Renewal theory for random variables with a heavy tailed distribution and finite variance

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Abstract

Let $X_1, X_2, \ldots X_n$ be independent and identically distributed (i.i.d.) non-negative random variables with common distribution function (d.f.) F with unbounded support and $\mathbb{E}X_1^2 < \infty$. We show that for a large class of heavy tailed random variables with a finite variance the renewal function U satisfies

$$U(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2} \sim -\frac{1}{\mu x} \int_x^\infty \int_s^\infty (1 - F(u)) du ds$$

as $x \to \infty$.

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1 Introduction and main results

Let $X_1, X_2, \ldots X_n$ be independent and identically distributed (i.i.d.) non-negative random variables with common distribution function (d.f.) $F = 1 - \overline{F}$ with unbounded support. We assume throughout this paper that F is non-singular. Let $S_0 = 0$, and $S_{n+1} = S_n + X_{n+1}$ $(n = 0, 1, \ldots)$. Renewal theory is focused on the study of the counting process $N(x),\ x \geq 0$, defined by the passage times for the random walk $S_n,\ n \geq 0$, i.e.

$$N(x) = n \quad \text{if and only if } S_{n-1} \leq x < S_n.$$

The expected value $U(x) = \mathbb{E}N(x),\ x \geq 0$ is the so-called renewal function. It is well-known that $U(x) = \sum_{n=0}^{\infty} \Pr\{S_n \leq x\},\ x \geq 0$.

The study of the asymptotic behavior for $x \to \infty$ of the renewal function has a long and celebrated history. It is well-known that if X_1 has finite mean $\mu = \mathbb{E}X_1$, $U(x) - x/\mu$ is non-negative and the renewal theorem states that

$$U(x)/x \to \mu^{-1} \text{ as } x \to \infty. \tag{1}$$

In this case the equilibrium distribution is the integrated tail function

$$F_I(x) := \frac{1}{\mu} \int_0^x \overline{F}(y)dy,\ x > 0 \tag{2}$$

which is not only of importance in renewal theory, but also in queueing theory and in ruin theory.

It is well-known that the renewal function U is the unique solution of the equation

$$\int_0^x \overline{F}(x-s)dU(s) = 1. \tag{3}$$

Integrating relation (3) shows that

$$\mu \int_0^x F_I(x-s)dU(s) = x,$$

hence

$$Z(x) = \int_0^x \overline{F}_I(x-s)dU(s), \tag{4}$$

where the function Z which is defined by

$$Z(x) = U(x) - \frac{x}{\mu}.$$

Refinements of (1) are usually proved using the key renewal theorem which gives the asymptotic behavior of the Stieltjes convolution

$$(U * Q)(x) \equiv \int_0^x Q(x-s)\ dU(s)$$

as $x \to \infty$ under suitable hypothesis on $Q(.)$ and $F(.)$. See for example Feller (1971).

In case $\mu_2 = \mathbb{E}(X_1^2) < \infty$, the key renewal theorem can be applied to (4) to obtain

$$Z(x) \to \frac{1}{\mu} \int_0^\infty \overline{F}_I(s)ds,$$

which is equivalent to the well-known relation

$$U(x) - \frac{x}{\mu} \to \frac{\mu_2}{2\mu^2} \quad \text{as} \quad x \to \infty. \tag{5}$$

The main result of this paper is the following refinement of (5):

$$U(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2} = -\frac{1}{\mu}\int_x^\infty \overline{F}_I(s)ds + O(\overline{F}_I(x)) \text{ as } x \to \infty. \tag{6}$$

For non-singular distributions with $\mu_2 < \infty$, the above refinement is valid under the condition $F_I \in \mathscr{S}^*$, a subclass of the subexponential distributions which was introduced by Klüppelberg (1988). See the relation (7) below. For related results the reader is referred to Frenk (1983).

Next we give an introduction to the results from subexponentiality which are needed. A d.f. F is subexponential (notation: $F \in \mathscr{S}$) if $P(X_1 + X_2 > x) \sim 2P(X_1 > x)$ as $x \to \infty$, equivalently, if $P(X_1 + X_2 > x) \sim P(\max(X_1, X_2) > x)$.

The theory of subexponential distributions is well established by now. Its relevance is obvious from applications in various areas of applied probability. For recent reviews of applications of subexponentiality the reader is referred to the books by Asmussen (2000), Embrechts et al. (1997) and Rolski et al. (1999). The class $\mathscr{S}$ is related to several other classes of functions. A well known result is the inclusion $\mathscr{S} \subset \mathcal{L}$, where $\mathcal{L}$ is the class of long tailed functions F satisfying $\overline{F}(x+a) \sim \overline{F}(x)$ as $x \to \infty$ (for $a \in \mathbb{R}$). (In this case convergence is uniform on compact subsets of $\mathbb{R}$.) There is a connection with functions of dominated variation as well: the inclusion $\mathcal{L} \cap \mathcal{D} \subset \mathscr{S}$. We write $F \in \mathcal{D}$ to denote that the tail function $\overline{F}$ is of dominated variation, i.e. if $\limsup_{x\to\infty} \overline{F}(ax)/\overline{F}(x) < \infty$ for $a > 0$.

In Embrechts and Omey (1984) the following sufficient condition for $F_I \in \mathscr{S}$ is given.

Lemma 1. *If $\overline{F}(x) \sim \exp\{-x\psi(x)\}$ as $x \to \infty$ where*

1. *for all $y \in \mathbb{R} : \psi(x+y) - \psi(x) = O(x^{-1})$ as $x \to \infty$;*
2. *$\psi(x) \downarrow 0$ and $x^2|\psi(x)| \uparrow \infty$ as $x \to \infty$;*
3. *$\int_0^\infty \exp\{-\frac{1}{2}x^2|\psi'(x)|\}dx < \infty$,*

then

$$F_I \in \mathscr{S}.$$

The property $F_I \in \mathscr{S}$ is also related to the class $\mathscr{S}^*$, a well known subclass of $\mathscr{S}$ introduced in Klüppelberg (1988).

Definition 1. *F belongs to the class $\mathscr{S}^*$ if F has finite expectation μ and*

$$\lim_{x\to\infty} \int_0^x \frac{\overline{F}(x-y)}{\overline{F}(x)} \overline{F}(y)dy = 2\mu. \tag{7}$$

Writing the integral above as two times the integral over $(0, x/2)$ and applying dominated convergence, it follows that if F has a finite mean, $F \in \mathcal{L} \cap \mathcal{D}$ implies that $F \in \mathscr{S}^*$.

Recent papers in which the class $\mathscr{S}^*$ plays a role are Asmussen et al. (2002) and Schmidli (1999). Embrechts and Omey (1984) used a result of Ney (see Lemma 2 below) in order to prove a rate of convergence result for the elementary renewal theorem. They show that under the assumption $F \in \mathcal{L} \cap \mathcal{D}$, relation (6) holds with the O-term on the right-hand side replaced with $o(x\overline{F}_I(x))$. We give (see Theorem 1 below) a more precise estimate, under more general conditions since for $\mu < \infty$, $F \in \mathcal{D}$ implies $F_I \in \mathcal{L} \cap \mathcal{D}$. See Embrechts and Omey (1984). As a consequence, it follows that when $\mu_2 < \infty$ we have $F_I \in \mathscr{S}^*$ so that Theorem 1 below can be applied.

Theorem 1. *Suppose F is non-singular and $F_I \in \mathscr{S}^*$. Then (6) holds.*

Under the conditions of the Theorem we have $F_I \in \mathscr{S}^* \subset \mathscr{S}$ and it follows that $\overline{F}_I(x) = o(\int_x^\infty \overline{F}_I(s)ds)$ as $x \to \infty$ by Proposition 2.3 in Asmussen (2000).

As a consequence, the asymptotic relation

$$U(x) - \frac{x}{\mu} - \frac{\mu_2}{2\mu^2} \sim -\frac{1}{\mu}\int_x^\infty \overline{F}_I(s)ds$$

follows from (6). If F has a regularly varying tail and a finite variance this result coincides with the one given in the Corollary of Embrechts and Omey (1984).

It should be emphasized that there exist examples with $F \notin \mathscr{S}, F_I \in \mathscr{S}$, as well as examples where $F \in \mathscr{S}, F_I \notin \mathscr{S}$. However in the standard examples of subexponential distributions with a finite variance such as the lognormal, Weibull with $\overline{F}(x) = \exp(-x^\alpha)$, $\alpha < 1$ and generalized Pareto distributions, one has $F \in \mathscr{S}^* \subset \mathscr{S}$ as well as $F_I \in \mathscr{S}^*$ (hence $F_I \in \mathscr{S}$) which can be verified by showing that the sufficient condition given in Klüppelberg (1988) is satisfied.

A common feature of the above mentioned standard examples of subexponential distributions is that they have a d.f. F for which $\overline{F}/\overline{F}_I$ is O-regularly varying. O-variation is the non-monotone variant of dominated variation, defined as follows.

Definition 2. *A positive measurable function f is O-regularly varying (notation: $f \in \mathcal{RO}$) if*

$$\limsup_{t\to\infty} \frac{f(tx)}{f(t)} < \infty \textit{ for } x > 0.$$

It is well-known (Klüppelberg (1988)) that $F \in \mathscr{S}^*$ implies that $F \in \mathscr{S}$ and $F_I \in \mathscr{S}$. Thus for subexponential distributions with a finite expectation, the class $\mathscr{S}^*$ gives a convenient criterion to ensure that both F and the integrated tail function F_I are in $\mathscr{S}$. Theorem 2 below shows that in case the hazard rate of F_I is O-regularly varying, the converse statement $F \cap F_I \in \mathscr{S} \Rightarrow F \in \mathscr{S}^*$ holds as well.

Theorem 2. *Suppose X is a positive random variable with c.d.f. F, $\mu = \mathbb{E}(X) < \infty$ and $\overline{F}/\overline{F}_I \in \mathcal{RO}$. Then the following statements are equivalent.*

1. *$F_I \in \mathscr{S}$ and $F \in \mathcal{L}$*
2. *$F \in \mathscr{S}^*$*

Note that $F_I \in \mathscr{S}$ does not imply $F \in \mathcal{L}$. For an example see Klüppelberg (1988), Example 4.2.

Finally we give some closure results related to the class $\mathscr{S}^*$. We will write $H = F * G$ (K) for the d.f. of the sum (maximum) of two independent random variables X and Y with d.f.'s F and G.

Theorem 3. *Suppose X, Y are independent random variables with d.f.'s $F, G \in \mathscr{S}^*$. Then the minimum $X \wedge Y$ has a distribution function in $\mathscr{S}^*$. Moreover the following are equivalent:*

1. $$K \in \mathscr{S}^*$$

2. $$H \in \mathscr{S}^*$$

3. $$\int_0^x \overline{F}(x-y)\overline{G}(y)dy \sim \mathbb{E}(Y)\overline{F}(x) + \mathbb{E}(X)\overline{G}(x).$$

Theorem 4. *Suppose F, $G \in \mathscr{S}^*$ and $\overline{F}/\overline{G}$ is in $\mathcal{RO}$. Then $F * G$ is in $\mathscr{S}^*$.*

2 Proofs

For the proof of Theorem 1 we need the following two Lemmas. The first result is from Ney's (1981) paper.

Lemma 2. *Suppose F is a distribution function on $(0, \infty)$ with an absolutely continuous convolution power $F^{(n)}$ and $\mu := \int_0^\infty y dF(y) < \infty$. If $F_I \in \mathscr{S}$, then the renewal function U satisfies*

$$|U(x+y) - U(x) - \frac{y}{\mu}| = O(\overline{F}_I(x)) \ (x \to \infty). \tag{8}$$

Lemma 3. *Under the conditions of Theorem 1, it follows that*

$$Z(x) - \frac{1}{\mu}\int_0^x \overline{F}_I(s)ds = \overline{F}_I(x)Z(x) + \frac{1}{\mu}\int_0^x (Z(x) - Z(s))\overline{F}(x-s)ds. \tag{9}$$

Proof of Lemma 3. From equation (4), it follows that

$$Z(x) - \frac{1}{\mu}\int_0^x \overline{F}_I(s)ds = \int_0^x \overline{F}_I(x-s)dZ(s).$$

Substitution of (2) on the right-hand side and application of Fubini's Theorem completes the proof. □

Proof of Theorem 1. We will prove the equivalent statement

$$U(x) - \frac{x}{\mu} - \frac{1}{\mu}\int_0^x \overline{F}_I(s)ds = O(\overline{F}_I(x)), (x \to \infty).$$

Note that $F_I \in \mathscr{S}^*$ implies that $\frac{1}{2}\mu_2 = \int_0^\infty x\overline{F}(x)dx = \int_0^\infty \int_s^\infty \overline{F}(t)dtds = \mu \int_0^\infty \overline{F}_I(s)ds < \infty$, hence F has a finite second moment μ_2.

By (8) there exists a constant $c > 0$ such that for all $x \geq 0$

$$|Z(x+y) - Z(x)| \leq c\overline{F}_I(x). \tag{10}$$

Moreover for $i \geq 1$, $\overline{F}_I(i+s) \leq \int_{i+s-1}^{i+s} \overline{F}_I(t)dt$, hence $\sum_{i=1}^{[x-s]} \overline{F}_I(i+s) \leq \int_s^x \overline{F}_I(t)dt$. Using Lemma 3, it follows that

$$
\begin{aligned}
&|U(x) - \frac{x}{\mu} - \frac{1}{\mu}\int_0^x \overline{F}_I(s)ds| \\
&\leq \overline{F}_I(x)|Z(x)| + \frac{1}{\mu}\int_0^x |Z(x) - Z(s)|\overline{F}(x-s)ds \\
&\leq \overline{F}_I(x)|Z(x)| + \frac{1}{\mu}\int_0^x \sum_{i=0}^{[x-s]} \{|Z(i+1+s) - Z(i+s)| \\
&\qquad\qquad + |Z(x) - Z([x-s]+1+s)|\}\overline{F}(x-s)ds.
\end{aligned}
$$

Using (10) the right-hand side can be dominated by

$$
\begin{aligned}
&\overline{F}_I(x)|Z(x)| + \frac{c}{\mu}\int_0^x \sum_{i=0}^{[x-s]} \{\overline{F}_I(i+s) + \overline{F}_I(x)\}\overline{F}(x-s)ds \\
&\leq \overline{F}_I(x)|Z(x)| + \frac{c}{\mu}\int_0^x \{\int_s^x \overline{F}_I(t)dt + \overline{F}_I(s) + \overline{F}_I(x)\}\overline{F}(x-s)ds \\
&= \overline{F}_I(x)|Z(x)| + c\int_0^x \overline{F}_I(t)\{\overline{F}_I(x-t) - \overline{F}_I(x)\}dt \\
&\qquad + c\{\int_0^x \overline{F}_I(x-s)dF_I(s) + \overline{F}_I(x)F_I(x)\} \\
&=: I_1 + I_2 + I_3.
\end{aligned}
$$

Note that $I_1/\overline{F}_I(x) = |Z(x)| \to \mu_2/2\mu^2$ as $x \to \infty$. Moreover, since $F_I \in \mathscr{S}^*$, $\int_0^x \overline{F}_I(t)\overline{F}_I(x-t)dt \sim 2c_1\overline{F}_I(x)$, hence $I_2 \sim cc_1\overline{F}_I(x)$, where $c_1 = \int_0^\infty \overline{F}_I(x)dx = \mu_2(2\mu)^{-1}$.

Finally, since $F_I \in \mathscr{S}^* \subset \mathscr{S}$, we have $\int_0^x \overline{F}_I(x-s)dF_I(s) \sim \overline{F}_I(x)$, hence $I_3 \sim (c+1)\overline{F}_I(x)$. Combination of the estimates completes the proof. □

Proof of Theorem 2. $1 \Rightarrow 2$

We have to prove that $F_I \in \mathscr{S}$ and $F \in \mathcal{L}$ implies

$$
\lim_{x\to\infty}\int_0^{x/2} \frac{\overline{F}(x-y)}{\overline{F}(x)}F(y)dy = \mu \lim_{x\to\infty}\int_0^\infty \frac{\overline{F}(x-y)}{\overline{F}(x)}\mathbf{1}_{(0,x/2)}(y)dF_I(y) = \mu. \tag{11}
$$

Since $\overline{F}/\overline{F}_I \in \mathcal{RO}$, using the uniform convergence theorem for $\mathcal{RO}$ varying functions (see e.g. Bingham et al. (1987), Ch. 3), it follows that there exists a constant $c > 0$ such that

$$
\frac{\overline{F}(x-y)}{\overline{F}(x)} \leq c\frac{\overline{F}_I(x-y)}{\overline{F}_I(x)} \text{ for } y \in (0, x/2),\ x > 0.
$$

Since $F_I \in \mathscr{S}$, we have $\lim_{x\to\infty}\int_0^\infty \frac{\overline{F}_I(x-y)}{\overline{F}_I(x)}\mathbf{1}_{(0,x/2)}(y)dF_I(y) = 1$ and $\int_0^\infty \lim_{x\to\infty}\frac{\overline{F}_I(x-y)}{\overline{F}_I(x)}\mathbf{1}_{(0,x/2)}(y)dF_I(y) = 1$ (since $F_I \in \mathscr{S} \subset \mathcal{L}$) . Application of Pratt's lemma (see Pratt (1960)) to the second integral in (11) completes the proof.

$2 \Rightarrow 1$ The implication $F \in \mathscr{S}^* \Rightarrow F, F_I \in \mathscr{S}$ is proved in Klüppelberg (1988). Since $\mathscr{S} \subset \mathcal{L}$, the implication $2 \Rightarrow 1$ follows. □

In order to prove Theorem 4 we need Theorem 3, which is related to a result by Klüppelberg and Villasenor (1991). In order to prove Theorem 3, we list the properties we need in two Lemmas.

Lemma 4. *The following statements are equivalent:*

- $F \in \mathscr{S}^*$
- $F \in \mathcal{L}$ *and*

$$\lim_{v\to\infty} \limsup_{x\to\infty} \int_v^{x-v} \frac{\overline{F}(x-y)\overline{F}(y)}{\overline{F}(x)} dy = 0. \tag{12}$$

Proof of Lemma 4. Write

$$\int_0^x \overline{F}(x-y)\overline{F}(y)\ dy = 2\int_0^v \overline{F}(x-y)\overline{F}(y)\ dy + \int_v^{x-v} \overline{F}(x-y)\overline{F}(y)\ dy.$$

□

Lemma 5. *For $F \in \mathscr{S}$, there exists a constant $c > 0$ such that*

$$\frac{\overline{F}(x-y)\overline{F}(y)}{\overline{F}(x)} \le c \text{ for all } x > 0, \text{ uniformly in } y \in (0, x).$$

Proof of Lemma 5. When X_1, X_2 are i.i.d. with d.f. $F \in \mathscr{S}$, then for any $y \in (0, x)$

$$\overline{F}(x-y)\overline{F}(y) = P(X_1 > x-y, X_2 > y) \le P(X_1 + X_2 > x) \sim 2\overline{F}(x).$$

□

Proof of Theorem 3. First we show that $X \wedge Y$ has a d.f. in $\mathscr{S}^*$. In order to prove this, we show that the equivalent of the necessary and sufficient condition (12) holds for the tail $\overline{F}(x)\overline{G}(x)$ of the d.f. of the minimum $X \wedge Y$. Since $F \in \mathscr{S}^* \subset \mathscr{S}$, with the constant c as in Lemma 5, we have

$$\begin{aligned} \lim_{v\to\infty} \limsup_{x\to\infty} & \int_v^{x-v} \frac{\overline{F}(x-y)\overline{F}(y)}{\overline{F}(x)} \frac{\overline{G}(x-y)\overline{G}(y)}{\overline{G}(x)} dy \\ & \le c \lim_{v\to\infty} \limsup_{x\to\infty} \int_v^{x-v} \frac{\overline{G}(x-y)\overline{G}(y)}{\overline{G}(x)} dy. \end{aligned} \tag{13}$$

Since $G \in \mathscr{S}^*$, the right-hand side in (13) is zero by Lemma 4. Since the product $\overline{F}.\overline{G}$ is the tail of a d.f. in $\mathcal{L}$ (since $F, G \in \mathscr{S}^* \subset \mathcal{L}$), the conditions of the second part of Lemma 4 are satisfied, which completes the proof of the first statement.

(1) ⇔ (2) From Theorem 1 in Geluk (2009), it follows that $K \in \mathscr{S}$ if and only if $H \in \mathscr{S}$, and as a consequence, by Theorem 2 in Embrechts and Goldie (1980) that $\overline{H}(x) \sim \overline{K}(x)$. Since $\mathscr{S}^* \subset \mathscr{S}$ and $\mathscr{S}^*$ is closed under asymptotic tail equivalence (see Klüppelberg (1988)), it follows that $H \in \mathscr{S}^*$ and $K \in \mathscr{S}^*$ are equivalent.

(1) ⇔ (3) See Klüppelberg and Villasenor (1991). □

Proof of Theorem 4. We write the integral $\int_0^x \overline{F}(x-y)\overline{G}(y)dy$ as $I_1 + I_2$, where $I_1 = \int_0^{x/2} \overline{F}(x-y)\overline{G}(y)dy$ and $I_2 = \int_0^{x/2} \overline{G}(x-y)\overline{F}(y)dy$. Define $a(x) = \overline{F}(x)/\overline{G}(x)$ and write $M = \sup_{x>0} \sup_{0\leq y\leq x/2} a(x-y)/a(x)$. It follows that

$$\frac{I_1}{\overline{F}(x)} = \int_0^{x/2} \frac{\overline{F}(x-y)}{\overline{F}(x)}\overline{G}(y)dy \leq M \int_0^{x/2} \frac{\overline{G}(x-y)}{\overline{G}(x)}\overline{G}(y)dy$$

An application of Pratt's Lemma then shows that $I_1 \sim \mathbb{E}(Y)\overline{F}(x)$. Since I_2 can be treated similarly, we may apply Theorem 3 in order to complete the proof. □

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