On preferences over subsets and the lattice structure of stable matchings

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Abstract. This paper studies the structure of stable multipartner matchings in two-sided markets where choice functions are *quotafilling* in the sense that they satisfy the substitutability axiom and, in addition, fill a quota whenever possible. It is shown that (i) the set of stable matchings is a lattice under the common revealed preference orderings of all agents on the same side, (ii) the supremum (infimum) operation of the lattice for each side consists componentwise of the join (meet) operation in the revealed preference ordering of the agents on that side, and (iii) the lattice has the polarity, distributivity, complementariness and full-quota properties.

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1 Introduction

In this paper we study matching between two sets of agents when each agent may have *multiple* partners. We refer to the agents as *men* and *women*. Our interest is in the structural properties of the set of matchings which are *stable* in the sense that they are not blocked by any individual or man-woman pair.[1]

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[1] This may be criticized since the set of multipartner matchings that are stable in this sense is not necessarily the core. See Sotomayor (1999). It is a fact on the other hand that, allowing blocking coalitions of bigger size may leave no allocation that is stable. We are therefore tacitly assuming, as widely done in the coalition formation literature for example, that there are inherent costs or structural reasons which forbid the formation of bigger coalitions.

As is well known, in the monogamous case, stable matchings have the structure of a lattice under the common preferences of all agents of the same sex and further remarkable properties.[2] It was shown, recently, by Baiou and Balinski (1998) and Alkan (1999), that these properties hold for multipartner matching as well, when preferences over partner-sets are *classical* in the sense that they are consistent with an ordering on individual partners. We show in this paper that all the properties generalize and hold over a substantially broader domain, namely that of (what we call) *quotafilling* preferences.

A somewhat novel aspect of our model is that agents' preferences are not given completely. We actually decribe each agent by a *choice function* that picks from any set of potential partners the subset (*team)* he prefers the most. We then adopt the definition that an agent *prefers* Team A to Team B if and only if he chooses Team A given all the individuals in Team A or Team B. This natural definition endows each agent with a (*revealed*) preference relation which, although incomplete, happens to be sufficiently complete for all our interest.

We place two assumptions on choice functions. One is the standard *substitutability* axiom according to which a partner who is chosen given a set of individuals is also chosen given any subset of the same individuals. It is a fact that, under a mild consistency axiom in addition to substitutability, revealed preference relations happen to be partial orders with further properties that we will make much use of. In particular, every pair of teams has a least upper bound (*join*), which coincides with the team chosen from their union, and a greatest lower bound (*meet*). Let us mention that these two axioms - substitutability and consistency - are together equivalent to the well-known *path independence* condition due to Plott (1973).[3]

Our second assumption on choice functions is that they fill a *quota* whenever there are sufficiently many potential partners available. We call choice functions that satisfy both of our assumptions *quotafilling*. They constitute a subclass of path independent choice functions.

Our findings in summary say the following: *The set of stable matchings in two-sided markets where choice functions are quotafilling is a lattice under the common revealed preference orderings of all agents on the same side. The supremum (infimum) operation of the lattice for each side consists componentwise of the join (meet) operation in the revealed preference ordering of the agents on that side. The lattice has the polarity, distributivity, complementariness and full-quota properties.*

Let us describe the four properties last mentioned above: The polarity and distributivity properties are the same as in monogamous matching: Polarity says that the supremum of two stable matchings with respect to one side is identical with their infimum with respect to the other. Distributivity says that the join and meet operations are distributive on the set of stable teams for each agent.

[2] See Roth and Sotomayor (1990).

[3] See for instance Aizerman and Aleskerov (1995) where the substitutability and consistency axioms are called the Heritage and Outcast conditions respectively. The equivalence result is due to Aizerman and Malishevski (1979).

The property we have called complementariness holds trivially in the monogamous case and, as we will point out below, has a key role in how we obtain our results. It says that the meet (join) of two stable teams is precisely the set of all partners who are in one of the two teams but *not* in their join (meet), united with those partners who are in both. The full-quota property says that the teams an agent is matched with in all stable matchings are either all full-quota or all identical. It is a generalization of the monogamous-matching property that an agent unmatched in one stable matching is unmatched in every stable matching.

Our results hold, as we already mentioned, on the classical domain where preferences are given by an ordering on individual partners. On this domain, in fact, the set of stable matchings has the following additional property: Given any two stable teams for an agent, one is always their join and the other their meet, or equivalently, every partner in one team is of a higher rank than every partner in the other team but not in the former (see Alkan 1999).[4] There is, consequently, a complete ordering on all stable teams. We show by an example that this need not be the case with quotafilling choice functions: Join of a pair of stable teams may be distinct from either team.[5]

For the broader domain of path-independent choice functions, on the other hand, Blair (1988) had shown that stable matchings always exist and form a lattice under the common preferences of all agents on one side of the market. However, as one sees upon checking the examples provided by Blair (1988), the properties that we cited above practically all fail to hold on this domain.[6] Notably, the supremum or infimum of stable matchings, as defined here, may be unstable.

Our findings thus establish quotafilling choice functions as constituting a substantially broad intermediate domain, between the classical and the path-independent, where stable matchings form a lattice with fine properties.[7] It is worth mentioning that these properties might make a difference that is of interest for economic design. For example, the distributivity property ensures that agents are able to evaluate different stable matchings by assigning them *scores*, which in turn render possible the definition of *sex-equal* stable matchings, as Gusfield and Irving (1989) had suggested for monogamous matching.

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[4] Theorem 5.2 in Roth and Sotomayor (1990) states this result for the case of one-to-many matchings.

[5] It has been posed, by one referee, whether a multipartner matching market under substitutable or quotafilling preferences is reducible to a monogamous matching market. The answer appears to be negative: Otherwise, preferences that achieve the reduction would be classical, hence join of stable teams could not be distinct, contradicting the example just mentioned.

[6] Blair remarked that "the multipartner lattice is not necessarily distributive, although the monogamous ones are. This suggests the two situations are fundamentally different, but some further insight would be helpful."

[7] In their study on *many-to-one* matchings, Martinez et al. (1999) prove the full-quota property for *complete* preferences that satisfy substitutability plus the "q-separability" property that they introduce in their paper. This preference domain appears to be the domain one would obtain by "completing" the *quotafilling* preferences we have introduced here. Let us add, as pointed out by one of our referees, that "filling a quota" is not a novel criterion and, for example, a basic feature of responsive preferences.

We close our introduction with a partial preview of the way we obtain our results. It will be helpful to first see this in the monogamous case. To this end, take any two stable matchings. Let each man choose the *more* preferable of his two partners. It is a fact that every man will choose a distinct woman and that it is feasible therefore to match every man with his choice. Call this matching the *supremum* of the two initial matchings. It is a further interesting fact that, in the supremum, each woman is matched with her *less* preferable partner. One recognizes that these two facts give the polarity property, namely that male supremum coincides with female *infimum* for stable matchings, under the (natural) definition that the female infimum of two matchings is the matching where every woman gets her less preferable partner.

In our exploration of multipartner matching here, we follow the same route above and aim first to obtain the polarity property. The definition we adopt for the supremum operation is a straightforward and natural extension of the definition in the monogamous case. Thus, given two matchings, we let each man choose and be matched with his most preferable team among all his partners in the two matchings. In other words, we define the supremum operation on matchings via the join operation of agents. The definition we adopt for the infimum, on the other hand, is via an operation that extends the definition in the monogamous case in not an evident way: Given two matchings, we let each man be matched with (what we call) the *pseudomeet* of the two teams he is matched with initially, namely the set of all partners who are in one of the two teams but not in their join, united with those partners who are in both. Pseudomeet of teams in general need not be their lower bound. As we show, on the other hand, the polarity property holds with supremum infimum so defined and pseudomeet of stable teams is their greatest lower bound (meet). One recognizes that these facts give the complementariness property we described earlier.

2 Basic definitions

A *matching market* $(M, W; \mathscr{C}_M, \mathscr{C}_W)$ consists of two finite sets M, W of agents, say *men* and *women*, where each man m is described by a *choice function* $\mathscr{C}_m : 2^W \longrightarrow 2^W$, satisfying

$$\mathscr{C}_m(T) \subset T$$

for all $T \subset W$, and the analogous description holds for each woman w. A *matching* is a map $\mu : M \cup W \longrightarrow 2^W \cup 2^M$ such that

$$\mu(m) \subset W, \mu(w) \subset M,$$

and

$$m \in \mu(w) \text{ if and only if } w \in \mu(m),$$

for all m, w. We use the notation m to denote an element of M as well as the set $\{m\}$.

A matching μ is *individually rational* if

$$\mathscr{C}_m(\mu(m)) = \mu(m), \mathscr{C}_w(\mu(w)) = \mu(w)$$

and *pairwise stable* if

$$w \in \mathscr{C}_m(\mu(m) \cup w) - \mu(m) \text{ implies } m \notin \mathscr{C}_w(\mu(w) \cup m)$$

for all m, w. Thus, a matching is individually rational if no one would disassociate with any current partner and pairwise stable if there is no man-woman pair who are not partners but who would each choose the other in the presence of all current partners.

We call a matching *stable* if it is individually rational and pairwise stable.

3 Quotafilling choice and lattice of teams

In this section we consider a single agent, say w. We call $S \subset M$ a *team* for w if S is in the range of $\mathscr{C}_w$, that is to say, if $S = \mathscr{C}_w(T)$ for some $T \subset M$. Let k_w denote the *maximum* cardinality of a team among all teams for w. Let $k = k_w$, $\mathscr{C} = \mathscr{C}_w$.

3.1 Preliminaries

The choice function $\mathscr{C}$ is said to satisfy *substitutability* if

$$a \in \mathscr{C}(T) \text{ implies } a \in \mathscr{C}(S \cup a) \text{ for all } S \subset T.$$

We call teams with cardinality k *full-quota*. We assume $\mathscr{C}$ is *quotafilling* in the sense that it satisfies substitutability and $\mathscr{C}(T)$ is full-quota for all T with cardinality at least k. It is easy to see that, under this assumption, the set of all teams is the collection

$$\mathscr{K} = \{S \subset M \mid |S| \leq k\}.$$

Observe that $\mathscr{C}$ is *idempotent*, i.e., $\mathscr{C}(S) = S$ for all $S \in \mathscr{K}$. Note also that $\mathscr{C}$ is *path independent* in the sense

$$\mathscr{C}(\mathscr{C}(T) \cup T') = \mathscr{C}(T \cup T') \text{ for all } T, T'.$$

(*Proof*. By substitutability $\mathscr{C}(T \cup T') \subset \mathscr{C}(\mathscr{C}(T) \cup T')$. If $\mathscr{C}(T \cup T')$ is full-quota then the inclusion is equality by quotafillingness. If on the other hand $\mathscr{C}(T \cup T')$ is not full-quota, then neither is T, so $\mathscr{C}(T) = T$, whence the inclusion is equality once again.)

We define the *revealed preference* binary relation $\succeq$ over $\mathscr{K}$ by

$$S \succeq S' \text{ if and only if } \mathscr{C}(S \cup S') = S.$$

Observe that $\succeq$ is *reflexive* since $\mathscr{C}$ is idempotent and *antisymmetric* since $\mathscr{C}(T)$ is unique for all T. Also, as is well known, it follows from $\mathscr{C}$ being path independent that $\succeq$ is *transitive*.[8] Thus $\{\mathscr{K}, \succeq\}$ is a partially ordered set.

As one easily sees, in fact, $\mathscr{C}(S \cup S')$ is the *least upper bound* of S, S'.

(*Proof.* Suppose $\mathscr{C}(S \cup S'') = \mathscr{C}(S' \cup S'') = S''$. Then by path independence $\mathscr{C}(\mathscr{C}(S \cup S') \cup S'') = \mathscr{C}(S \cup S' \cup S'') = \mathscr{C}(S \cup \mathscr{C}(S' \cup S'')) = \mathscr{C}(S \cup S'') = S''$.) Thus $\mathscr{K}$ is a semilattice with the *join* operation $\vee$ given by

$$S \vee S' = \mathscr{C}(S \cup S').$$

Furthermore, since $\mathscr{K}$ has a minimum element, namely the empty set, it follows that every pair in $\mathscr{K}$ also has a *greatest lower bound*. Thus $\mathscr{K}$ is endowed with a *meet* (greatest lower bound) operation $\wedge$ as well, i.e., $\{\mathscr{K}, \vee, \wedge\}$ is a *lattice*.[9]

In Sect. 6.1, we give an example of a quotafilling choice function and the diagram of the lattice of teams chosen by this function.

3.2 Pseudomeet

Let S, S' be any pair of teams. We introduce a binary operation $\triangle$ on $\mathscr{K}$, that we call *pseudomeet*, by defining

$$S \triangle S' := ((S \cup S') - (S \vee S')) \cup (S \cap S'). \tag{1}$$

Thus, pseudomeet of two teams is the set of all partners who belong to either one of the teams but *not* to their join, united with those partners who belong to both. This operation will serve as the main tool in obtaining our results in the next section. We now make three observations that we will make use of.

Pseudomeet of teams is not in general their lower bound.[10] Our first observation gives a sufficient condition under which pseudomeet is greatest lower bound (meet).

Lemma 1. *If $S \triangle S'$ is full-quota and a lower bound for S, S', then $S \triangle S' = S \wedge S'$.*

Proof. Let L be any lower bound for S, S' and denote $Q = S \triangle S'$. Take $a \in Q$. Say $a \in S$. Then $a \in \mathscr{C}(S \cup Q)$ since $S = \mathscr{C}(S \cup Q)$ by assumption. So $a \in \mathscr{C}(\mathscr{C}(L \cup S) \cup Q)$ since $S = \mathscr{C}(L \cup S)$. Therefore $a \in \mathscr{C}(L \cup S \cup Q)$ by path independence, hence $a \in \mathscr{C}(L \cup Q)$ by substitutability. That is, $Q \subset \mathscr{C}(L \cup Q)$. Since Q is full-quota, $Q = \mathscr{C}(L \cup Q)$. Thus, Q is an upper bound for L, hence, the greatest lower bound of S, S'. $\square$

[8] *Proof.* For any $S, S', S'' \in \mathscr{K}$, if $S \succeq S'$ and $S' \succeq S''$, then $\mathscr{C}(S \cup S'') = \mathscr{C}(\mathscr{C}(S \cup S') \cup S'') = \mathscr{C}(S \cup S' \cup S'') = \mathscr{C}(S \cup \mathscr{C}(S' \cup S'')) = \mathscr{C}(S \cup S') = S$.

[9] More generally, the range of a path independent choice function is a lattice and has interesting properties. See Koshevoy (1999) and Monjardet and Raderanirina (1999).

[10] For example, the choice function $\mathscr{C}$ such that $\mathscr{C}\{a,b,c,d\} = \mathscr{C}\{a,b,c\} = \mathscr{C}\{a,b,d\} = \{a,b\}, \mathscr{C}\{a,c,d\} = \{a,c\}, \mathscr{C}\{b,c,d\} = \{b,c\}$ is quotafilling, but pseudomeet of $\{a,c\}, \{b,d\}$ is $\{c,d\}$ which is not a lower bound of $\{b,d\}$.

We next observe that pseudomeet of a pair of teams is never greater than their join in cardinality: To see this, first note from (1) that $|S \triangle S'| = |S \cup S'| - (|S \vee S'| - |(S \vee S') \cap (S \cap S')|) = |S| + |S'| - |S \cap S'| - (|S \vee S'| - |(S \vee S') \cap (S \cap S')|)$. Say $|S| \leq |S'|$. Upon rearrangement,

$$|S \triangle S'| = |S| + (|S'| - |S \vee S'|) - (|S \cap S'| - |(S \vee S') \cap (S \cap S')|). \tag{2}$$

Now, since $|S'| \leq |S \vee S'|$ by quotafillingness and $|S \cap S'|$ is never greater than $|(S \vee S') \cap\ (S \cap S')|$, it follows that

$$|S \triangle S'| \leq \min\{|S|, |S'|\} \leq \max\{|S|, |S'|\} \leq |S \vee S'|. \tag{3}$$

Our third observation describes when pseudomeet of a pair of teams is full-quota: We call a pair of teams S, S' *concordant* if $S \vee S'$ contains $S \cap S'$.

Lemma 2. *Let S, S' be any pair of distinct teams. The following three conditions are equivalent: (i) $|S \triangle S'| = |S \vee S'|$. (ii) S, S' are full-quota and concordant. (iii) $S \triangle S'$ is full-quota.*

Proof. Assume (i). Then $|S| = |S'|$ by (3). So $|S| < |S \cup S'|$ since $S \neq S'$. But then S (likewise S') must be full-quota, for otherwise $|S| < |S \vee S'|$ by quotafillingness, implying $|S \triangle S'| < |S \vee S'|$ by (3); contradiction. Also, $S \vee S'$ must contain $S \cap S'$, for otherwise $|S \cap S'| < |(S \vee S') \cap (S \cap S')|$, implying $|S \triangle S'| < |S \vee S'|$ by (2); contradiction again. Thus (i) implies (ii). (ii) implies (iii) by (2). (iii) implies (i) by (3). □

We shall call a set of teams concordant if every pair of teams in the set is concordant.

4 Lattice structure and other properties of stable matchings

Let $(M, W; \mathscr{C}_M, \mathscr{C}_W)$ be a matching market with quotafilling choice functions $\mathscr{C}_m, \mathscr{C}_w$. We define two binary operations, *supremum* and *infimum*, on the set of all matchings: The *male* supremum of a pair μ_1, μ_2 is the matching μ^M where

$$\mu^M(m) = \mu_1(m) \vee_m \mu_2(m),$$

and the *male* infimum of μ_1, μ_2 is the matching μ_M where

$$\mu_M(m) = \mu_1(m) \triangle_m \mu_2(m).$$

Female supremum and infimum are defined analogously.

We will show that the matchings μ^M, μ_M are stable, and that

$$\mu_M(m) = \mu_1(m) \wedge_m \mu_2(m),$$

when μ_1, μ_2 are stable.

Say that a set of matchings Ψ has the *polarity* property if $\mu^M = \mu_W$ and $\mu^W = \mu_M$ for all μ_1, μ_2 in Ψ. We start by showing that the set of stable matchings has this property.

Let μ_1, μ_2 be any pair of stable matchings.

Proposition 1. *The set of stable matchings has the polarity property.*

Proof. We first show

$$\mu^M \subset \mu_W. \tag{4}$$

Take any m and any $w \in \mu^M(m)$, that is

$$w \in \mathscr{C}_m(\mu_1(m) \cup \mu_2(m)) \subset \mu_1(m) \cup \mu_2(m).$$

If $w \in \mu_1(m) \cap \mu_2(m)$ then $m \in \mu_1(w) \cap \mu_2(w)$ so by definition of pseudomeet $m \in \mu_W(w)$ hence $w \in \mu_W(m)$ affirming (4). So say $w \in \mu_2(m) - \mu_1(m)$. Then

$$w \in \mathscr{C}_m(\mu_1(m) \cup w) - \mu_1(m)$$

by substitutability, so $m \notin \mathscr{C}_w(\mu_1(w) \cup m)$ by stability. Hence, by substitutability, $m \notin \mathscr{C}_w(\mu_1(w) \cup \mu_2(w)) = \mu^W(w)$. By definition of pseudomeet then $m \in \mu_W(w)$ so $w \in \mu_W(m)$ proving (4).

From (4), $|\mu^M| \leq |\mu_W|$. Also $|\mu^W| \leq |\mu_M|$ by symmetry. From (3), on the other hand, $|\mu_W(w)| \leq |\mu^W(w)|$ for all w, so

$$|\mu_W| = \sum_w |\mu_W(w)| \leq \sum_w |\mu^W(w)| = |\mu^W|. \tag{5}$$

Also $|\mu_M| \leq |\mu^M|$ by symmetry. Thus $|\mu^M| \leq |\mu_W| \leq |\mu^W| \leq |\mu_M| \leq |\mu^M|$. Hence,

$$|\mu^M| = |\mu_W| = |\mu^W| = |\mu_M|. \tag{6}$$

From (4) therefore $\mu^M = \mu_W$. □

It follows from (5) and (6), in fact, that supremum and infimum of stable matchings match an agent with an equal number of partners, namely

$$|\mu_W(w)| = |\mu^W(w)| \text{ for all } w. \tag{7}$$

Corollary 1 and Lemma 3 stated below now directly follow from Lemma 2:

Given a set of matchings Ψ, denote Ψ_m the set of all $\mu(m)$ where μ is in Ψ. We will say that Ψ has (i) the *concordance* property if every Ψ_m, Ψ_w is concordant and (ii) the *full-quota* property if every Ψ_m, Ψ_w is either a set of full-quota teams or a singleton.

Corollary 1. *The set of stable matchings has the full-quota and concordance properties.*

Say that S is a *stable team* for an agent if he or she is matched with S by some stable matching. Lemma 3 below says that pseudomeet of distinct stable teams is full-quota:

Lemma 3. $\mu_W(w)$ *is full-quota for all* w *such that* $\mu_1(w) \neq \mu_2(w)$.

We next show that pseudomeet of stable teams is their lower bound.

Lemma 4. $\mathscr{C}_w(\mu_1(w) \cup \mu_W(w)) = \mu_1(w)$ *for all* w.

Proof. If $m \in \mu_w(w) - \mu_1(w)$ then $w \in \mu_w(m) - \mu_1(m)$, hence $w \in \mu^M(m) - \mu_1(m)$ by polarity. Thus $w \in \mathscr{C}_m(\mu_1(m) \cup \mu_2(m)) - \mu_1(m)$ and $w \in \mu_2(m)$. So $w \in \mathscr{C}_m(\mu_1(m) \cup w) - \mu_1(m)$ by substitutability, therefore $m \notin \mathscr{C}_w(\mu_1(w) \cup m)$ by stability, so $m \notin \mathscr{C}_w(\mu_1(w) \cup \mu_w(w))$ by substitutability. Thus $\mathscr{C}_w(\mu_1(w) \cup \mu_w(w)) \subset \mu_1(w)$. By quotafillingness, the inclusion is equality. □

It follows from Lemmas 1, 3 and 4 that pseudomeet is meet over stable teams. Thus

Proposition 2. $\mu_w(w) = \mu_1(w) \wedge_w \mu_2(w)$ *for all* w.

It also follows from the pseudomeet-meet equivalence just noted and the concordance property (Corollary 1) that join and meet of stable teams are symmetric "complements" in the following sense:

Corollary 2. $\mu_1(w) \wedge_w \mu_2(w) = ((\mu_1(w) \cup \mu_2(w)) - (\mu_1(w) \vee_w \mu_2(w))) \cup (\mu_1(w) \cap \mu_2(w))$ *and* $\mu_1(w) \vee_w \mu_2(w) = ((\mu_1(w) \cup \mu_2(w)) - (\mu_1(w) \wedge_w \mu_2(w))) \cup (\mu_1(w) \cap \mu_2(w))$ *for all* w.

We will refer to Corollary 2 as the *complementariness* property.

Our next result says that the set of stable matchings is closed under supremum and infimum:

Proposition 3. *The supremum and infimum of stable matchings are stable.*

Proof. Take any agent say w. Since $\mu^M(w) = \mu_w(w)$ by polarity and $|\mu_w(w)| = |\mu^W(w)|$ (see (7)), $\mu^M(w)$ is in $\mathscr{K}_w$. So the supremum is individually rational by idempotency. We show in the paragraph below that μ^M is pairwise stable. Proposition then follows by polarity.

Take any m and any

$$w \in \mathscr{C}_m(\mu^M(m) \cup w) - \mu^M(m). \tag{8}$$

We need to show

$$m \notin \mathscr{C}_w(\mu^M(w) \cup m). \tag{9}$$

From (8), $w \in \mathscr{C}_m(\mu_1(m) \cup \mu_2(m) \cup w)$ by path independence, moreover $w \notin \mu_1(m) \cup \mu_2(m)$ (since $w \notin \mu^M(m)$), in particular $w \in \mathscr{C}_m(\mu_1(m) \cup w) - \mu_1(m)$ by substitutability, so $m \notin \mathscr{C}_w(\mu_1(w) \cup m)$ by stability, hence

$$\mathscr{C}_w(\mu_1(w) \cup m) = \mu_1(w).$$

Therefore, using path independence and the fact that pseudomeet is lower bound (Lemma 2), $\mathscr{C}_w(\mu_1(w) \cup \mu_w(w) \cup m) = \mathscr{C}_w(\mathscr{C}_w(\mu_1(w) \cup m) \cup \mu_w(w)) = \mathscr{C}_w(\mu_1(w) \cup \mu_w(w)) = \mu_1(w)$. By substitutability then $m \in \mathscr{C}_w(\mu_w(w) \cup m)$ for all $m \in \mu_1(w) \cap \mu_w(w)$, that is

$$\mu_1(w) \cap \mu_w(w) \subset \mathscr{C}_w(\mu_w(w) \cup m).$$

Symmetrically, $\mu_2(w) \cap \mu_w(w) \subset \mathscr{C}_w(\mu_w(w) \cup m)$. Thus

$$\mu_w(w) \subset \mathscr{C}_w(\mu_w(w) \cup m).$$

Since $\mu_w(w)$ is full-quota (Lemma 3), the inclusion above must be equality. In particular $m \notin \mathscr{C}_w(\mu_w(w) \cup m)$ which gives (9) by polarity. □

We now put all our findings together: Say that a set of matchings Ψ has the *distributivity* property if supremum and infimum are distributive over Ψ.

Theorem 1. *The set of stable matchings in any market with quotafilling choice functions is a lattice under the revealed preference orderings of all agents on one side of the market. The supremum (infimum) operation of the lattice for each side consists componentwise of the join (meet) operation in the revealed preference ordering of the associated agent. The lattice has the polarity, full-quota, complementariness and distributivity properties.*

Proof. By Propositions 2 and 3, stable matchings are a lattice under the supremum infimum operations whose coordinates are the join meet operations in agents' revealed preference orderings. The polarity property was shown in Proposition 1. The full-quota property was noted in Corollary 1 and the complementariness property noted in Corollary 2. It only remains to show the distributivity property. We do so by showing that the join and meet operations are distributive on the set of stable teams for each agent:

Take any three teams S, S', S'' for an agent. Let $\vee, \wedge$ denote the join meet operations of the agent. Suppose

$$S \vee S' = S \vee S'' \text{ and } S \wedge S' = S \wedge S''.$$

We claim $S' = S''$ from which distributivity follows (by Corollary to Theorem 13, Birkhoff (1973)):

Take $a \in S'$. Note $a \in S' \subset S \cup S' = (S \vee S') \cup (S \wedge S') = (S \vee S'') \cup (S \wedge S'') = S \cup S''$. So if $a \notin S$ then $a \in S''$. If on the other hand $a \in S$, then $a \in S \cap S'$ so by definition $a \in S \Delta S' = S \wedge S'$ and by concordance $a \in S \vee S'$. Hence $a \in (S \vee S'') \cap (S \wedge S'') = S \cap S''$ proving $a \in S''$ again, thus our claim. □

5 Example

5.1 An agent with a quotafilling choice function

Consider an agent w who has the choice function $\mathscr{C}$ over the set $\{A, B, C, a, b, c\}$ such that $\mathscr{C}(T) = \{A, B, C\}$ for all $T \supset \{A, B, C\}$ and

$$\mathscr{C}\{A, B, a, b, c\} = \mathscr{C}\{A, B, a, c\} = \mathscr{C}\{A, B, b, c\} = \{A, B, c\},$$

$$\mathscr{C}\{A, C, a, b, c\} = \mathscr{C}\{A, C, a, c\} = \mathscr{C}\{A, C, a, b\} = \{A, C, a\},$$

$$\mathscr{C}\{B, C, a, b, c\} = \mathscr{C}\{B, C, a, c\} = \mathscr{C}\{B, C, a, b\} = \{B, C, a\},$$

$$\begin{aligned}
\mathscr{C}\{A,a,b,c\} &= \{A,a,c\},\\
\mathscr{C}\{B,a,b,c\} &= \{B,a,c\},\\
\mathscr{C}\{C,a,b,c\} &= \{C,a,b\},\\
\\
\mathscr{C}\{A,B,a,b\} &= \{A,B,b\},\\
\mathscr{C}\{A,C,b,c\} &= \{A,C,b\},\\
\mathscr{C}\{B,C,b,c\} &= \{B,C,c\},
\end{aligned}$$

while $\mathscr{C}(S) = S$ for all $S \subset A$ with $|S| \leq 3$. It is routine to check that $\mathscr{C}$ is quotafilling.

The lattice of teams for w can be seen in diagram in Fig. 1. It is worth pointing out that, for instance, $a > b$ in the presence of A, C but $b > a$ in the presence of A, B. In particular, there is no partial order on $\{A, B, C, a, b, c\}$ which would rationalize $\mathscr{C}$.

5.2 A matching market

Consider a market with six men A, B, C, a, b, c and four women w, x, y, z where w is described in the previous subsection. All other agents have quotas equal to 1. Agent A regards x his best mate and w his second-best mate. The matrix below expresses this and the best and second-best mates for the others:

A	**B**	**C**	**a**	**b**	**c**	**x**	**y**	**z**
x	y	z	w	w	w	a	b	c
w	w	w	x	y	z	A	B	C

It is straightforward to check that the six matchings listed below are stable:

$$\mu^w(w,x,y,z) = (\{A,B,C\},\{a\},\{b\},\{c\}),$$

$$\mu_1(w,x,y,z) = (\{A,B,c\},\{a\},\{b\},\{C\}),$$

$$\mu_2(w,x,y,z) = (\{B,C,a\},\{A\},\{b\},\{c\}),$$

$$\mu_3(w,x,y,z) = (\{C,a,b\},\{A\},\{B\},\{c\}),$$

$$\mu_4(w,x,y,z) = (\{B,a,c\},\{A\},\{b\},\{C\}),$$

$$\mu_w(w,x,y,z) = (\{a,b,c\},\{A\},\{B\},\{C\}).$$

It is also not difficult to see that any matching μ where $\mu(w)$ contains $\{A,a\}$, $\{B,b\}$, $\{C,c\}$ is unstable and that $\{a,w\}$ would block μ if $\mu(w)$ were $\{A,b,c\}$ or $\{A,b,C\}$. Thus the set of stable matchings Σ consists of the six matchings listed above.

We observe that μ^w, μ_w are the female supremum and infimum of Σ respectively, that

$$\mu^w = \mu_1 \vee_w \mu_2 = \mu_1 \vee_w \mu_3,$$

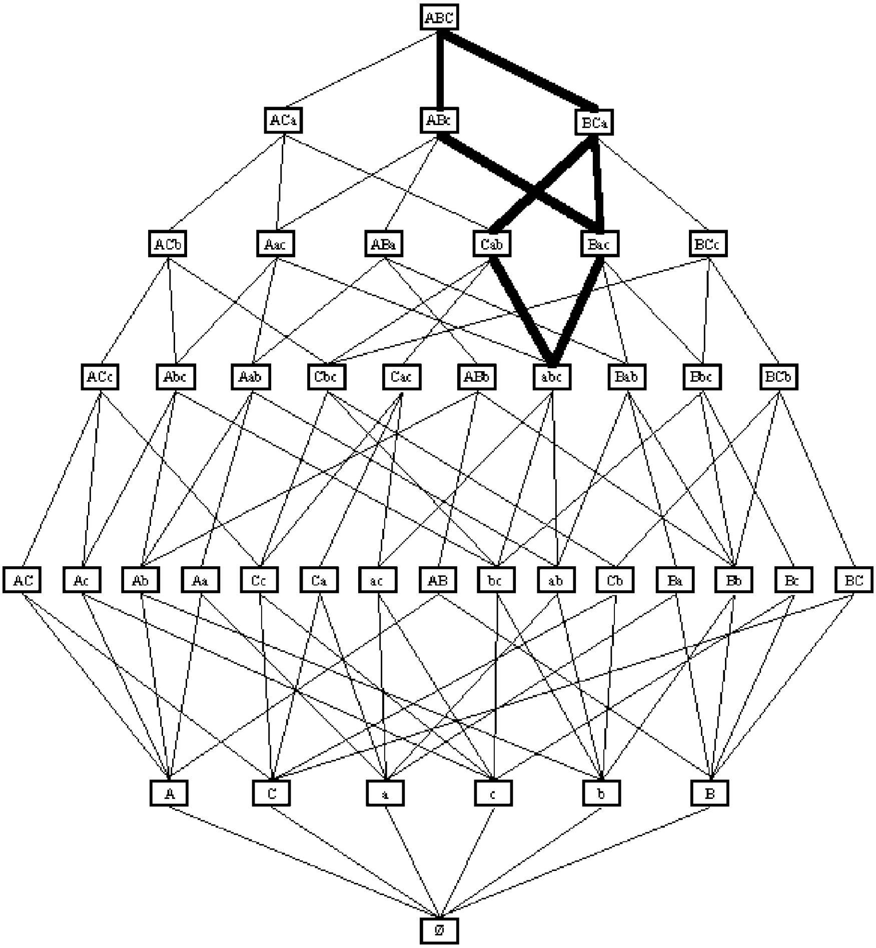

Fig. 1. The lattice of teams of a quotafilling choice function

$$
\begin{aligned}
\mu_2 &= \mu_3 \vee_w \mu_4, \\
\mu_4 &= \mu_1 \wedge_w \mu_2,
\end{aligned}
$$

$$\mu_w = \mu_1 \wedge_w \mu_3 = \mu_3 \wedge_w \mu_4,$$

and that Σ owns the two sublattices $\{\mu^w, \mu_1, \mu_2, \mu_4\}$,$\{\mu_2, \mu_3, \mu_4, \mu_w\}$.

We have displayed the stable teams for w in bold in Fig. 1. We note that there are stable teams, for instance $\{A, B, c\}$ and $\{B, C, a\}$, which are mutually incomparable, a feature that would never occur with classical choice functions, as we mentioned in the introduction to our paper.

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