theoretical justification for the use of linear contracts by showing that among the optimal contracts there exist one that is linear in the final output and susceptible to neither collusion nor renegotiation among agents. To that regard, we consider an N-agent Brownian agency model: The principal and N agents, all having CARA (constant absolute risk aversion) utilities, interact over a time interval; and, agents jointly determine the drift and the diffusion rates of a stochastic process governed by a Brownian motion, and can exploit all collusion and renegotiation opportunities at every instant.

In this setting, we prove that there is an optimal stationary and linear sharing rule which is also immune to collusion and renegotiation. Thus, it is as if agents, who can exploit all collusion and renegotiation opportunities, were to choose the mean and the variance only once and the principal were restricted to employ stationary and linear sharing rules.

The nature of the strategic interaction among agents is important. We assume that agents can perfectly observe and verify others' behavior, and can engage in perfectly enforceable side contracts. Hence, agents observing all of the history of their previous choices, results in a *bargaining* among agents at every instance, the outcome of which may be implemented by the agents with a feasible date and state dependent side contracting scheme. Hence, agents, in our model, are able to not only collectively coordinate their efforts, but also share risk optimally; and, our model

[1] For instance, with two agents one could be the sales manager, and the other the finance manager, both affecting the mean and the variance at different rates and with different marginal costs.

does not involve the free-rider concerns of Holmstrom (1982). [2] [3]

On the other hand, the principal knows that this bargaining (induced by her own offer) must result in an ex-ante efficient outcome, yet she does not need to know the particular bargaining weights of the agents. Hence, the requirement is: The optimal contract, an individually rational sharing rule and control laws (for drift and diffusion rates), must be on the Pareto frontier (i.e. be ex-ante efficient) of the bargaining problem induced by itself, at every instance and history.

It is appropriate to point out that ex-ante efficiency implies a collective incentive compatibility condition on the set of all agents, which we will refer to as the *team*. It turns out that thanks to the use of CARA utilities, the specific form of the bargaining rule and the "real" values of agents' bargaining coefficients are not relevant. This is because, whenever all individuals' preferences are CARA, Bone (1998) shows that efficient risk sharing (implied by ex-ante efficiency) entails that agents' utilities can be aggregated with a CARA utility function in which the CARA coefficient of the team (alternatively, the representative agent) is given by the inverse of the summation of agents' CARA coefficients (hence, is less than each one of the CARA coefficients of agents). Moreover, by using bargaining weights where each agent's weight is given by the team's CARA coefficient divided by his CARA coefficient, the share of an agent in the team's state dependent compensation level is simply given by this particular bargaining weight of his. This, in turn, implies that an agent's utility in any date and state under this sharing rule equals to that of the team. In other words, in such a situation interests of the team and each one of the agents' are perfectly aligned.

Thus, when using these particular bargaining weights, it will be as if the principal

[2] The effects of agents forming a team through side contracting on principal's welfare is not clear. See Varian (1990), Itoh (1993), Holmstrom and Milgrom (1990) and Barlo and Özdoğan (2008) for a detailed discussion on this issue.

[3] We refer the reader to Koo, Shim, and Sung (2008) in order to point to a different type of strategic interaction among agents, involving free-rider considerations of Holmstrom (1982).

faces a team with a CARA utility and members with perfectly aligned interests. This, then, provides us with the ability to reduce agents' instantaneous bargaining problem to a form which is suited to use techniques given in Sung (1995), and to prove that there is an optimal and stationary linear contract for the team even when the team can control both the mean and the variance.

Finally, we show that the allocation of the team's stationary and linear compensation to each agent according to the ratio of the team's CARA coefficient divided by his one, is linear and ex-ante efficient. Hence, principal's optimal contract is ex-ante efficient in the bargaining problem that it creates, stationary, and linear, thus is immune to collusion and renegotiation.

The rest of this section will discuss the related literature. In section 2 we present the ingredients of the model and the principal's problem. Section 3 states our main result and the rest of it is devoted to the proof of our main theorem.

1.1 Related Literature

It is well known that static agency models lead to optimal contracts with very complicated shapes even in simple environments.[4] The continuous-time approach, on the

[4]Most of the early literature analyzes static agency settings often by the use of "first-order" approach that replaces incentive constraints of the agents with the first-order conditions for the optimality of agents' problem, ignoring the possible interaction that could take place among the agents. However, as is well known and first pointed out by Mirrlees (1975), the equivalence of the original problem and the problem with this weaker constraint is not immediate. The sufficiency conditions for the validity of first-order approach, as given by Rogerson (1985), involve monotone likelihood ratio condition and convexity of the distribution function. Jewitt (1988) replace convexity of the distribution function with a weaker condition, but the requirements are still quite strong. Grossman and Hart (1983) propose another method to avoid the drawbacks of the first-order approach. They convert principal's problem to a convex programming problem by first finding a cost-minimizing way of implementing a certain action profile and then determining which action should be implemented. On the other hand, both approaches lead to optimal contracts having very complicated shapes even in simple environments. However, it is observed that the real life contracts generally have much simpler forms. As far as empirical evidence is concerned Lafontaine (1992) reports that "franchise contracts generally involve the payment, from the franchisee to the franchisor, of a lump-sum franchise fee as well as a proportion of sales in royalties, with the latter usually constant over all sales

other hand, offers more tractable principal-agent models and simpler optimal contracts than static and discrete-time counterparts. The pioneer work displaying the optimality of linear contracts in a repeated agency setting with exponential utilities is Holmstrom and Milgrom (1987). In this study, they consider a principal-agent pair involved in a repeated agency relation where the agent determines the drift rate of a Brownian motion governing the returns of an asset belonging to the principal who is able to observe only the time-path of the outcome process (at any instant). Due to the lack of income effects with exponential utility functions and time-state independent cost functions, Holmstrom and Milgrom (1987) establishes that the optimal control the agent chooses is time-state independent. Stationarity of the environment, and thus, the stationarity of the optimal control by the agent imply that among all possible compensation schemes, an optimal one is stationary and linear in the final output.[5] Therefore, it is as if the agent were to choose the mean of a normal distribution only once and the principal were restricted to employ linear sharing rules.[6] On the other hand, Schättler and Sung (1993) generalizes the continuous-time principal-agent problem with exponential utility to a larger class of stochastic processes, in which Holmstrom and Milgrom (1987)'s Brownian model is a special case. They use martingale methods to derive necessary conditions for optimality of the agent's problem, and also provide sufficiency conditions for the validity of the first-order approach to the continuous-time principal agent problems, which are simpler compared to the

levels." Furthermore, another observation is given in Slade (1996) where the author notes that only linear contracts are used by the oil companies engaged in franchising in retail-gasoline markets in the city of Vancouver.

[5]More precisely, they prove that an optimal compensation scheme is a linear function of a finite set of time-aggregated "accounts," each describing the number of times a certain outcome occurs (can be different tasks, sales, profits etc). Further aggregation over accounts is not generally possible. In particular, if the profit is aggregation of these accounts, then optimal sharing rule may not be linear in profits.

[6]This gives a resolution for Mirlees (1974) nonexistence result obtained by approximating the first-best solution arbitrarily closely via "two-wage" scheme.

ones given for the sufficiency of the first-order approach in static models.[7] The key restriction of the model presented by Holmstrom and Milgrom (1987) and also by Schättler and Sung (1993) is that the agent is *not* allowed to control the *variance* of the outcome process. Sung (1995), on the other hand, extends Holmstrom and Milgrom (1987)'s Brownian model to the case where the agent can also control the diffusion rate of the Brownian motion governing the returns. Again, time-state independent technology and exponential utility functions imply that at every instant the agent's best responses are time-state independent even when the agent can control the diffusion rate privately. The resulting problem becomes similar to that in Holmstrom and Milgrom (1987) with an additional time-state independent constraint (thus, features a similar stationary decision-making environment), and he proves that the linearity in outcome result holds.[8]

In a study related to our analysis, Koo, Shim, and Sung (2008) present a continuous-time principal-agent model under moral hazard with many agents. Their model is a continuous-time counterpart of Holmstrom (1982) discrete-time model and an extension of Holmstrom and Milgrom (1987) with a team of finitely many agents. Indeed, an important difference from our analysis is that in theirs, the principal has N production tasks one for each one of the N agents. They use Schättler and Sung (1993)'s martingale method to model both the principal and agents' problems, in which all

[7]Sufficiency of the first-order approach to the continuous-time model was conjectured, but not proved in Holmstrom and Milgrom (1987).

[8]For more on the continuous-time approach to principal-agent problems with a single agent, we refer the reader Cvitanic, Wan, and Zhang (2005), Sannikov (2008) and Williams (2003). On the other hand, whether or not the linearity results are due to the specific nature of continuous-time models is analyzed by Schättler and Sung (1997) and Hellwig and Schmidt (2002). Schättler and Sung (1997) relate the first-order approaches for discrete and continuous-time problems by considering continuous-time agency problems as limiting cases of discrete-time formulations. On the other, Hellwig and Schmidt (2002) derive the continuous-time model as a limit of discrete-time models and prove that Holmstrom and Milgrom (1987)'s result, optimal compensation scheme being linear in accounts in continuous-time Brownian motion model, can be approximated by a sequence of optimal compensation schemes in discrete models.

agents jointly choose probability distributions of given outcome processes. However, agents, in their model, cannot observe each other, thus, cannot write binding side contracts. Each one of their agents chooses an effort level as if all other agents' effort choices have already been made. Then, they show that optimal team contracts with finitely many agents are also linear in all outcomes (produced separately by each agent). For their linearity result, the formulation involving the simultaneous-move game played by agents is important to preserve Holmstrom and Milgrom (1987)'s stationary decision making environment for the principal, which, in turn, is crucial to obtain linear contracts as optimal ones. In our model, we do not restrict attention to separate production processes, and our agents can perfectly observe each other and can engage in renegotiable side contracting.

2 Model

We consider a variant of the model given by Sung (1995), a generalization of the Brownian model of Holmstrom and Milgrom (1987). In our version, there are N agents who can control not only the mean but also the variance of output, and instantaneously exploit all collusion opportunities knowing that an arrangement made at an instance may be renegotiated in the future. In this setting, we prove that among optimal sharing rules there exists a stationary and linear one.

The principal and N agents interact over the time interval $t \in [0, 1]$ during which agents jointly determine (with strictly convex costs) the instantaneous drift and diffusion rates of a stochastic process governed by a Brownian motion defined by $\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t$. As usual, the intermediate outcome X_t should be thought of as the total returns up to period $t \in [0, 1]$ where μ_t and σ_t^2 are the instantaneous mean and variance of the accumulated return at time t, and B_t is the standard Wiener process. We keep the standard restriction that the mean, the variance and intermediate accumulated returns are neither observable nor verifiable by the principal. However, X_1, the level of accumulated returns at the end of the project, is both observable and verifiable by the principal.

At the beginning of the project, the principal and agents agree upon a contract, i.e. salary rules S_i, $i = 1, ..., N$, [9] and control laws $(\mu(.), \sigma(.))$. Salaries are payable at time 1, the end of the project, according to the salary functions agreed upon at time 0 which depend solely on principal's observation of X_1.[10]

It is assumed that the probability space is given by $(\Omega, \mathcal{F}, P)$ where Ω is the space $C = C([0,1])$ of all continuous functions on the interval $[0,1]$ with values in $\Re$. Hence, a particular event $w \in \Omega$ is of the form $w : [0,1] \to \Re$. The control laws μ and σ are $\mathcal{F}_t$-predictable mappings, $\mu : [0,1] \times \Omega \to U$ and $\sigma : [0,1] \times \Omega \to \mathbb{S}$, where U is a bounded open subset of $\Re$ and $\mathbb{S}$ is a compact subset of $\Re_{++}$.[11] These control laws, μ and σ, determine the instantaneous values of μ_s and σ_s at each date $s \in [0,t]$ as functions of the history of the process X up to time t, for every $t \in [0,1]$. Moreover, we adopt the notation of calling the controls determined at time t, by μ_t and σ_t, i.e. $\mu_t = \mu(t, X)$ and $\sigma_t = \sigma(t, X)$. Furthermore, we assume σ satisfies a uniform Lipschitz condition in $Z, \bar{Z} \in C[0,1]$ and there exists a constant K such that $|\sigma(t, Z) - \sigma(t, \bar{Z})| \leq K \sup_{0 \leq s \leq t} \|Z(s) - \bar{Z}(s)\|$.

[9] With an abuse of notation, we also denote the set of agents by N.

[10] It is appropriate to point out that this formulation is consistent with our hypothesis of the mean and variance being unobservable and/or nonverifiable by the principal. Indeed, if $(S_i)_{i \in N}$ were to depend on the entire process $\{X_t\}_t$, implying the requirement that $\{X_t\}_t$ is both observable and verifiable by the principal, then the principal could infer $\{\mu_t\}_t$ and/or $\{\sigma_t\}_t$. For more on this issue, we refer the reader to Sung (1995) footnote 7 and 8.

[11] Y is $\mathcal{F}_t$-predictable: Suppose $Z : [0,1] \times \Omega \to \Re$ is measurable, i.e. for any Borel set B in $\Re$, the inverse image $\{(t, w) | Z_t(w) \in B\}$ is in $\mathcal{B}([0,1]) \times \mathcal{F}_1$ ($\mathcal{B}([0,1])$ is the σ-algebra of Borel sets on $[0,1]$); and let $\mathcal{L}$ be the family of such measurable processes Z whose sample paths are left-continuous and Z_t is $\mathcal{F}_t$-measurable for all $t \in [0,1]$ ($\mathcal{F}_t$-adapted). Let $\mathcal{G}$ be the smallest σ-algebra of subsets of $[0,1] \times \Omega$ such that all the processes in $\mathcal{L}$ are $\mathcal{G}$-measurable. Then a process Y is $\mathcal{F}_t$-predictable if $Y(t, w)$ is $\mathcal{G}$-measurable. Any $\mathcal{F}_t$-adapted measurable process with left-continuous sample paths is predictable.

We let agents' instantaneous time-state independent cost functions be given by $c_i(\mu_t, \sigma_t)$, where $c_i : U \times \mathbb{S} \to \Re$, $i \in N$ and is assumed to be twice continuously differentiable. We assume c_i and $c_{i\mu}$ (derivative with respect to mean) are bounded and are strictly increasing, i.e. $c_{i\mu}, c_{i\mu\mu} > 0$. The total costs incurred by agent i is given by $\int_0^1 c_i(\mu_t, \sigma_t)\mathrm{d}t$, $i \in N$.

All the parties involved are von Neumann-Morgenstern (henceforth, abbreviated as vNM) utility maximizers having exponential utilities with constant absolute risk aversions. We denote the coefficient of absolute risk aversion for the principal by R and those for the agents by r_i, $i \in N$. The reservation certainty equivalent figures for the agents are given by W_{i0}, $i \in N$.

We assume that at each instance $t \in [0, 1]$, every agent observes $\{X_s, \mu_s, \sigma_s\}_{s \leq t}$. Therefore, when modeling a dynamic interaction among agents (choosing effort levels with the set of information they possess) a consistent formulation is one that involves the use of perfect information among agents: At any instance each agent observes all the others' previous choices and all the previous levels of the instantaneous accumulated returns. Therefore, because that at any instance $t \in [0, 1]$ the whole history $\{X_s, \mu_s, \sigma_s\}_{s \leq t}$ (along with all the previous effort choices) are observable and verifiable by all agents and salary functions $(S_i)_{i \in N}$ are determined by the principal at the beginning of the project, a *bargaining problem* among agents emerges due to collusion at each instance $\tau \in [0, 1]$ for a given history $\{X_s, \mu_s, \sigma_s\}_{s \leq \tau}$. Given any state, the outcome of this bargaining then can be implemented via a state contingent binding contract, i.e. the salary rules and control laws, drafted and agreed upon in period zero, specifying the arrangement among agents for each possible date and state. [12]

Ex-ante efficiency implied by agents' bargaining necessitates that there should not

[12]Because that this state contingent contract is drafted and agreed upon at date zero, we will have the participation constraint in the agents' problem formulated with only the date zero information.

be any history and state and any other feasible contract that every agent prefers to the one that was agreed upon. This, then, brings about optimal risk sharing among agents. And, given any history, any utilitarian bargaining rule and bargaining weights $\theta = (\theta_1, ..., \theta_N) \in \Delta$ (where Δ denotes the N dimensional simplex), the ex-ante efficiency requirement can be handled by Bone (1998) thanks to the use of CARA utility functions. The details about how this task is accomplished will be presented in the next section. Nevertheless, it is appropriate to point out that Bone (1998) considers a group (alternatively, team) of CARA agents involved in a static utilitarian bargaining (with given bargaining weights) over a state contingent financial returns. It establishes that the collective behavior of agents can be represented by a vNM utility function which, in fact, is CARA, with a coefficient given by $r_c \equiv \left(\sum_{i \in N} \frac{1}{r_i}\right)^{-1}$. Furthermore, Bone (1998) shows that agent i's share in a given state from the team's compensation involves (1) a fixed, state-independent constant payment (determined by the profile of agents' CARA coefficients and bargaining weights: $\frac{r_c}{r_i} \sum_{j \in N} \frac{\ln(\theta_i r_i) - \ln(\theta_j r_j)}{r_j}$), plus (2) a uniform proportion (depending only on the profile of agents' CARA coefficients: $\frac{r_c}{r_i}$) of the team's state contingent compensation. Across all individuals, the proportions sum to unity, while the fixed payments sum to zero. Hence, various efficient state contingent distributions of team's compensation must differ only in the fixed payments, and to any zero-sum vector of fixed payments, there corresponds an efficient contingent distribution.

The principal is aware of the bargaining among agents, and knows that it is induced by her own offer. Yet, she does not know the particular bargaining rule or weights of the agents. She simply knows that the bargaining problem must produce an ex-ante efficient outcome. Hence, principal's optimal contract, a sharing rule for each agent and control laws (for drift and diffusion rate), must be on the Pareto frontier of the agents' bargaining problem starting from any given history.

Definition 1 (Principal's Problem) *Principal chooses salary functions* $(\hat{S}_i)_{i\in N}$*, which depend only on* X_1*, and control laws* $(\hat{\mu}, \hat{\sigma})$ *such that*

$$\left((\hat{S}_i)_{i\in N}, \hat{\mu}, \hat{\sigma}\right) \in \operatorname{argmax}_{((S_i)_{i\in N}, \mu, \sigma)} E\left[-\exp\left\{-R\left(X_1 - \sum_{i=1}^{N} S_i(X_1)\right)\right\}\middle| \mathcal{F}_0\right]$$

subject to

(i) *(Feasibility) For all* $t \in [0,1]$

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t,$$

(ii) *(Individual rationality) For all* $i = 1, ..., N$,

$$E\left[-\exp\left\{-r_i\left(S_i - \int_0^1 c_i(\mu_t, \sigma_t)\mathrm{d}t\right)\right\}\middle| \mathcal{F}_0\right] \geq -\exp\{r_i W_{i0}\}.$$

(iii) *(Bargaining: Agents' problem) Salary functions* $(S_i)_{i\in N}$*, and control laws* (μ, σ) *must be such that for all* $t \in [0,1]$ *and* $\{X_s, \mu_s, \sigma_s\}_{s\leq t}$*,* $((S_i)_{i\in N}, \mu, \sigma)$ *is on the Pareto frontier of the agents' bargaining problem, i.e. for some some* $\theta \in \Delta$*,*

$$E\left[\theta_i r_i \left(\exp\left\{-r_i\left(\tilde{S}_i(X) - \int_0^1 c_i(\tilde{\mu}_\tau, \tilde{\sigma}_\tau)\mathrm{d}\tau\right)\right\}\right)\middle| \mathcal{F}_t\right] \quad (1)$$
$$= E\left[\theta_j r_j \left(\exp\left\{-r_j\left(\tilde{S}_j(X) - \int_0^1 c_j(\tilde{\mu}_\tau, \tilde{\sigma}_\tau)\mathrm{d}\tau\right)\right\}\right)\middle| \mathcal{F}_t\right]$$

for all $i, j \in N$*, where for all* $i \in N$ *salary functions and control laws given by* $\left((\tilde{S}_i)_{i\in N}, \tilde{\mu}, \tilde{\sigma}\right)$*,* $\tilde{S}_i : [0,1] \times \Omega \to \Re$ *(where* $\tilde{S}_i(t, X)$ *is interpreted as the compensation of agent* i *in period 1 formed with the information in period* t*) and* $\tilde{\mu} : [0,1] \times \Omega \to U$ *and* $\tilde{\sigma} : [0,1] \times \Omega \to \mathbb{S}$*, are all* $\mathcal{F}_t$*-predictable mappings* *and*

$$\mathrm{d}X_\tau = \tilde{\mu}_\tau \mathrm{d}\tau + \tilde{\sigma}_\tau \mathrm{d}B_\tau, \quad \tau \geq t \tag{2}$$

$$\sum_{i=1}^{N} \tilde{S}_{i,t}(X_1) \leq \sum_{i=1}^{N} S_i(X_1), \quad \textit{for all } t \in [0,1] \textit{ and every } X_1 \in \Omega \tag{3}$$

$$\begin{aligned} E\left[-\exp\left\{-r_i\left(\tilde{S}_i - \int_0^1 c_i(\tilde{\mu}_\tau, \tilde{\sigma}_\tau)\mathrm{d}\tau\right)\right\}\middle|\, \mathcal{F}_0\right] \\ \geq E\left[-\exp\left\{-r_i\left(S_i - \int_0^1 c_i(\mu_\tau, \sigma_\tau)\mathrm{d}\tau\right)\right\}\middle|\, \mathcal{F}_0\right] \end{aligned} \tag{4}$$

The feasibility and individual rationality requirement in the above definition is standard. On the other hand, collusion concerns result in some modifications to the standard case, and they are discussed below.

Agents' ability to employ binding side contracts and to observe each others' choices and history of accumulated returns defines their interaction as a bargaining problem. We assume that agents are rational, and thus, their agreement is ex-ante efficient given the principal's offer. This implies that at any instance $t \in [0,1]$ given the history $\{X_s, \mu_s, \sigma_s\}_{s \leq t}$ the control laws for the salaries (specified for instance t and the particular history at that instance, and these salaries are to be paid at the end of the project depending on the particular realization of X_1) the mean and variance must be chosen in order to provide optimal risk sharing among agents (formed with agent specific and time-invariant bargaining weights, i.e. Equation 1), with the following constraints: (1) feasibility concerns given by the physical form of the stochastic return, i.e. Equation 2; (2) any salary arrangements specified for any further histories, as a result of the bargaining at history $\{X_s, \mu_s, \sigma_s\}_{s \leq t}$, has to be a feasible redistribution of salaries specified by the principal, i.e. Equation 3; and (3) voluntary participation of all the agents into this arrangement, meaning that under this arrangement each agent obtains at least as much as he would obtain without this arrangement, i.e. Equation

4. [13]

An important aspect that needs to be pointed out is that agents' problem is restricted to be solved at the beginning of the project, but in every instance based on the observations of the past. This arrangement, then, can be implemented by employing state contingent binding side contracts among agents. That is, agents are required to commit to these side contracts and control laws via the use of binding side contracts which are drafted at the beginning of the project for every possible history of the project. Hence, our formulation of collusion among agents involves renegotiation concerns, because for every given history agents' arrangement drafted and agreed upon at the beginning of the project ensures optimality from that state onwards. The only requirement is that the bargaining weights are stationary. In fact, this restriction can be replaced with one where agents' bargaining weights are given by a measurable function mapping histories in the interior of the N dimensional simplex.

Finally, the principal knows agents' collusion capabilities, and hence, is aware that the control laws associated with her offer may be altered unless it is ex-ante efficient in the bargaining problem that starts at every history possible. Therefore, the principal knows that she is restricted to offer contracts which are ex-ante efficient in any possible history that might arise during the life-span of the project. On the other hand, she does not know the specific bargaining weights that are to be employed in this bargaining. These, then, imply that for the principal's salary offers to be immune against such arrangements, they have to be robust against such bargaining

[13] In fact in this formulation of the voluntary participation constraint, we only consider the grand coalition and not sub-coalitions of agents. Due to that regard our formulation can be interpreted as one in which each agent has the ability to report/present his observable and verifiable information/evidence regarding the past to the principal, whenever he obtains lower payoffs due to some other agents' (a sub-coalition's) arrangements. Therefore, in essence, each agent has a veto power over any one of the feasible side contracting schemes, where the disagreement payoff levels are the ones obtained from the principal's salary offers.

given any feasible bargaining weight agents may employ and any history possible.

3 Optimality of Linearity under Collusion

The following is our main result:

Theorem 1 *There exists a stationary and linear optimal collusion proof and renegotiation proof contract.*

That is, $(S_i^*)_{i\in N}$ are optimal compensation schemes and (μ^*, σ^*) are stationary and optimal control laws solving the principal's problem given in Definition 1, where (μ^*, σ^*) determine $(S_i^*)_{i\in N}$ as follows:

$$S_i^*(\mu^*, \sigma^*)(X_1) = c_i(\mu^*, \sigma^*) + W_{i0} + \theta_i^* \left[S_c^*(\mu^*, \sigma^*)(X_1) - \sum_{j\in N} (c_j(\mu^*, \sigma^*) + W_{j0}) \right],$$

where

$$\begin{aligned} S_c^*(\mu^*, \sigma^*) &= \sum_{i\in N} W_{i0} + \sum_{i\in N} c_i(\mu^*, \sigma^*) + \left[\sum_{i\in N} c_{i\mu}(\mu^*, \sigma^*) \right] (X_1 - X_0) \\ &\quad + \left[\sum_{i\in N} c_{i\mu}(\mu^*, \sigma^*) \right] \mu^* + \frac{r_c}{2} \left[\sum_{i\in N} c_{i\mu}(\mu^*, \sigma^*) \right]^2 \sigma^{*2}, \end{aligned}$$

$$\begin{aligned} r_c &= \frac{\prod_{i\in N} r_i}{\sum_{i\in N} \prod_{j\neq i} r_j}, \\ \theta_i^* &= \frac{r_c}{r_i} = \frac{\prod_{j\neq i} r_j}{\sum_{i\in N} \prod_{j\neq i} r_j} \end{aligned}$$

and the control pair $(\mu^*, \sigma^*) \in U \times \mathbb{S}$ solves the following "static" maximization problem,

$$\Phi^p(\hat{\mu}, \hat{\sigma}) \equiv \hat{\mu}_t + R\left[\sum_{i\in N} c_{i\mu}(\hat{\mu}_t, \hat{\sigma}_t) \right] \hat{\sigma}_t^2 - \sum_{i\in N} c_i(\hat{\mu}_t, \hat{\sigma}_t) - \frac{1}{2}(R + r_c) \left(\sum_{i\in N} c_{i\mu}(\hat{\mu}_t, \hat{\sigma}_t) \right)^2 \hat{\sigma}_t^2 - \frac{R}{2}\hat{\sigma}^2$$

subject to $(\hat{\mu}, \hat{\sigma})$ *maximizing*

$$\Phi^a(\mu_t, \sigma_t; \hat{\mu}_t, \hat{\sigma}_t) \equiv \left[\sum_{i\in N} c_{i\mu}(\hat{\mu}_t, \hat{\sigma}_t)\right]\mu_t - \sum_{i\in N} c_i(\mu_t, \sigma_t) - \frac{1}{2}r_c\left(\sum_{i\in N} c_{i\mu}(\hat{\mu}_t, \hat{\sigma}_t)\right)^2 \sigma_t^2.$$

This Theorem displays that among collusion proof and renegotiation proof arrangements, non-trivial sharing rules contingent on the realization of the accumulated returns at the end of the project, a linear and stationary contract must be among the optimal ones. The main reason of this observation is the lack of income effects and the well behaved nature of the bargaining among agents (both of which are due to CARA utilities), and the use of a Markovian stochastic process as in Holmstrom and Milgrom (1987), Schättler and Sung (1993), and Sung (1995). Thus, our main Theorem displays that this result is robust against colluding and renegotiating agents even when the variance of the stochastic process can be controlled by agents.

The representation of the total salary function S_c involves the following economics interpretation: The first two terms provide agents with their reservation certainty equivalents and actual costs incurred to implement (μ^*, σ^*); the second two terms represent the compensation error as salaries depend on the realized outcomes rather than the actual performances of agents; and finally, the last term involves the risk premia given to agents because of compensation errors. These risk premia are formed with the team's CARA coefficient, r_c, which is less than any of the agents' individual CARA coefficients.

Then, this total compensation is divided among agents as follows: Each one gets his costs and reserve payoffs and some additional compensation which is proportional to the inverse of their coefficients of risk aversions (which, indeed, is the particular bargaining weight with which this contract achieves ex-ante efficiency) from the total compensation net of total costs and total reserve utilities. Recall that when splitting this total share, the principal is not aware of the "real" bargaining powers of the agents. Yet, she knows that for this given contract, the stationary nature of the optimal solutions of the agents' problem for any given date and state (due to CARA utilities and the use of Brownian Motion) implies: Because that principal's stationary offer formed with bargaining weight vector θ^* is on the Pareto frontier of agents' bargaining problem for any date and state, that particular offer must be the optimal one in this very same date that state, even if agents were to employ different bargaining weights in the formation of the date and state contingent side contracts. Thus, this representation shows that the principal sees the agents as a single entity with the cost function given by the sum of individual cost functions, and the CARA coefficient given by $r_c = \frac{\prod_{i\in N} r_i}{\sum_{i\in N}\prod_{j\neq i} r_j} = \left(\sum_{i\in N}\frac{1}{r_i}\right)^{-1}$. The principal knows that, then, the particular sharing of the salaries among agents will be done according to the particular bargaining weight profile θ^*. Moreover, thanks to stationarity and the particular choice of bargaining weights given by θ^* the following compensation scheme is ex-ante efficient due to Bone (1998): First, $\theta_i^* = \frac{r_c}{r_i} = \left(\frac{\prod_{j\neq i} r_j}{\sum_{i\in N}\prod_{j\neq i} r_j}\right)$, implies that an agent's relative bargaining power against the other must equal to inverse ratios of coefficients of risk aversion, i.e. $\frac{\theta_i^*}{\theta_j^*} = \frac{r_j}{r_i}$ for all $i, j \in N$, and $\theta^* \in \Delta$ because $\theta_i^* \in [0, 1]$ and $\Sigma_i \theta_i^*$ equals

$$\sum_i \left(\frac{1}{r_i}\frac{\prod_{j\in N} r_j}{\sum_{k\in N}\frac{1}{r_k}\prod_{j\in N} r_j}\right) = \sum_i \left(\frac{1}{r_i}\frac{1}{\sum_{k\in N}\frac{1}{r_k}}\right) = \frac{\sum_{i\in N}\frac{1}{ri}}{\sum_{k\in N}\frac{1}{r_k}} = 1.$$

Second, agent i's compensation net of agent i's incurred costs and his reserve payoffs from the team's state contingent compensation net of the total costs and total reserve payoffs, does not involve any state-independent constant terms, because

$$\frac{r_c}{r_i}\sum_{j\in N}\left(\frac{\ln(\theta_i^* r_i) - \ln(\theta_j^* r_j)}{r_j}\right) = \frac{r_c}{r_i}\sum_{j\in N}\left(\frac{\ln(\frac{r_c}{r_i} r_i) - \ln(\frac{r_c}{r_j} r_j)}{r_j}\right) = 0.$$

Third, agent i having been paid his incurred costs and reserve payoffs, agent i's uniform proportion of $\frac{r_c}{r_i}$ from the team's state contingent compensation net of the total costs and total reserve payoffs, constitutes the rest of his compensation.

3.1 Proof of Theorem 1

First, we reduce the principal's problem in Definition 1 into the following one where she interacts with the team, rather than with each agent separately, by using the results of Bone (1998) that were discussed above.

Definition 2 *Principal chooses a total salary function for the team $\hat{S}_c$, which depend only on X_1, and control laws $(\hat{\mu}, \hat{\sigma})$, such that*

$$\left(\hat{S}_c, \hat{\mu}, \hat{\sigma}\right) \in \operatorname{argmax}_{(S_c, \mu, \sigma)} E\left[-\exp\left\{-R\left(X_1 - S_c(X_1)\right)\right\} \middle| \mathcal{F}_0\right]$$

subject to

(i) *(Feasibility) For all $t \in [0,1]$*

$$\mathrm{d}X_t = \mu_t \mathrm{d}t + \sigma_t \mathrm{d}B_t,$$

(ii) *(The team's participation)*

$$E\left[-\exp\left\{-r_c\left(S_c - \sum_{i \in N} \int_0^1 c_i(\mu_t, \sigma_t)\mathrm{d}t\right)\right\} \middle| \mathcal{F}_0\right] \geq -\exp\{r_c \sum_{i \in N} W_{i0}\}.$$

(iii) *(The team's problem) Total salary function S_c and control laws (μ, σ) must be* *such that for all* $t \in [0,1]$ *and* $r_c = \frac{\prod_{i\in N} r_i}{\sum_{i\in N}\prod_{j\neq i} r_j}$, (S_c, μ, σ) *must be in*

$$\operatorname{argmax}_{(\tilde{S}_c,\tilde{\mu},\tilde{\sigma})} E\left[-\left(\exp\left\{-r_c\left(\tilde{S}_c(X)-\sum_{i\in N}\int_0^1 c_i(\tilde{\mu}_\tau,\tilde{\sigma}_\tau)\mathrm{d}\tau\right)\right\}\right)\middle|\mathcal{F}_t\right] \quad (5)$$

subject to total salary function and control laws $\left(\tilde{S}_c, \tilde{\mu}, \tilde{\sigma}\right)$ *are all* $\mathcal{F}_t$*-predictable mappings and*

$$\mathrm{d}X_\tau = \tilde{\mu}_\tau \mathrm{d}\tau + \tilde{\sigma}_\tau \mathrm{d}B_\tau, \quad \tau \geq t \quad (6)$$

The principal uses bargaining weights θ^* in the computation of the ex-ante efficient outcome of agents' bargaining problem and proposes contracts implementing that particular solution. The resulting technical convenience is that this method simplifies agents' instantaneous bargaining problem to a form which is suited to use techniques given in Sung (1995). Then, we can apply Sung (1995)'s Theorem to obtain an optimal total salary rule for the team and optimal control laws. In the last step, we divide this optimal total salary among the agents according to the discussion presented at the end of the previous paragraph.

The problem stated in Definition 2 belongs to the class of problems studied by Schättler and Sung (1993) and Sung (1995). Schättler and Sung (1993) provide the first-order approach in the continuous-time setting where the agent controls only the mean of the process. They relate the principal's problem to a stochastic optimal control problem by replacing agent's incentive compatibility condition with the first-order conditions of the agent's problem. These first-order necessary conditions for optimality lead to a semi-martingale representation of agent's salary function. The principal's relaxed problem is formulated by replacing the salary function in the principal's maximization with this semi-martingale representation. Following the same lines, we can come up with a representation rule for the total salary function proposed to the representative agent, i.e. agents are treated as a team by the use of the first-order approach on the representative agent's problem given in definition 2. The sufficiency conditions for the validity of the first-order approach for a general class of continuous-time models provided in Schättler and Sung (1993), are met in our Brownian setting. Then, following the direct implication of Proposition 2 of Sung (1995) which allows the agents control the variance as well as the mean in the Brownian model, we can come up with the following Proposition for the representation of the total salary function and optimal control laws:

Proposition 1 *There exists a stationary and linear optimal contract for the team. That is, S_c^* optimal compensation of the team and (μ^*, σ^*) are stationary and optimal control laws solving the principal's problem given in Definition 2, where (μ^*, σ^*) determine S_c^* as follows:*

$$\begin{aligned} S_c^*(\mu^*, \sigma^*) &= \sum_{i \in N} W_{i0} + \sum_{i \in N} c_i(\mu^*, \sigma^*) + \left[\sum_{i \in N} c_{i\mu}(\mu^*, \sigma^*) \right] (X_1 - X_0) \\ &\quad + \left[\sum_{i \in N} c_{i\mu}(\mu^*, \sigma^*) \right] \mu^* + \frac{r_c}{2} \left[\sum_{i \in N} c_{i\mu}(\mu^*, \sigma^*) \right]^2 \sigma^{*2}, \end{aligned}$$

and the control pair $(\mu^*, \sigma^*) \in U \times \mathbb{S}$ solves the following "static" maximization problem,

$$\Phi^p(\hat{\mu}, \hat{\sigma}) \equiv \hat{\mu}_t + R \left[\sum_{i \in N} c_{i\mu}(\hat{\mu}_t, \hat{\sigma}_t) \right] \hat{\sigma}_t^2 - \sum_{i \in N} c_i(\hat{\mu}_t, \hat{\sigma}_t) - \frac{1}{2}(R + r_c) \Big(\sum_{i \in N} c_{i\mu}(\hat{\mu}_t, \hat{\sigma}_t) \Big)^2 \hat{\sigma}_t^2 - \frac{R}{2} \hat{\sigma}^2$$

subject to $(\hat{\mu}, \hat{\sigma})$ maximizing

$$\Phi^a(\mu_t, \sigma_t; \hat{\mu}_t, \hat{\sigma}_t) \equiv \left[\sum_{i \in N} c_{i\mu}(\hat{\mu}_t, \hat{\sigma}_t) \right] \mu_t - \sum_{i \in N} c_i(\mu_t, \sigma_t) - \frac{1}{2} r_c \Big(\sum_{i \in N} c_{i\mu}(\hat{\mu}_t, \hat{\sigma}_t) \Big)^2 \sigma_t^2.$$

The last step of the proof entails the following allocation of the total salary among agents:

$$\begin{aligned} S_i^*(\mu^*,\sigma^*)(X_1) &= c_i(\mu^*,\sigma^*) + W_{i0} + \theta_i^* \left[S_c^*(\mu^*,\sigma^*)(X_1) - \sum_{j\in N} (c_j(\mu^*,\sigma^*) + W_{j0}) \right] \\ &= c_i(\mu^*,\sigma^*) + W_{i0} + \frac{r_c}{r_i} \left(A_1(X_1 - X_0) + A_2 \right), \end{aligned}$$

where

$$A_1 = \left(\sum_{j\in N} c_{j\mu}(\mu^*,\sigma^*) \right)$$

and

$$A_2 = \left(\sum_{j\in N} c_{j\mu}(\mu^*,\sigma^*) \right) \mu^* + \frac{r_c}{2} \left(\sum_{j\in N} c_{j\mu}(\mu^*,\sigma^*) \right)^2 \sigma^{*2}.$$

Recall that because $c_{i\mu}, c_{i\mu\mu}$ are strictly positive, both A_1 and A_2 are strictly positive.

Next, we show that the individual rationality constraint of agent i, i.e. condition (ii) in Definition 1, is satisfied. First, $E_0 \left[-\exp \left\{ -r_i \left(S_i^* - \int_0^1 c_i(\mu^*,\sigma^*)\mathrm{d}t \right) \right\} \right]$ equals

$$\begin{aligned} & E_0 \left[-\exp \left\{ -r_i \left(W_{i0} + \frac{r_c}{r_i} \left(A_1(X_1 - X_0) + A_2 \right) \right) \right\} \right] \\ & = -\exp\{-r_i W_{i0}\} \, E_0 \left[-\exp\{-r_c \left(A_1(X_1 - X_0) + A_2 \right)\} \right] \\ & = -\exp\{-r_i W_{i0}\} \, E_0 \left[-\exp \left\{ -r_c \left(S_c^*(\mu^*,\sigma^*) - \sum_{j\in N} W_{j0} - \sum_{j\in N} c_j(\mu^*,\sigma^*) \right) \right\} \right]. \end{aligned}$$

Second, because of the individual rationality constraint of the team (condition (ii) in Definition 2) we have

$$E_0 \left[-\exp \left\{ -r_c \left(S_c - \sum_{j\in N} W_{j0} - \sum_{j\in N} c_j(\mu^*,\sigma^*)\mathrm{d}t \right) \right\} \right] \geq 0,$$

and this implies that

$$E_0 \left[- \exp \left\{ -r_i \left(S_i^* - c_i(\mu^*, \sigma^*) \right) \right\} \right] \geq - \exp \left\{ -r_i W_{i0} \right\}$$

establishing that this arrangement is individually rational for agent i, i.e. condition (ii) in Definition 1, is satisfied.

Ex-ante efficiency condition in the agents' bargaining problem in Definition 1 requires that after agents are compensated for their individual costs and reserve payoffs, they have to share the total excess profit (total salary net of total costs and total reserve payoffs) proportional to $\theta_i^* = \frac{r_c}{r_i}$. Therefore, as was discussed at the end of the previous section, the principal's offer is also ex-ante efficient. Because that the total salary is linear in the final outcome and each individual's compensation net of his own costs and reserve payoffs only depends on the team's compensation net of total costs and total reserve payoffs and the profile of agent's CARA coefficients, $(S_i^*)_{i \in N}$ are linear in X_1 as well.

This completes the proof.

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