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Minimizing Value-at-Risk in the Single-Machine Total Weighted Tardiness Problem

Semih Atakan · Birce Tezel · Kerem Bülbül · Nilay Noyan

Abstract The vast majority of the machine scheduling literature focuses on deterministic problems, in which all data is known with certainty a priori. This may be a reasonable assumption when the variability in the problem parameters is low. However, as variability in the parameters increases incorporating this uncertainty explicitly into a scheduling model is essential to mitigate the resulting adverse effects. In this paper, we consider the celebrated single-machine total weighted tardiness (TWT) problem in the presence of uncertain problem parameters. We impose a probabilistic constraint on the random TWT and introduce a risk-averse stochastic programming model. In particular, the objective of the proposed model is to find a non-preemptive static job processing sequence that minimizes the value-at-risk (VaR) measure on the random TWT at a specified confidence level. Furthermore, we develop a lower bound on the optimal VaR that may also benefit alternate solution approaches in the future. In this study, we implement a tabu-search heuristic to obtain reasonably good feasible solutions and present results to demonstrate the effect of the risk parameter and the value of the proposed model with respect to a corresponding risk-neutral approach.

Keywords single-machine; weighted tardiness; stochastic processing times; stochastic scheduling; value-at-risk; probabilistic constraint; stochastic programming.

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1 Introduction

The weighted tardiness objective is a classical due date related performance measure in make-to-order environments. The goal is to find a job (order) processing sequence in order to minimize the total cost incurred due to missed due dates. For a given job, the cost is directly proportional to the associated tardiness. The unit tardiness cost (weight) may either be associated with the perceived penalty due to a loss of customer goodwill or may represent actual contractual penalties. The interested reader is referred to Sen et al (2003) for a relatively recent survey on the topic.

In the traditional single-machine TWT problem described above, all processing times, release dates, due dates, and weights are known in advance at time zero with certainty when the job sequencing decision is taken. However, in many practical settings the exact values of one or several of these parameter types may not be available at the time the dispatcher determines a job processing sequence. In particular, possible machine breakdowns, variable sequence-independent setup times, inconsistency of the worker performance, or changes in tool quality may introduce uncertainty into the processing times. The uncertainty in the processing time of a job is resolved at the time of the job completion. The models developed in this paper are sufficiently general to incorporate randomness into all parameters. However, from a practical point of view it is reasonable to presume that a due date is quoted as a result of a mutual agreement with the customer, and the unit tardiness weight associated with a customer is also known based on either the internal priority of the customer or the contractual agreement. Therefore, in our computational experiments the due dates and the unit tardiness weights are deterministic. Furthermore, we assume that all jobs are ready to be released at time zero. Consequently, we focus on the uncertainty in the processing times which leads to uncertain completion times and tardiness values. Our objective is to determine a risk-averse fixed job processing sequence at time zero that hedges against the uncertainty in the processing times. In the stochastic scheduling terminology (see Pinedo (2008)), we construct a non-preemptive static list policy.

Traditional models for decision making under uncertainty define optimality criteria based on expected values and disregard variability inherent in the system. Following this mainstream risk-neutral approach, most of the classical stochastic scheduling puts a lot of effort into analyzing the expected performance by assuming that uncertain parameters such as processing times follow specific distributions. See Pinedo (2008) for an excellent overview of conventional stochastic scheduling. However, variability typically implies a deterioration in performance, and risk-neutral models may provide solutions that perform poorly under certain realizations of the random data. Capturing the effect of variability can be accomplished by incorporating the appropriate risk measures into the model that reflect the preferences of the decision maker. Several criteria to select risk measures have been discussed in the literature (see, e.g., Ogryczak and Ruszczyński (1999, 2002); Artzner et al (1999)). Considering the wide range of criteria, there is no universally accepted single risk measure appropriate for all decision making contexts. In this study, we consider the VaR measure which is a very popular and widely applied risk measure in the finance literature. For the studies related to VaR we refer to the chapter by Larsen et al (2002). In our context, we focus on the TWT as the random outcome associated with a fixed job processing sequence selected at time zero. The goal is to specify the smallest possible upper bound on the random TWT that will be exceeded with at most a pre-specified small probability. Here, the selected upper bound is the VaR of the random TWT at the desired probability level, and we minimize

VaR. The concept of VaR is closely related to probabilistic constraints. Stochastic programming models with probabilistic constraints were introduced by Charnes et al (1958) and have been employed successfully in a variety of fields. The interested reader is referred to Prékopa (1995) and Dentcheva (2006) for reviews and a comprehensive list of references. Our proposed approach is an intuitive and practical way of modeling a service level requirement for the TWT under the stochastic setup and leads to a novel risk-averse stochastic programming model. To the best of our knowledge, this is a first in the machine scheduling literature.

It is well known that models incorporating VaR exhibit a non-convex structure even if the underlying deterministic problem is convex. The existing solution methods primarily deal with VaR integrated into a linear program (LP). Thus, the decision variables are continuous, and VaR is introduced on a random outcome expressed as a linear function of the decision variables. Larsen et al (2002) provides a review of the algorithms available for solving such problems. Note that these studies are generally concerned with portfolio optimization problems. Larsen et al (2002) also introduces two heuristic algorithms which solve a series of problems involving a related risk measure known as conditional-value-at-risk (CVaR). In contrast to VaR, the problem of minimizing CVaR can be formulated as an LP if the uncertainty is represented by a set of scenarios, and the proposed heuristics use LP techniques iteratively. However, in our study the underlying problem involves sequencing decisions that can only be expressed by employing binary variables; and therefore, even minimizing CVaR is hard. Consequently, the existing solution methods do not apply in our case.

We characterize the randomness associated with the uncertain parameters by a finite set of scenarios, where a scenario represents a joint realization of all random parameters. It is important to point out that the scenario approach allows us to generate data from any distribution and to model, for instance, the correlation of the random processing times among different jobs by considering their joint realizations. In this sense, a scenario-based approach is more general than assuming specific distributions. On the down side, the computational complexity of solving the problem is closely affected by the number of scenarios. There are only a few studies utilizing a scenario-based approach for machine scheduling problems. For example, Gutjahr et al (1999) minimize the expected TWT with stochastic processing times and propose a stochastic branch-and-bound technique, where a sampling approach is embedded into their bounding schemes. Alternatively, other existing scenario-based studies develop robust optimization models in order to optimize the worst-case performance over all scenarios. Such a worst-case analysis does not require the probabilities of the scenarios. The sum of completion times is employed in Daniels and Kouvelis (1995); Yang and Yu (2002), and the weighted sum of completion times is considered by de Farias et al (2010), while Kasperski (2005) focuses on the maximum lateness as the random performance criterion. One or several of the robustness measures known as the maximum deviation from optimality, the maximum relative deviation from optimality, and the maximum value over all scenarios are incorporated in these papers. Except de Farias et al (2010), all these studies design specialized algorithms for the robustness measure and random performance criterion of interest. de Farias et al (2010) identify a family of valid inequalities to strengthen the mixed-integer formulation of their problem. Furthermore, Aloulou and Croce (2008) provide several complexity results in the domain of robust scheduling. In contrast to robust approaches adopting a conservative worst-case view, we define our optimality criterion based on VaR which is a quantile of the random outcome at a specified probability level. That is, we utilize probabilistic information and develop a risk-averse stochastic programming model alternative to existing robust optimization models. Note that setting the required probability level to one subsumes the robust optimization problem of minimizing the maximum TWT over all scenarios. However, when the required probability level is specified as $\alpha < 1$, we minimize the maximum TWT over a subset of scenarios with an aggregate probability of at least α. Our risk-averse model identifies the optimal subset of scenarios with the specified minimum aggregate probability level and minimizes the maximum TWT over this subset. Thus, it is less conservative than the robustness approach which considers all scenarios.

The main contribution of this study is to develop a risk-averse model that is novel in machine scheduling. We analyze the behavior of the proposed model in comparison to that of the risk-neutral model and provide insights on the impact of the risk preference. Furthermore, in all papers on robust scheduling mentioned above the corresponding deterministic single-machine problems are polynomially solvable. However, the single-machine TWT problem is strongly $\mathcal{NP}$-hard (Lenstra et al (1977)), and incorporating VaR poses additional computational difficulties. Thus, we also implement a tabu search algorithm to solve the proposed model heuristically.

In the next section, we formally define the risk-averse TWT problem and present a mathematical programming formulation. In Section 3, we introduce a lower bounding scheme for our problem and also briefly discuss the implementation details of the tabu search. Computational results are presented in Section 4, and we conclude in Section 5 with further research directions.

2 Stochastic Programming Model

In this section, we first present the underlying deterministic model of the stochastic TWT problem. Then, we discuss how to model the uncertainty inherent in the system and develop our risk-averse stochastic programming model.

2.1 Underlying deterministic model

For single-machine scheduling problems, four frequently used alternate deterministic formulations appear in the literature (see Keha et al (2009)): disjunctive (DF), time-indexed (TIF), linear ordering (LOF), and the assignment and positional date formulations (APDF). TIF has a tight LP relaxation and is the best contender among these four formulations if the processing times are small. TIF, however, cannot be adapted to our stochastic setting directly, because it infers the sequence from the completion times represented by binary decision variables. Recall that our goal is to find a non-preemptive static job processing sequence at time zero. That is, the decisions are independent of the random realizations of data, and therefore, relying on completion time information that is contingent on the random processing times (and random release dates if applicable) is not appropriate to construct a static job processing sequence. Our preliminary results indicate that DF is outperformed by LOF and APDF. This observation is also supported by the extensive computational study presented in Keha et al (2009). Thus, among the common formulations only LOF and APDF are viable options for our proposed risk-averse model. In this study, we work with LOF but note that the proposed modeling framework would apply to APDF in a similar way.

We define the set of jobs to be processed as $N := \{1, \ldots, n\}$, where n denotes the number of jobs. Associated with each job $j \in N$ are several parameters: a processing time p_j, a due date d_j, and a tardiness cost per unit time w_j if job j completes processing after d_j. The completion time of job j is represented by C_j, and the tardiness T_j of job j is then expressed by $T_j = \max(0, C_j - d_j)$. The binary variable δ_{jk} takes the value 1, if job j precedes job k in the processing sequence, and is zero otherwise. By convention, we set $\delta_{jj} = 1$ for all $j \in N$. Assuming zero release dates, the deterministic single-machine TWT problem, described as $1//\sum_j w_j T_j$ following the common three field notation of Graham et al (1979), is formulated below:

$$\min \quad \sum_{j=1}^{n} w_j T_j$$

subject to

$$\delta_{jk} + \delta_{kj} = 1, \qquad 1 \leq j < k \leq n, \qquad (1)$$

$$\delta_{jk} + \delta_{kl} + \delta_{lj} \leq 2, \qquad \forall j,\ k,\ l \in N \ :\ \ j \neq k,\ k \neq l,\ l \neq j, \qquad (2)$$

$$C_j = \sum_{k \in N} p_k \delta_{kj}, \qquad \forall j \in N, \qquad (3)$$

$$T_j \geq C_j - d_j, \qquad \forall j \in N, \qquad (4)$$

$$T_j \geq 0, \qquad \forall j \in N, \qquad (5)$$

$$\delta_{jk} \in \{0, 1\}, \qquad \forall j, k \in N. \qquad (6)$$

Constraints (1) ensure that for each pair of jobs j and k either job j precedes job k or vice versa. Constraints (2) represent the transitivity requirements for a linear ordering of the jobs. In other words, they guarantee that for any triplet of jobs j, k, l, if job j precedes job k and job k precedes job l then job j precedes job l. The completion time C_j of job j is the sum of the processing times of all of its predecessors (recall that $\delta_{jj} = 1$ by convention) as prescribed by (3), and T_j is related to C_j by constraints (4) and (5). Constraints (6) are the binary variable restrictions required for the sequencing decisions.

In our setting, the actual values of the processing times are not certain at the time we determine the job processing sequence, and the processing times can be represented by random variables. This implies that the completion times and the tardiness values associated with a sequence are also random variables, since they are functions of the random processing times. In this case, comparing alternate candidate sequences requires comparing their respective random TWT values. In this paper, we propose a risk-averse approach which evaluates a sequence with respect to a certain quantile of the distribution of the associated random TWT. Let Υ and ξ_j denote the random TWT and the random processing time of job $j \in N$, respectively. The random variable Υ is a random outcome associated with a sequence $\delta \in \{0, 1\}^{n \times n}$. We can represent it as a function of the decision vector $\delta \in \{0, 1\}^{n \times n}$ as below:

$$\Upsilon = \sum_{j=1}^{n} w_j \max\left(\sum_{k=1}^{n} \xi_k \delta_{kj} - d_j, 0\right). \qquad (7)$$

We intend to model the risk associated with the variability of the random outcome Υ by introducing the following probabilistic constraint:

$$P(\Upsilon \leq \theta) \geq \alpha, \qquad (8)$$

where α is a specified large probability such as 0.90 or 0.95. Here θ denotes an upper bound on the TWT that is exceeded with at most a small probability of $1-\alpha$. If $\alpha = 1$, $\Upsilon \leq \theta$ holds almost surely. As discussed in more depth in Section 1, such a probabilistic constraint is intuitive and allows us to model a service level requirement for the TWT under the stochastic setup. We refer to α as the risk parameter which reflects the level of risk-aversion of the decision maker. Clearly, increasing α results in allowing a higher value of the upper bound θ. We propose not to specify the value of θ as an input, but consider it as a decision variable with the purpose of identifying the sequence with the smallest possible value of θ given the risk aversion of the decision maker. Thus, in our model we minimize θ for a specified parameter α, which is equivalent to minimizing the α-quantile of the random TWT. The α-quantile has a special name in risk theory as presented in the next definition.

Definition 1 Let X be a random variable. The α-quantile

$$\inf\{\eta \in \mathbb{R} : \; F_X(\eta) \geq \alpha\}$$

is called the Value at Risk (VaR) at the confidence level α and denoted by $\mathrm{VaR}_\alpha(X)$, $\alpha \in (0, 1]$.

The probabilistic constraint (8) can equivalently be formulated as a constraint on the VaR of the random TWT:

$$\mathrm{VaR}_\alpha(\Upsilon) \leq \theta. \tag{9}$$

In other words, by considering the proposed probabilistic constraint (8) we specify the VaR as the risk measure on the random TWT, and minimizing θ corresponds to seeking the sequence with the smallest possible VaR measure for a specified α value.

A model with a probabilistic constraint similar to that in (8) with randomness on the left hand side was first studied by de Panne and Popp (1963) and Kataoka (1963). Kataoka introduces a transportation type model and Van de Panne and Popp present a diet (cattle feed) optimization model with a single probabilistic constraint. In these studies, the random outcome of interest is a linear function of the decision vector, and in both studies the solution methods are specific to random coefficients with a joint normal distribution. In contrast, the random outcome Υ in our work is not a linear function of the decision vector as evident from (7), and we do not assume that it has a special distribution.

We characterize the random processing times by a finite set of scenarios denoted by S, where a scenario represents a joint realization of the processing times of all jobs. To develop our stochastic programming formulation, previously introduced parameters and variables are augmented with scenario indices:

p_j^s: processing time of job j under scenario s, $s \in S$.
T_j^s: tardiness of job j under scenario s, $s \in S$.
C_j^s: completion time of job j under scenario s, $s \in S$.
π^s: probability of scenario s, $s \in S$.

Then, we formulate the problem of minimizing the VaR in the single-machine TWT problem as follows:

$$\min \quad \theta \tag{10}$$

$$\text{subject to} \quad \delta_{jk} + \delta_{kj} = 1, \qquad 1 \leq j < k \leq n, \tag{11}$$

$$\delta_{jk} + \delta_{kl} + \delta_{lj} \le 2, \qquad \forall j,\ k,\ l \in N \ : \quad j \ne k,\ k \ne l,\ l \ne j, \quad (12)$$

$$C_j^s = \sum_{k \in N} p_k^s \delta_{kj}, \qquad \forall j \in N, s \in S, \quad (13)$$

$$T_j^s \ge C_j^s - d_j, \qquad \forall j \in N, s \in S, \quad (14)$$

$$T_j^s \ge 0, \qquad \forall j \in N, s \in S, \quad (15)$$

$$\sum_{j \in N} w_j T_j^s - \theta \le T_{\max}^s \beta^s, \qquad \forall s \in S, \quad (16)$$

$$\sum_{s \in S} \pi^s \beta^s \le 1 - \alpha, \quad (17)$$

$$\beta^s \in \{0, 1\}, \qquad \forall s \in S, \quad (18)$$

$$\delta_{jk} \in \{0, 1\}, \qquad \forall j, k \in N. \quad (19)$$

We emphasize that the constraints (11), (12), and (19) in the model above are identical to the constraints (1), (2), and (6) in the deterministic model, respectively. That is, the sequencing decisions are independent of the uncertainty. Constraints (13)-(15) are the counterparts of the constraints (3)-(5) in the deterministic model, respectively, and calculate the resulting job completion times under each joint realization of the processing times. The parameter $T_{\max}^s$ is a scenario-specific sufficiently large number which guarantees that the binary variable β^s is set to 1 by the corresponding constraint (16) if the realization of the TWT under scenario s exceeds the threshold value θ. Constraint (17) mandates that the probability of exceeding the threshold value θ for the random TWT is no more than $1 - \alpha$. At an optimal solution of the formulation above, T_j^s may be strictly larger than $\max\{C_j^s - d_j, 0\}$ for some scenario $s \in S$ because the tardiness values are not associated with positive cost coefficients in the objective. Obviously, we preserve optimality by setting $T_j^s = \max\{C_j^s - d_j, 0\}$. For the validity of the formulation (10)-(19), we must also ensure that $T_{\max}^s$ is no smaller than the maximum possible TWT under scenario s. In order to obtain a reasonably tight formulation, we sort the processing times under scenario s in non-increasing order and denote the jth largest processing time under scenario s by $p_{[j]}^s$. Then, the maximum possible completion time of the kth job in the sequence, $k = 1, \ldots, n$, under scenario s is computed as $C_{[k]}^s = \sum_{j=1}^{k} p_{[j]}^s$. Next, the due dates and the unit tardiness weights are assigned to the completion times in non-increasing and non-decreasing order, respectively. A standard pairwise interchange argument (not necessarily adjacent) demonstrates that the resulting TWT is an upper bound on the TWT of any job processing sequence under scenario s.

Uncertainty in the due dates and/or the unit tardiness weights may be incorporated in our formulation in a straightforward manner by replacing the parameters d_j and w_j by d_j^s and w_j^s in the constraints (14) and (16), respectively. This modification does not affect the number of variables and constraints. However, if the release dates are not known in advance, then the completion time constraints (13) must be replaced by the following set of constraints adapted from the deterministic formulation in Nemhauser and Savelsbergh (1992):

$$C_j^s \ge r_i^s \delta_{ij} + \sum_{\{k \,:\, r_k^s < r_i^s,\ k \ne j\}} p_k^s (\delta_{ik} + \delta_{kj} - 1) + \sum_{\{k \,:\, r_k^s \ge r_i^s\}} p_k^s \delta_{kj}, \quad \forall i, j \in N.$$

In the remainder of the paper, we refer to the formulation (10)-(19) as VaR-TWT.

3 On the Solution Method

The main intent of our work is to argue and illustrate the value of the risk-averse single-machine TWT problem. Therefore, our efforts are focused primarily on modeling and the insights provided by the computational results. Clearly, developing an effective solution method to solve VaR-TWT is an ambitious endeavor that we will delve into in the future. In this larger scheme, we first develop a lower bound on the optimal VaR that may benefit prospective studies as well. This lower bounding mechanism is employed for computing the optimality gaps in our computational study in Section 4 when it is too expensive to solve VaR-TWT to optimality. Then, we briefly discuss some implementation details of a standard tabu search heuristic that yields reasonably high-quality feasible solutions to the risk-averse single-machine TWT problem.

3.1 Lower Bounding Scheme

The relation of stochastic dominance is one of the fundamental concepts to compare random variables (Mann and Whitney (1947); Lehmann (1955)). It introduces a pre-order in the space of real random variables. We refer to Muller and Stoyan (2002) for a detailed and comprehensive discussion on stochastic dominance relations. In a stochastic dominance based approach, random variables are compared by a point-wise comparison of some performance functions constructed from their distribution functions. In this study, we utilize the first-order stochastic dominance (FSD) which considers the cumulative distribution function itself as the performance function. Let F_X and F_Y denote the distribution functions of the random variables X and Y, respectively. The FSD relation between X and Y is defined as below:

Definition 2 A random variable X dominates another random variable Y in the first order; that is, X is stochastically larger than Y, if

$$F_X(\eta) \leq F_Y(\eta) \quad \text{for all} \quad \eta \in \mathbb{R}. \tag{20}$$

This ordering is denoted by $X \succeq_{(1)} Y$.

It is easy to see that by the definition of the FSD relation we have

$$\Big[X \succeq_{(1)} Y\Big] \Leftrightarrow \Big[\mathrm{VaR}_\alpha(X) \geq \mathrm{VaR}_\alpha(Y) \quad \text{for all } 0 < \alpha \leq 1\Big]. \tag{21}$$

We leverage on this fundamental relation between the concepts of VaR and FSD in order to obtain a lower bound on the optimal objective value of VaR-TWT. We consider a finite probability space where the sample space is given by $\Omega = \{\omega_1, \ldots, \omega_N\}$ with corresponding probabilities $p_1, \ldots, p_N$. Let $y_i = Y(\omega_i),\ i = 1, \ldots, N$, and $x_i = X(\omega_i),\ i = 1, \ldots, N$, denote the realizations of the random variables Y and X, respectively. In our study, we are interested in the random TWT. In particular, the realizations of the random variable Y are obtained by solving a single-machine TWT problem independently for each scenario. On the other hand, the random variable X denotes the random TWT associated with the optimal sequence of the problem VaR-TWT. Next, we state formally that $\mathrm{VaR}_\alpha(Y)$ is a lower bound on the optimal VaR obtained by solving VaR-TWT for any given fixed α.

Proposition 1 *Let Y represent a random variable, where the realization $Y(\omega_i)$ is equal to the TWT associated with the sequence that minimizes the TWT under scenario $i, i = 1, \ldots, N$. Furthermore, the random variable X denotes the random TWT associated with the optimal sequence δ^* of the problem VaR-TWT. Then, $VaR_\alpha(Y) \leq VaR_\alpha(X)$ for all $0 < \alpha \leq 1$.*

Proof $X(\omega_i)$ is the TWT associated with the sequence δ^* under scenario i. Since δ^* is a feasible sequence for the problem of minimizing the TWT under scenario i, we have $X(\omega_i) \geq Y(\omega_i)$ for all $i = 1, \ldots, N$. It trivially follows that $P(X \leq \eta) \leq P(Y \leq \eta)$ for all $\eta \in R$, i.e., X dominates Y in the first-order. Consequently, $\mathrm{VaR}_\alpha(Y) \leq \mathrm{VaR}_\alpha(X)$ for all $0 < \alpha \leq 1$ by (21).

Note that the random variable Y does not have a special interpretation in the context of our problem. It only serves the purpose of obtaining a valid lower bound on the optimal objective function value of our problem.

Calculating the lower bound in Proposition 1 is strongly $\mathcal{NP}$-hard because it requires solving $|S|$ instances of the deterministic single-machine TWT problem. A remedy to this issue is constructing a lower bound on the optimal TWT under each scenario. Then, the realizations $Y(\omega_i), i = 1, \ldots, N$, in Proposition 1 are replaced with these lower bounds. The interested reader is referred to Tanaka et al (2009); Pan and Shi (2007); Sen (2010) for a variety of lower bounds for the deterministic single-machine TWT problem.

3.2 Tabu Search

For many combinatorial optimization problems in scheduling and routing, local search algorithms, and in particular metaheuristics, provide us with established and viable solution methods. Given the additional complexity stemming from the consideration of risk in the single-machine TWT problem, it is of interest to gauge the potential of local search algorithms for this problem. Thus, in this study we implemented a basic tabu search algorithm for a preliminary assessment. A further important consideration is whether this tabu search algorithm fares well compared to a risk-neutral optimization model to be introduced in Section 4. In other words, we intend to figure out whether we can do at least as good as an expectation-based model even with a simple metaheuristic. In the rest of this section we briefly discuss some details of the tabu search implementation.

The feasible region of the risk-averse single-machine TWT problem is identical to that of its deterministic counterpart with $n!$ feasible sequences. Therefore, it suffices to conduct a search over the job processing sequence. In the deterministic version, a well-known dominance rule (see Pinedo (2008)) mandates that there exists an optimal sequence in which job i precedes job j if $d_i \leq d_j$, $p_i \leq p_j$, and $w_i \geq w_j$. We adapt this rule to our problem by replacing the processing times by their expected values, where the expected processing time of job j is given by $\bar{p}_j = \sum_{s \in S} \pi^s p_j^s$. We determine the priority of job j as $\frac{w_j}{\bar{p}_j \times d_j}$ and pick the jobs in non-increasing order of their priorities to construct an initial feasible sequence. Clearly, in the future we need to conceive and incorporate dominance rules that are more tailored toward the risk-averse nature of our problem.

The neighborhood generator is adjacent pairwise interchange. Note that function evaluations are expensive for our problem. Even for this basic neighborhood, the randomness in the processing times (and possibly in the other parameters of the problem) does not allow to calculate the objective value of a neighboring solution in a straightforward way based on the objective value of the current solution. Therefore, the small size of this neighborhood makes it attractive from a computational speed point of view. Features such as approximate move evaluations may need to be incorporated for an effective local search heuristic for larger instances.

The search is guided with a best-improve strategy. That is, at each iteration we pick the best of the $n-1$ neighboring solutions. The tabu list keeps track of the jobs not to be interchanged for a number of iterations. The size of the tabu list is limited to $\lceil 5n/2 \rceil$. A tabu move is accepted only if the associated objective value exceeds the best solution found so far. We terminate the algorithm at $500n$ iterations or if the best objective value does not improve for $15n$ iterations, or if all moves in the current neighborhood are tabu.

4 Computational Study

The goals of our computational study are two-fold. In the first part, we demonstrate that the lower bounding procedure developed in Section 3.1 provides good bounds in this preliminary computational study. Furthermore, the results indicate that the basic tabu search algorithm of Section 3.2 yields feasible solutions of reasonable quality for the risk-averse single-machine TWT problem for instances with up to 20 jobs. In the second part, the value of the proposed risk-averse model is investigated with respect to that of a risk-neutral model.

All runs were conducted on a single core of an HP Linux workstation with two Intel®Xeon®W5580 3.20GHz CPU's and 32 GB of memory. The integer programming formulations were solved by `CPLEX 11.2`, and the tabu search was implemented in C++.

4.1 Generation of problem instances

While our modeling framework allows for randomness in all problem parameters, we focus on the uncertainty in the processing times in our computational study as justified by the discussion in Section 1. For each instance, we generate a set of equally likely scenarios representing the joint realizations of the processing times by adding negative or positive perturbations to each estimated processing time $\hat{p}_j$, where $\hat{p}_j$ follows an integer uniform distribution $U[1, 100]$ for $j = 1, \ldots, n$. To this end, let ε_j denote the random perturbation for job j, where ε_j^s is the realization of ε_j for scenario s. Then, the processing time of job j under scenario s is given by $p_j^s = \hat{p}_j + \varepsilon_j^s$. In our first set of experiments, we set $\varepsilon_j \sim U(-\hat{p}_j/4, \hat{p}_j/3)$, which results in $E(\hat{p}_j + \varepsilon_j) = \hat{p}_j + \hat{p}_j/24$ and a coefficient of variation (CV) of 0.16. CV is a normalized measure of dispersion and is defined as $CV(\hat{p}_j + \varepsilon_j) = \textit{standard deviation}(\hat{p}_j + \varepsilon_j)/E(\hat{p}_j + \varepsilon_j)$ for the processing time of job j. Additional data with different CV values are generated to further analyze the value of the risk-averse model in Section 4.3. The details are presented in Table 2.

In the literature, it is well established that the tightness and the range of the due dates is a primary determinant of difficulty for due date related problems. Thus, by following the popular scheme of Potts and van Wassenhove (1982), we first generate the due dates from a discrete uniform distribution $[\lceil(1 - TF - RDD/2) \times \bar{P}\rceil, \lceil(1 - TF + RDD/2) \times \bar{P}\rceil]$, where $\bar{P}$ is the sum of the expected processing times, i.e., $\bar{P} = \sum_{j=1}^{n} \sum_{s \in S} \pi^s p_j^s$. The tardiness factor TF is a rough estimate of the proportion of jobs that might be expected to be tardy in an arbitrary sequence (Srinivasan (1971)) and is set to 0.3. Hard instances generally result from such small values of TF (see Bulbul et al (2007); Sen (2010)). The due date range factor RDD is set to 0.2 to increase the contention around the mean due date. The weights are drawn from an integer uniform distribution $U[1, 100]$.

4.2 Computational performance of the Lower Bound and the Tabu Search

In the first part of our study, we generate 5 instances for each combination of $n = 10, 20, 30$, and $|S| = 100, 150, 200, 250, 300$, as described in the previous section. The risk parameter $\alpha = 0.90$. For each instance, we calculate the lower bound presented in Section 3.1 and then solve it by both VaR-TWT and the tabu search implemented. The results averaged over 5 instances appear in Table 1.

Table 1 Effectiveness of the tabu search and the lower bound ($\alpha = 0.90$).

				| S |		
		100	**150**	**200**	**250**	**300**
n = 10	**UB on Gap - Tabu**	1.93%	0.00%	3.95%	1.29%	1.02%
	UB on Gap - Var-TWT	0.00%	0.00%	0.00%	0.37%	1.02%
	CPU - Tabu	0.16	0.24	0.34	0.41	0.48
	CPU - Var-TWT	439.54	446.44	1263.48	5231.41	5195.23
n = 20	**UB on Gap - Tabu**	6.88%	7.27%	10.55%	1.80%	4.63%
	UB on Gap - Var-TWT	0.80%	2.50%	3.13%	2.76%	4.05%
	CPU - Tabu	1.29	1.94	2.40	3.11	3.60
	CPU - Var-TWT	7194.83	7195.69	7195.35	7197.25	7197.70
n = 30	**UB on Gap - Tabu**	5.91%	25.25%	14.46%	12.47%	8.20%
	UB on Gap - Var-TWT	3.71%	6.45%	6.76%	16.20%	16.80%
	CPU - Tabu	4.39	6.49	9.04	10.07	12.17
	CPU - Var-TWT	7199.31	7199.43	7199.34	7199.53	7198.64

The time limit for VaR-TWT is set to 7200 seconds, and if optimality is not proven in the time allotted, then we record both the best lower bound and the incumbent solution available from `CPLEX`. For a given instance and an algorithm, the optimality gap is computed with respect to the optimal solution if it is available. Otherwise, the best known lower bound is determined by taking the maximum of our lower bound and the best lower bound retrieved from `CPLEX`, and the optimality gap is computed with respect to this lower bound. For each n, the first two rows specify the average upper bounds on the optimality gaps (“UB on Gap - Tabu” and “UB on Gap - VaR-TWT”), and the associated average CPU times in seconds are reported in the last two rows.

Several conclusions may be drawn from Table 1. First, solving VaR-TWT is very time consuming even for small n as the number of scenarios grows. Only half of the instances with $n = 10$ and $|S| = 250, 300$, and none of instances with $n = 20, 30$ can be solved to optimality within the time limit. Second, our lower bounding scheme is quite tight for this data set. Note that for $n = 20, 30$, `CPLEX` returns a trivial lower bound of 0 for all instances. That is, all optimality gaps for instances with 20 and 30 jobs are computed with respect to our lower bound. Furthermore, note that a standard time-indexed formulation is applied to the single-machine TWT problems to be solved for our lower bounding scheme, and if optimality is not achieved in 1000 seconds, then we use the best available lower bound instead of the optimal solution for calculating the lower bound based on Proposition 1. Third, the quality of solutions obtained from the tabu search is relatively high for $n = 10, 20$. However, the performance degrades considerably for $n = 30$.

4.3 Value of the Risk-Averse Model

The value of a risk-averse solution depends on the relative performance of the corresponding deterministic and risk-neutral solutions as a function of the risk appetite. Therefore, in this part, we benchmark VaR-TWT against corresponding deterministic and risk-averse models as the risk parameter α is varied. The deterministic counterpart of our problem is the conventional single-machine TWT problem, in which all processing times take on their expected values; that is, we have $p_j = \bar{p}_j = \sum_{s\in S} \pi^s p_j^s$. In the risk-neutral version of our problem, we minimize the expected TWT by solving the following formulation:

$$\min \quad \sum_{j=1}^{n} w_j \sum_{s\in S} \pi^s T_j^s \tag{22}$$

$$\text{subject to} \quad (11) - (15) \text{ and } (19). \tag{23}$$

In Figure 1, we zoom into two instances from Table 1 to illustrate how the VaR changes as α is varied. For this data set we obtain risk-averse solutions without sacrificing much from the expected TWT as α increases. Finally, we create data with different variability in the processing times. To this end, we consider the uniform distribution and a mixture of two exponential distributions to generate perturbations. We specify the parameters of these distributions such that the resulting processing times have different CV values. The details are summarized in Table 2. All scenarios are assumed to be equally likely. A total of 10 instances for $n = 10, 20$ are solved by the risk-neutral model and tabu search for $\alpha = 0.90$. For these 10 instances, the entries in Table 3 indicate the relative decrease in VaR and the relative increase in the expected TWT for the solution of the tabu search in comparison to that of the risk-neutral model. The risk-averse solution exhibits significant improvements over the risk-neutral solution, albeit at times at the expense of the expected TWT to hedge against the uncertainty.

	Random Perturbation	**Resulting Processing Times**	
		$E(\hat{p}_j + \varepsilon_j)$	$CV(\hat{p}_j + \varepsilon_j)$
DataSet 1	$\varepsilon_j \sim U(-\hat{p}_j/4, \hat{p}_j)$	$\hat{p}_j + 3\hat{p}_j/8$	0.26
DataSet 2	$\varepsilon_j \sim U(-\hat{p}_j/4, 2\hat{p}_j)$	$\hat{p}_j + 7\hat{p}_j/8$	0.35
DataSet 3	$\varepsilon_j \sim \begin{cases} exp(13\hat{p}_j/24) & \text{with prob of } 0.75 \\ exp(-\hat{p}_j/8) & \text{with prob of } 0.25 \end{cases}$	$\hat{p}_j + 3\hat{p}_j/8$	0.40
DataSet 4	$\varepsilon_j \sim \begin{cases} exp(6\hat{p}_j/5) & \text{with prob of } 0.75 \\ exp(-\hat{p}_j/10) & \text{with prob of } 0.25 \end{cases}$	$\hat{p}_j + 7\hat{p}_j/8$	0.63

Table 2 Summary of parameters for creating processing times.

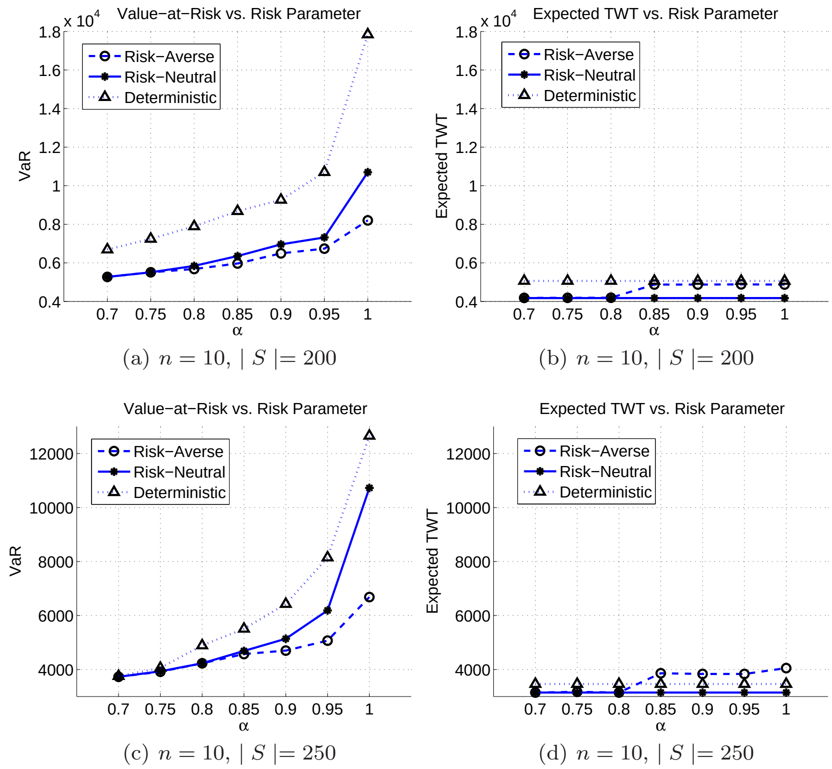

Fig. 1 Comparison of the risk-averse model to its deterministic and risk-neutral counterparts.

Table 3 The risk-averse model (tabu search) versus the risk-neutral model ($\alpha = 0.90$).

	n = 10							
	DataSet 1		**DataSet 2**		**DataSet 3**		**DataSet 4**	
| S |	θ	**E(TWT)**	θ	**E(TWT)**	θ	**E(TWT)**	θ	**E(TWT)**
100	-17.44%	3.08%	-14.82%	0.13%	-13.37%	5.54%	-21.39%	7.93%
150	-5.64%	21.21%	-9.72%	12.07%	-21.25%	9.10%	-8.73%	15.27%
200	-28.07%	5.40%	-7.27%	24.64%	-25.25%	13.84%	-15.83%	27.94%
250	-4.11%	0.32%	-5.40%	1.86%	-5.89%	0.39%	-11.32%	11.11%
300	-6.81%	18.77%	-5.47%	12.63%	-10.02%	6.38%	-11.90%	5.02%

	n = 20							
	DataSet 1		**DataSet 2**		**DataSet 3**		**DataSet 4**	
| S |	θ	**E(TWT)**	θ	**E(TWT)**	θ	**E(TWT)**	θ	**E(TWT)**
100	-17.98%	7.10%	-5.14%	16.99%	-25.77%	12.41%	-6.22%	15.27%
150	-15.81%	4.02%	-23.89%	0.88%	-12.07%	4.69%	-9.09%	5.36%
200	-2.62%	8.12%	-4.03%	4.06%	-2.99%	8.72%	-12.33%	22.70%
250	-3.87%	8.75%	-14.02%	12.81%	-5.16%	13.08%	-8.97%	5.93%
300	-14.19%	4.65%	-6.13%	7.33%	-12.97%	9.48%	-9.51%	11.43%

5 Conclusion

In this paper, we modeled the problem of minimizing VaR in the single-machine TWT problem under the presence of uncertainty and illustrated the value of the proposed risk-averse model. Furthermore, we presented a promising lower bounding scheme. As part of our ongoing research, we focus on developing an efficient mathematical programming based method to solve the proposed model VaR-TWT. In particular, we intend to incorporate a Lagrangian method to decompose the problem over scenarios.

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