On the Investment Implications of Bankruptcy Laws*

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Abstract

Axiomatic analysis of bankruptcy problems reveals three major principles: (*i*) *proportionality (PRO)*, (*ii*) *equal awards (EA)*, and (*iii*) *equal losses (EL)*. However, most real life bankruptcy procedures implement only the proportionality principle. We construct a noncooperative investment game to explore whether the explanation lies in the alternative implications of these principles on investment behavior. Our results are as follows (*i*) *EL* always induces higher total investment than *PRO* which in turn induces higher total investment than *EA*; (*ii*) *PRO* always induces higher egalitarian social welfare than both *EA* and *EL* in interior equilibria; (*iii*) *PRO* induces higher utilitarian social welfare than *EL* in interior equilibria but its relation to *EA* depends on the parameter values (however, a numerical analysis shows that on a large part of the parameter space, *PRO* induces higher utilitarian social welfare than *EA*).

Keywords: bankruptcy, noncooperative investment game, proportional, equal awards, equal losses, total investment, egalitarian social welfare, utilitarian social welfare.

JEL classification codes: C72, D78, G33

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1 Introduction

Following the seminal work of O'Neill (1982), a vast literature focused on the axiomatic analysis of "bankruptcy problems". As the name suggests, a canonical example to this problem is the case of a bankrupt firm whose monetary worth is to be allocated among its creditors. Each creditor holds a *claim* on the firm and the firm's liquidation value is less than the total of the creditors' claims. The axiomatic literature provided a large variety of "bankruptcy rules" as solutions to this problem. The most central of these rules are all based on one (or more) of three central principles: (*i*) **proportionality**, (*ii*) **equal awards**, and (*iii*) **equal losses**.[1]

Bankruptcy has also been a central topic in corporate finance where researchers analyze a large number of issues related to it (*e.g.* see Hotchkiss *et al* (2008)).[2] This literature shows that, in practice almost every country uses the following rule to allocate the liquidation value of a bankrupt firm.[3] First, creditors are sorted into different priority groups (such as secured creditors or unsecured creditors). These groups are served sequentially. That is, a creditor is not awarded a share until creditors in higher priority groups are fully reimbursed. Second, in each priority group, the shares of the creditors are determined in *proportion* to their claims.[4]

In this paper, we explore why in actual bankruptcy laws, *proportionality* has been preferred over the other two principles. Our starting observation is that alternative bankruptcy rules affect the investment behavior in a country in different ways. In a way, each rule induces a different noncooperative game among the investors. Comparing the equilibria of these games, in terms of

[1] As their names suggest, these principles suggest that the agents' shares should be chosen, respectively, (*i*) proportional to their investments, (*ii*) so as to equate their awarded shares, (*iii*) so as to equate their losses from initial investment. There are bankruptcy rules purely based on one of these principles (such as the Proportional, Constrained Equal Awards, Constrained Egalitarian, Constrained Equal Losses rules) as well as rules that apply different principles on different types of problems (such as the Talmud rule which uses both equal awards and equal losses principles).

[2] This is not surprising considering that in US between 1999 and 2009, more than 551000 firms filed for Chapter 7 bankruptcy and more than 22.16 billion USD were allocated in these cases (see http://www.justice.gov/ust/index.htm).

[3] Procedures on the liquidation of the firm and its allocation among creditors exist in bankruptcy laws of every country. For examples, see Chapter 7 of the U.S. Bankruptcy Code or the Receivership code in U.K. In some countries such as Sweden or Finland, these procedures provide the only option for the resolution of bankruptcy. Bankruptcy laws of some other countries, such as U.S., also offer procedures (such as Chapter 11) for reorganization of the bankrupt firm.

[4] This is a very old and common practice, referred to as a *pari passu* distribution; the term meaning "proportionally, at an equal pace, without preference" (see Black's Law Dictionary, 2004).

total investment or social welfare, might provide us ways of comparing alternative bankruptcy rules and thus, the principles underlying them, in a way that is not previously considered in either the axiomatic literature or the corporate finance literature on bankruptcy, both discussed at the end of this section.

As a representation of the proportionality principle, we use the *Proportional rule* (hereafter, **PRO**), which assigns each investor a share proportional to his investment. We then look at a class of rules that mix the proportionality principle with equal awards (hereafter, $\mathbf{AP}[\boldsymbol{\alpha}]$). These rules pick an α-weighted average of the proportional allocation and the (pure) equal division. For $\alpha = 0$, the rule $AP[\alpha]$ coincides with an "unconstrained equal awards rule" *(***EA***)* which always chooses equal division. For $\alpha = 1$, it coincides with PRO. Thirdly, we look at a class of rules that mix the proportionality principle with equal losses (hereafter, $\mathbf{LP}[\boldsymbol{\alpha}]$). These rules pick an α-weighted average of the proportional allocation and an allocation which equates the losses incurred by the investors. For $\alpha = 0$, the rule $LP[\alpha]$ coincides with an "unconstrained equal losses rule" *(***EL***)* which always equates the investors' losses. For $\alpha = 1$, it coincides with PRO.

For each one of these bankruptcy rules, we construct a simple game among n investors who simultaneously choose how much money to invest in a firm. The total of these investments determine the value of the firm. The firm is a lottery which either brings a positive return or goes bankrupt. In the latter case, its liquidation value is allocated among the investors according to the prespecified bankruptcy rule. For each bankruptcy rule, we analyze the Nash equilibria of the corresponding investment game. We then compare these equilibria.

In our model, agents have Constant Absolute Risk Aversion preferences and are weakly ordered according to their degrees of risk aversion. (This ordering is without loss of generality since the agents are identical in other dimensions.) The agents do not face liquidity constraints and thus, their income levels are not relevant. However, as is standard in the literature, it is possible to interpret the agents' risk aversion levels as a decreasing function of their income levels. (Thus, less risk averse agents can be thought of as richer, bigger investors.) Alternatively, each agent can be taken to represent an investment fund. In this case, the income level is irrelevant. The risk-aversion parameter attached to each investment fund then represents the type of that fund.

Since we do not restrict possible configurations of risk aversion, our model can be used to represent and compare societies with very different risk aversion (or income) distributions, ranging from symmetric to asymmetric distributions with different moments. This flexibility also allows us to compare the three principles in terms of how they treat different types of agents (such as big versus small investors) as well as how they react to changes in the risk-aversion distribution.

Our analysis compares bankruptcy rules in terms of two criteria that were not considered before. Our first criterion is **total equilibrium investment** which is a simple measure of how a bankruptcy rule affects investment behavior in the economy. It is reasonable to think that a government prefers bankruptcy rules that induce higher total investment in the economy. Thus, a bankruptcy rule that induces higher total investment than PRO might be considered a superior alternative to it. On the other hand, it is not clear that an increase in total investment will also increase the welfare of the investors. Thus, our second criterion is **equilibrium social welfare**. Egalitarianism and utilitarianism present two competing and central notions of measuring social welfare. We therefore compare bankruptcy rules in terms of both **egalitarian** and **utilitarian** social welfare that they induce in equilibrium.

A summary of our main results is as follows. The investment game has a unique Nash equilibrium for every parameter combination and for each bankruptcy rule. These equilibria are such that, at all parameters values (*i*) EL induces higher total investment than PRO which in turn induces higher total investment than EA; (*ii*) PRO induces higher egalitarian social welfare than both EA and EL in interior equilibria; (*iii*) PRO induces higher utilitarian social welfare than EL in interior equilibria but its relation to EA depends on the parameter values (however, a numerical analysis shows that on a large part of the parameter space, PRO induces higher utilitarian social welfare than EA). Thus, in the confines of our simple model, PRO outperforms EA in almost every criterion. Also, switching from PRO to EL increases total investment but decreases both egalitarian and utilitarian social welfare.

PRO is advantageous to the other rules also in the sense that only under PRO do the investors have dominant strategies (which are strictly dominant). Thus, for planning purposes, the government has a stronger prediction on investor behavior under PRO.

Finally, potential heterogeneity of the agents' risk attitudes plays an important role in our analysis. Bankruptcy rules are very different in terms of the incentives that they provide for big versus small investors. The equal losses principle offers relatively better protection to the bigger (i.e. less risk averse) investors. The equal awards principle does the opposite. The proportionality principle strikes a compromise by offering the same proportion of their investment to every agent. We also observe that under different rules an agent reacts very differently to changes in the others' risk attitudes: under EA (EL) his investment decreases (increases) as the other agents get more risk averse; under PRO, however, his investment remains constant. This once again makes the equilibrium prediction under PRO more reliable since under PRO, the agents, in determining their investment strategies, need not be informed about the risk-aversion (or income) profile of the other investors. A detailed summary of our findings as well as their possible policy implications is presented in *Section 6*.

In the axiomatic literature, the most common applications of the equal awards and the equal losses principles are the *Constrained Equal Awards rule (CEA)* (which equates the agents' awards subject to the constraint that no agent should receive a share higher than his initial investment) and the *Constrained Equal Losses rule (CEL)* (which equates the agents' losses subject to the constraint that no agent should receive a negative share).[5] Instead of using these rules, in the paper we use their unconstrained versions but later restrict the parameter space so that the aforementioned constraints will not be binding in equilibrium. As will be discussed later, we prefer this approach since the unconstrained rules induce games that are much better behaved than their constrained counterparts (which lead to existence and multiplicity problems). Additionally, we show in *Appendix B* that any equilibrium under the constrained CEA and CEL rules is also an equilibrium under their unconstrained versions.

The paper is organized as follows. In *Section 2*, we present the model. In *Section 3*, we calculate and analyze the Nash equilibrium induced by each rule. In *Section 4*, we compare bankruptcy rules in terms of the total investment they induce in equilibrium. In *Section 5*, we then compare them in terms of social welfare. We summarize our findings and conclude in *Section 6*. *Appendix A* contains the proofs. *Appendix B* is on *CEA* and *CEL* rules. Finally, *Appendix C* contains numerical welfare comparisons.

Related Literature.

The axiomatic literature on bankruptcy and taxation problems contains many studies that analyze the properties of alternative bankruptcy rules. For example, Dagan (1996), Schummer and Thomson (1997), Herrero and Villar (2002), and Yeh (2001) analyze properties of CEA. Yeh (2001a), Herrero and Villar (2002), and Herrero (2003) analyze properties of CEL. Aumann and Maschler (1985) and the following literature discuss properties of a Talmudic rule. O'Neill (1982), Moulin (1985a,b), Young (1988), Chun (1988a), de Frutos (1999), Ching and Kakkar (2000), Chambers and Thomson (2002), and Ju, Miyagawa, and Sakai (2007) analyze properties of PRO. Thomson (2003 and 2008) presents a detailed review of the extensive axiomatic literature.

[5] Additional to *CEA*, the axiomatic literature contains several rules that are based on the principle of equal awards, such as the *Piniles' rule*, and the *Constrained Egalitarian rule*.

The corporate finance literature also contains a large number of papers that study bankruptcy (e.g. see Bebchuck (1988), Aghion, Hart, and Moore (1992), Atiyas (1995), Hart (1999), Stiglitz (2001)). However, most of these papers study reorganization procedures such as Chapter 11 in the US. There are some papers that discuss liquidation procedures (and some, such as Baird (1986) argue that they are superior to reorganization procedures). For example, Bris, Welch, and Zhu (2006) use a comprehensive data set from the US to compare liquidation and reorganization procedures in terms of costs and efficiency. Stromberg (2000) uses Swedish data to evaluate liquidation procedures. Also, Hotchkiss *et al* (2008) summarize bankruptcy laws in different countries and as part of it, describe liquidation procedures (as these constitute the only resolution to bankruptcy in some countries). Finally, there are studies that discuss the implications of priority classes on investor behavior. However, all of these studies take the existing proportional allocation rule (*i.e.* PRO) as a given, nonchanging constant and does not discuss its merits in relation to alternative rules.

There are previous papers that employ game theoretical tools to analyze bankruptcy problems. Aumann and Maschler (1985), Curiel, Maschler, and Tijs, (1987), and Dagan and Volij (1993) relate bankruptcy rules to cooperative game theoretical solutions. Chun (1989) presents a noncooperative game that implements an egalitarian surplus sharing rule. Dagan, Serrano, and Volij (1997) present a noncooperative game that implements a large family of consistent bankruptcy rules by employing the rule's two-person version in the design. Chang and Hu (2008) carry out a similar analysis for a class of "f-just" rules. Herrero (2003) implements the CEA and CEL bankruptcy rules. Garcia-Jurado, Gonzalez-Diaz, and Villar (2006) present noncooperative games for a large class of "acceptable" rules. Finally, Eraslan and Yılmaz (2007) and the literature cited therein analyze negotiation games that arise during reorganization of the bankrupt firm. None of these paper however focus on investment implications of these bankruptcy rules.

This paper is closely related to Karagözoğlu (2010) who also designs a noncooperative game and analyzes investment implications of a class of rules that include PRO, CEA, and CEL. Aside from the fact that Karagözoğlu considers the constrained rules CEA and CEL, the main differences are as follows. In Karagözoğlu (2010) model, (*i*) there are two types of agents (high income and low income) who (*ii*) choose either zero or full investment of their income, and (*iii*) the agents are risk neutral. Due to these differences, our results are quite different. In Karagözoğlu (2010), PRO maximizes total investment whereas in our setting, the maximizer of total investment is the

EL (as seen in Section 4).[6] On the other hand, both studies find PRO to induce higher total investment than EA and CEA, respectively. Also, Karagözoğlu (2010) does not carry out a welfare analysis but additionally analyzes a class of rules that includes the Talmud rule (the TAL family by Moreno-Ternero and Villar, 2006) and discusses the case of two firms.

2 Model

Let $N = \{1, ..., n\}$ be the set of agents. Each $i \in N$ has the following **Constant Absolute Risk Aversion (CARA) utility function** $u_i : \mathbb{R}_+ \to \mathbb{R}$ on money: $u_i(x) = -e^{-a_i x}$. Assume that each $i \in N$ is risk averse, that is, $a_i > 0$. Also assume that $a_1 \leq ... \leq a_n$.

Each agent i **invests** $s_i \in \mathbb{R}_+$ units of wealth on a risky company. The company has value $\sum_N s_i$ after investments. With **success probability** $p \in (0, 1)$, this value brings a **return** $r \in (0, 1]$ and becomes $(1 + r) \sum_N s_i$. With the remaining probability $(1 - p)$, the company goes bankrupt and its value becomes $\beta \sum_N s_i$ where $\beta \in (0, 1)$ is the **fraction that survives bankruptcy**.

In case of bankruptcy, the value of the firm is allocated among the agents according to a prespecified bankruptcy rule. Formally, a **bankruptcy problem** is a vector of claims (*i.e.* investments) $s = (s_1, ..., s_n) \in \mathbb{R}^n_+$ and an endowment $E \in \mathbb{R}_+$ satisfying $\sum_N s_i \geq E$. In our model, the bankrupt firm retains β fraction of its capital.[7] Thus $E = \beta \sum_N s_i$ is a function of s. As a result, the vector s (together with β) is sufficient to fully describe the bankruptcy problem at hand. Thus in our setting, the class of all bankruptcy problems is simply $\mathbb{R}^n_+$.

A **bankruptcy rule** F assigns each $s \in \mathbb{R}^n_+$ to an allocation $x \in \mathbb{R}^n$ satisfying $\sum_N x_i = \beta \sum_N s_i$. In this paper, we will focus on the following bankruptcy rules. The **Proportional Rule (PRO)** is defined as follows: for each $i \in N$, $PRO_i(s) = \beta s_i$. The **Equal Awards rule (EA)** is defined as $EA_i(s) = \frac{\beta}{n} \sum_N s_i$. The **Constrained Equal Awards rule (CEA)** is defined as $CEA_i(s) = \min\{s_i, \rho\}$ where $\rho \in \mathbb{R}_+$ satisfies $\sum_N \min\{s_i, \rho\} = \beta \sum_N s_i$. The **Equal Losses rule (EL)** is defined as $EL_i(s) = s_i - \frac{1-\beta}{n} \sum_N s_j$. The **Constrained Equal Losses rule (CEL)** is

[6] In our opinion, this difference is due to two reasons. First, Karagözoğlu uses binary strategies and this limits the sensitivity of equilibrium total investment to the problem's parameters. Thus, when in binary strategies the two rules induce equal investment, it might be that EL exceeds PRO when we take into account how much the agents do invest. The second reason is the difference between EL and CEL.We show in *Appendix B* that CEL induces more types of equilibria than EL and in some of them, PRO induces more total investment than CEL.

[7] This assumption is supported by empirical evidence from Bris, Welch, and Zhu (2006) who note that the firm scale is fairly unrelated to percent value changes in bankruptcy.

defined as $CEL_i(s) = \max\{s_i - \rho, 0\}$ where $\rho \in \mathbb{R}_+$ satisfies $\sum_N \max\{s_i - \rho, 0\} = \beta \sum_N s_i$.

We will also be interested in the following families of rules. For each $\alpha \in [0,1]$, the **EA-PRO mixture rule with weight α** is

$$\begin{aligned} AP[\alpha]_i(s) &= \alpha PRO_i(s) + (1-\alpha) EA_i(s) \\ &= \frac{1+(n-1)\alpha}{n}\beta s_i + (1-\alpha)\frac{\beta}{n}\sum_{N\setminus i} s_j. \end{aligned}$$

For $\alpha = 1$, this rule becomes equal to PRO and for $\alpha = 0$, it becomes equal to EA. Similarly, for each $\alpha \in [0,1]$, the **EL-PRO mixture rule with weight α** is

$$\begin{aligned} LP[\alpha]_i(s) &= \alpha PRO_i(s) + (1-\alpha) EL_i(s) \\ &= \frac{n\alpha\beta + (n-1+\beta)(1-\alpha)}{n} s_i - \frac{(1-\alpha)(1-\beta)}{n}\sum_{N\setminus i} s_j. \end{aligned}$$

For $\alpha = 1$, this rule becomes equal to PRO and for $\alpha = 0$, it becomes equal to EL.

For each bankruptcy rule F, we analyze the following **investment game** it induces over the agents. Each $i \in N$ has the **strategy set** $S_i = \mathbb{R}_+$ from which he chooses an investment level s_i. Let $S = \prod_N S_i$. A strategy profile $s \in S$ corresponds for agent i to the lottery that brings the net return $(1+r)s_i - s_i = rs_i$ with probability p and the net return $F_i(s) - s_i$ with the remaining probability $(1-p)$. Note that $F_i(s) - s_i \leq 0$. The interpretation is that the agent initially borrows s_i at an interest rate normalized to 0. If the investment is successful, he receives $(1+r)s_i$, pays back s_i, and is left with his profit rs_i. In case of bankruptcy, he only receives back $F_i(s)$ and has to pay back s_i, so his net profit becomes $F_i(s) - s_i$. The same lottery is obtained from an environment where each agent i allocates his monetary endowment between a riskless asset (whose return is normalized to 0) and the risky company. In this second interpretation, assume that the agent does not have a liquidity constraint. That is, he is allowed to invest more than his endowment. This assumption only serves to rid us from (the rather unrealistic) boundary cases where some agents spend all their monetary endowment on the risky firm. Alternatively, one can impose a liquidity constraint but focus on equilibria which are in the interior of the strategy spaces.

Agent i's expected payoff from strategy profile $s \in S$ is thus

$$\begin{aligned} U_i^F(s) &= pu_i(rs_i) + (1-p)u_i(F_i(s) - s_i). \qquad (1) \\ &= -pe^{-a_i r s_i} - (1-p)e^{-a_i F_i(s_i, s_{-i}) + a_i s_i} \end{aligned}$$

Let $U^F = \left(U_1^F, ..., U_n^F\right)$. The **investment game induced by the bankruptcy rule F** is then defined as $G^F = \langle S, U^F \rangle$. Let $\epsilon(G^F)$ denote the set of **Nash equilibria** of G^F.

3 Equilibria Under Alternative Bankruptcy Rules

We start by analyzing the Nash equilibria and dominant strategy equilibria of each game. This section serves as a preliminary for our comparisons of total investment (in Section 4) and welfare (in Section 5).

Proportional Rule (PRO):

The following proposition shows that under PRO, the investment game has a unique dominant strategy equilibrium and no other Nash equilibria.

Proposition 1 *If* $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) \leq 0$, *the investment game under the rule* PRO *has a unique dominant strategy equilibrium* $(0,...,0)$. *Otherwise, the game has a unique dominant strategy equilibrium* s^* *in which each agent* i *chooses a positive investment level* s_i^* *given by*

$$s_i^* = \frac{1}{a_i\,(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right).$$

There is no other Nash equilibria.

It is useful to note that the strategies in Proposition 1 are *strictly dominant.*

Note that if $pr > (1-p)(1-\beta)$, all agents choose a positive investment level at the dominant strategy equilibrium. This condition simply compares the return on unit investment in case of success, r, weighted by the probability of success, p, with the loss incurred on unit investment in case of failure, $(1-\beta)$, weighted by the probability of failure, $(1-p)$. Investing in the firm is optimal if the returns in case of success outweigh the losses incurred in case of failure.

Equilibrium investment levels are ordered as $s_1^* \geq ... \geq s_n^*$. Also s_i^* is increasing in the probability of success p and the fraction of the firm that survives bankruptcy β and it is decreasing in the agent's degree of risk aversion a_i. It does not have a fixed relation to the rate of return in case of success, r.

Mixtures of Proportionality and Equal Awards ($AP[\alpha]$):

The following proposition determines the form of the unique Nash equilibrium under $AP[\alpha]$. We would like to exclude parameter values for which at the Nash equilibrium an agent's compensation in case of bankruptcy is more than his investment. Thus, we also identify the parameter values under which $AP[\alpha]_i\,(s^*) \leq s_i^*$ for each $i \in N$.

Proposition 2 *If* $\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right) \leq 0$, *the investment game under the rule* $AP[\alpha]$ *has a unique Nash equilibrium* $(0, ..., 0)$. *Otherwise, the game has a unique Nash equilibrium* s^* *in which each agent* i *chooses a positive investment level* s_i^* *given by*

$$s_i^* = \frac{n\left(1+r-\beta\right)+\beta\left(1-\alpha\right)+\beta\left(1-\alpha\right)a_i\sum_{N\backslash i}\frac{1}{a_j}}{a_i n\left(1+r-\beta\right)\left(1+r-\alpha\beta\right)}\ln\left(\frac{npr}{\left(1-p\right)\left(n-\beta-\left(n-1\right)\alpha\beta\right)}\right).$$

In the latter case, the unique Nash equilibrium s^* *satisfies* $AP[\alpha]_i\left(s^*\right) \leq s_i^*$ *for each* $i \in N$ *if and only if*

$$\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}} \geq \frac{r\beta\left(1-\alpha\right)}{n\left(1-\alpha\beta\right)\left(1+r-\beta\right)}. \tag{2}$$

Note that if $pr > (1-p)(1-\frac{1+(n-1)\alpha}{n}\beta)$, all agents choose a positive investment level at the Nash equilibrium. This condition simply compares the return on unit investment in case of success, r, weighted by the probability of success, p, with the loss incurred on unit investment in case of failure, $(1-\frac{1+(n-1)\alpha}{n}\beta)$, weighted by the probability of failure, $(1-p)$.[8] Investing in the firm is optimal if the returns in case of success outweigh the losses incurred in case of failure.

Equilibrium investment levels are ordered as $s_1^* \geq ... \geq s_n^*$. Also s_i^* is increasing in the probability of success p and the fraction of the firm that survives bankruptcy β and it is decreasing in the agent's degree of risk aversion a_i. It does not have a fixed relation to the rate of return in case of success, r.

For $\alpha = 0$, $AP[\alpha]$ becomes the Equal Awards rule, EA. This is the unconstrained version of a well-known rule from the axiomatic literature, called the Constrained Equal Awards rule, CEA, defined in Section 2. In *Appendix B*, we first show that, under CEA, a Nash equilibrium does not exist for every parameter combination. We then show in *Proposition 9* that if a Nash equilibrium exists under CEA, it is unique and more importantly, it is identical to the unique equilibrium under EA. Thus, Proposition 9 implies that analyzing the Nash equilibrium under EA also tells us about CEA.

Mixtures of Proportionality and Equal Losses ($LP[\alpha]$):

The following proposition shows that the Nash equilibrium under $LP[\alpha]$ is of the form $s_1^* \geq ... \geq s_n^*$ where agents up to some $k \in N$ choose positive investment and the rest chooses zero investment. For $\alpha < 1$, that is, for $LP[\alpha] \neq PRO$, there are parameter values under which, at the

[8] The term $(1-\frac{1+(n-1)\alpha}{n}\beta)$ is equal to $\alpha\left(1-\beta\right)+\left(1-\alpha\right)\left(1-\frac{\beta}{n}\right)$. The α weighted part of this expression is the loss incurred in case of PRO and the $(1-\alpha)$ weighted part is the loss incurred in case of EA.

Nash equilibrium, $LP[\alpha]$ proposes a negative share for some agents. We would like to exclude such parameter values. That, is we restrict ourselves to cases where $LP[\alpha]_i(s^*) \geq 0$ for each $i \in N$. Such equilibria are also identified in the next proposition.

Proposition 3 *If* $\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right) \leq 0$, *the investment game under* $LP[\alpha]$ *has a unique Nash equilibrium* $(0,...,0)$. *Otherwise, there is* $k \in N$ *such that the unique Nash equilibrium is* $s^* = (s_1^*,...,s_k^*,0,...,0)$ *where for each* $i \in \{1,...,k\}$, $s_i^* > 0$ *and is given by*

$$s_i^* = \left(\frac{1}{a_i} - \frac{(1-\alpha)(1-\beta)}{(1-\beta)(1-\alpha)k + n(\alpha(1-\beta)+r)}\sum_{j=1}^{k}\frac{1}{a_j}\right)\frac{\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right)}{r+\alpha(1-\beta)}.$$

In the latter case, the unique Nash equilibrium s^* *satisfies* $LP[\alpha]_i(s^*) \geq 0$ *for each* $i \in N$ *if and only if*

$$\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}} \geq \frac{(r+1)(1-\alpha)(1-\beta)}{n(1-\beta+r)(1-\alpha+\alpha\beta)}. \tag{3}$$

Under Inequality 3, $k = n$. *That is,* $s^* = (s_1^*,...,s_n^*) > 0$.

Under *Inequality 3*, if $pr > \frac{(1-\beta)}{n}(1-p)(1+(n-1)\alpha)$, all agents choose a positive investment level at the Nash equilibrium. This condition simply compares the return on unit investment in case of success, r, weighted by the probability of success, p, with the loss incurred on unit investment in case of failure, $\frac{(1-\beta)}{n}(1+(n-1)\alpha)$, weighted by the probability of failure, $(1-p)$.[9] Investing in the firm is optimal if the returns in case of success outweigh the losses incurred in case of failure.

Equilibrium investment levels are ordered as $s_1^* \geq ... \geq s_n^*$. Also s_i^* is increasing in the probability of success p and the fraction of the firm that survives bankruptcy β and it is decreasing in the agent's degree of risk aversion a_i. It does not have a fixed relation to the rate of return in case of success, r.

For $\alpha = 0$, $LP[\alpha]$ becomes the Equal Losses rule, EL. This is the unconstrained version of a well-known rule from the axiomatic literature, called the Constrained Equal Losses rule, CEL. In *Appendix B*, we first show that, under CEL, a Nash equilibrium does not exist for every parameter combination. We then show in *Proposition 10* that if a Nash equilibrium exists under CEL, for two agents, it can be one of four types two of which are identical to the equilibria under EL. Thus, Proposition 10 implies that analyzing the Nash equilibrium under EL also tells us about CEL.

[9] The term $\frac{(1-\beta)}{n}(1+(n-1)\alpha)$ is equal to $\alpha(1-\beta)+(1-\alpha)\left(\frac{1-\beta}{n}\right)$. The α weighted part of this expression is the loss incurred in case of PRO and the $(1-\alpha)$ weighted part is the loss incurred in case of EL.

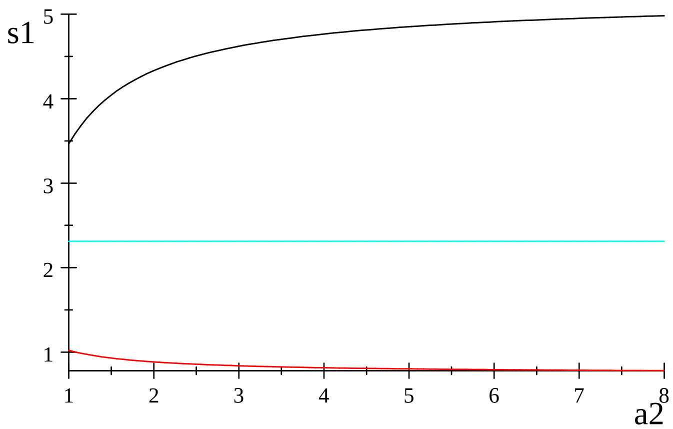

Figure 1: Agent 1's equilibrium investment s_1 as a function of the other agent's risk aversion a_2 under EA (red), PRO (green), and EL (black) where $r = 0.3$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 1$.

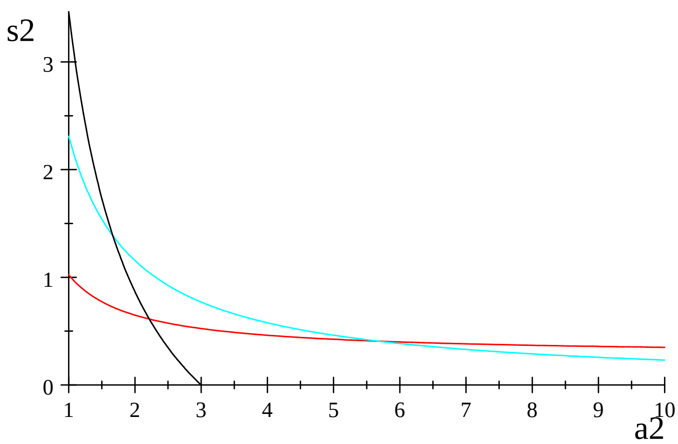

Figure 2: Agent 2's equilibrium investment s_2 as a function of his risk aversion a_2 under EA (red), PRO (green), and EL (black) where $r = 0.3$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 1$.

4 Comparisons of Equilibrium Investment

In this section, we compare bankruptcy rules in terms of total investment that they induce in equilibrium. Let us first check individual investment levels in a numerical example for two investors where $r = 0.3$, $p = 0.8$, $\beta = 0.7$, and $a_1 = 1$ and for which *figures 1* and *2* respectively demonstrate s_1^* and s_2^* as a function of a_2 for the three extreme rules: PRO, EA, and EL.

As can be seen in *Figure 1*, in terms of s_1^*, the three rules are ordered as $EL > PRO > EA$. Also, s_1^* is independent of a_2 under PRO but it is increasing (decreasing) in a_2 under EL (EA). This demonstrates a general phenomenon. The bigger investor, that is, the relatively less risk averse agent 1, faces very different incentives under the three rules. In case of bankruptcy, he is protected best by EL and worst by EA whereas his share under PRO is independent of the other agents. This reflects on his investment choices.

Figure 2 looks at the smaller investor, the more risk averse agent 2 and shows that under all three rules, his equilibrium choice s_2^* is decreasing in a_2. Also, the three rules do not have a fixed order in terms of s_2^*. For low risk aversion levels (*i.e.* when agent 2 is not too different than agent 1), the ordering of the three rules in terms of s_2^* is $EL > PRO > EA$, same as s_1^*. But it is reversed for high risk aversion levels. For this case, agent 2 is protected best under EA and worst under EL and this reflects to his equilibrium investment choices under them. It is also interesting to note that, for risk aversion levels in between the two extremes, it is PRO that induces the highest investment level s_2^* on agent 2.

Looking at individual investment levels, one does not observe a clear ordering of the three rules. However, in terms of total investment, we obtain a strong result. The following two theorems establish that, in terms of total investment, the rules analyzed in the previous section are ordered as

$$EL > LP[\alpha] > PRO > AP[\alpha] > EA.$$

Theorem 1 *Equilibrium total investment under the EA-PRO mixture rule, $AP[\alpha]$, is weakly increasing in the weight of PRO, α, and it is strictly increasing whenever $AP[\alpha]$ induces positive investment in equilibrium. Thus, in the class $\{AP[\alpha] \mid \alpha \in [0,1]\}$, PRO maximizes total investment and EA minimizes total investment.*

Theorem 2 *Equilibrium total investment under the EL-PRO mixture rule, $LP[\alpha]$, is weakly decreasing in the weight of PRO, α, and it is strictly decreasing whenever $LP[\alpha]$ induces positive*

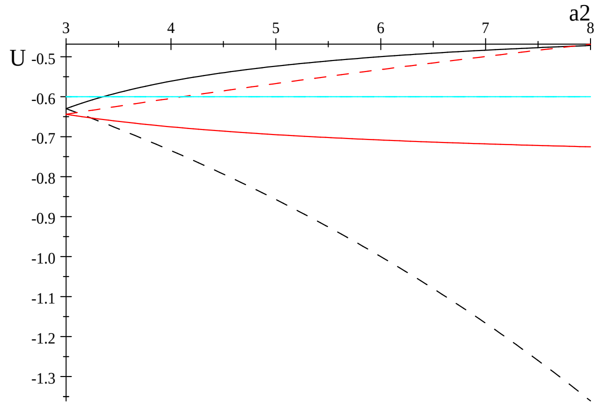

Figure 3: Individual welfare levels under PRO (green), EA (red), and EL (black). The solid line represents the first agent and the dashed line represents the second agent. The parameter values are $p = 0.8$, $r = 0.6$, $\beta = 0.7$, $a_1 = 3$.

investment in equilibrium. Thus, in the class $\{LP[\alpha] \mid \alpha \in [0,1]\}$, *PRO minimizes total investment and EL maximizes total investment.*

It is interesting to note that, even when all agents are identical in terms of risk aversion, the ordering of the rules in terms of total investment is as above. Particularly, EL still induces more total investment than the other rules. This means that these rules not only differ in terms of how they treat big versus small investors (as discussed at the beginning of this section), but they also differ in terms of the investment incentives that they provide in a symmetric game where all agents are identical in terms of risk aversion. This can be observed in *figures 1* and *2* by choosing $a_2 = a_1 = 1$.

5 Comparisons of Equilibrium Welfare

In this section, we look at the individual and social welfare levels induced by the Nash equilibria under the PRO, EA, and EL rules. We compare these three rules in terms of both egalitarian and utilitarian social welfare.

For analytical tractability, we focus on the two-agent case and the three main rules. We also assume that inequalities (2) and (3) hold, that is, in equilibrium EA does not award an agent a share greater than his investment level and EL does not award an agent a negative share.

In *Figure 3*, we fix $p = 0.8$, $r = 0.6$, $\beta = 0.7$, $a_1 = 3$ and demonstrate individual welfare levels as a function of a_2. As noted above, an agent's welfare under PRO is independent of a_2. Thus, it remains constant at -0.6 for both agents. The individual welfare under both EA and EL depends on a_2. At the symmetric case (when $a_1 = a_2 = 3$) agents 1 and 2 receive identical welfare levels. They receive the highest common payoff under PRO, then under EL, then under EA.[10] An increase in a_2, under both EA and EL, has opposite effects on the two agents. Under EA, the more risk averse agent 2 always receives a higher payoff than the less risk averse agent 1 and his welfare increases in a_2. Exactly the opposite holds for EL: the less risk averse agent 1 always receives a higher payoff than agent 2 and his payoff increases in a_2. This observation is summarized in the following lemma. Note that, under EA and EL, the two agents receive the same payoff if and only if the game is symmetric ($a_1 = a_2$) or the equilibrium investments are both zero.

Lemma 4 *Assume $a_1 \leq a_2$. Then,*
(i) $U_1^{PRO}(\epsilon\left(G^{PRO}\right)) = U_2^{PRO}(\epsilon\left(G^{PRO}\right))$,
(ii) $U_1^{EA}\left(\epsilon\left(G^{EA}\right)\right) \leq U_2^{EA}\left(\epsilon\left(G^{EA}\right)\right)$, *equality achieved iff* $a_1 = a_2$ *or* $\ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) \leq 0$,
(iii) $U_1^{EL}\left(\epsilon\left(G^{EL}\right)\right) \geq U_2^{EL}\left(\epsilon\left(G^{EL}\right)\right)$, *equality achieved iff* $a_1 = a_2$ *or* $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) \leq 0$.

5.1 Egalitarian Social Welfare Levels

The **egalitarian social welfare level induced by a rule F** is the minimum utility an agent obtains at the Nash equilibrium of the investment game induced by F:

$$EG^F\left(p, r, \beta, a_1, a_2\right) = \min\left\{U_1^F(\epsilon\left(G^F\right)), U_2^F(\epsilon\left(G^F\right))\right\}.$$

We next compare the Egalitarian social welfare levels induced by PRO, EA, and EL for interior equilibria (where both agents choose positive investment levels). Other cases mostly employ numerical comparisons and are presented in detail in *Appendix C.*

Lemmas 6 and 7 in *Appendix A* respectively show that the egalitarian social welfare under both EA and EL is maximized when $a_1 = a_2$. We use them to show that PRO induces higher egalitarian social welfare than both EA and EL.

[10] If $a_1 = a_2$, PRO always induces higher payoff than the other two rules. The ordering between EA and EL, however, depends on the parameter values. It is easy to construct another numerical example where EA induces higher payoff than EL in the symmetric case.

Theorem 3 *For parameter values where all three rules induce an interior equilibrium (where all agents choose a positive investment level), that is, when* $\ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0$*, PRO induces strictly greater egalitarian social welfare than both EA and EL.*

Note that if $a_1 = a_2$, for each one of our three rules, $U_1^F(\epsilon\left(G^F\right)) = U_2^F(\epsilon\left(G^F\right)) = EG^F\left(p, r, \beta, a\right).$ Thus, we have the following corollary to Theorem 3.

Corollary 5 *If* $a_1 = a_2$*, the Nash equilibrium payoff profile induced by PRO, that is,* $\left(U_1^{PRO}(\epsilon\left(G^{PRO}\right)), U_2^{PRO}(\epsilon\left(G^{PRO}\right))\right)$*, Pareto dominates the Nash equilibrium payoff profiles induced by EA and EL,* $\left(U_1^{EA}(\epsilon\left(G^{EA}\right)), U_2^{EA}(\epsilon\left(G^{EA}\right))\right)$ *and* $\left(U_1^{EL}(\epsilon\left(G^{EL}\right)), U_2^{EL}(\epsilon\left(G^{EL}\right))\right).$

We thus conclude that PRO always induces a higher egalitarian social welfare than the other two rules. The relationship between EL and EA changes with the parameters but a numerical analysis shows that the number of parameter values for which EA induces higher welfare than EL is almost three times as much as the parameter values for which EL induces higher social welfare than EA. The details of this comparison as well as other cases are presented in *Appendix C*.

5.2 Utilitarian Social Welfare Levels

The **utilitarian social welfare level induced by a rule F** is the total utility the two agents obtain at the Nash equilibrium of the investment game induced by F:

$$UT^F\left(p, r, \beta, a_1, a_2\right) = U_1^F(\epsilon\left(G^F\right)) + U_2^F(\epsilon\left(G^F\right)).$$

We next compare the Utilitarian social welfare levels induced by PRO, EA, and EL for interior equilibria (where both agents choose positive investment levels). Other cases mostly employ numerical comparisons and are presented in detail in *Appendix C*.

Lemma 8 in *Appendix A* shows that the utilitarian social welfare under EL is maximized when $a_1 = a_2$. We next use it to show that PRO induces higher utilitarian social welfare than EL.

Theorem 4 *For parameter values where both PRO and EL induce an interior equilibrium (where all agents choose a positive investment level), that is, when* $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0$*, PRO induces strictly greater utilitarian social welfare than EL.*

We know by *Corollary 5* that in every symmetric game (that is, where $a_1 = a_2$), PRO induces higher utilitarian social welfare than EA.

For $a_1 < a_2$, we run a numerical analysis and show that the number of parameter values for which PRO induces higher welfare than EA is almost one and a half times as much as the parameter values for which EA induces higher social welfare than PRO. The numerical comparison of EA and EL reveals that EA outperforms EL almost twice as much as EL outperforms EA. The details of this comparison as well as other cases are presented in *Appendix C*.

6 Conclusion

Our analysis compares the proportionality, equal awards, and equal losses principles in terms of two criteria (total investment and social welfare induced in equilibrium) that were not considered before. Our findings are as follows:
(i) In terms of total investment, there is a constant ranking of these rules which is independent of the parameter values considered:

$$\text{Total Investment: } EL > LP[\alpha] > PRO > AP[\alpha] > EA.$$

For a mixture of EL and PRO, $LP[\alpha]$, equilibrium total investment is decreasing in α, the weight of PRO. Also, for a mixture of EA and PRO, $AP[\alpha]$, equilibrium total investment is increasing in α, the weight of PRO. We also get a similar ordering for CEA, PRO, and CEL.[11]
(ii) Independent of the parameter values considered, PRO always induces a higher egalitarian social welfare than both EA and EL in an interior equilibrium.:

$$\text{Egalitarian social welfare: } PRO > \{EA, EL\}.$$

The ranking between EA and EL depends on the parameter values. However, a numerical analysis shows that EA exceeds EL three times as much as EL exceeds EA.
(iii) Independent of the parameter values considered, PRO always induces a higher utilitarian social welfare than EL in an interior equilibrium.:

$$\text{Utilitarian social welfare: } PRO > EL.$$

Also, a numerical analysis shows that size of the parameter space where PRO induces higher utilitarian social welfare than EA is one and a half times as much as the size of the part where EA exceeds PRO. We also numerically compared EA and EL and observed that EA exceeds EL two times as much as EL exceeds EA.

[11] One exception is a rather unrealistic equilibrium under CEL.

(iv) In symmetric games (where $a_1 = a_2$), we obtain a very strong welfare comparison: PRO Pareto dominates both EA and EL.

(v) There always is a unique dominant strategy equilibrium under PRO (where agents play strictly dominant strategies). No other rule induces dominant strategies. However, under both $AP[\alpha]$ and $LP[\alpha]$, a unique Nash equilibrium always exists.

Overall, the almost universal principle of proportionality does not maximize total investment in the economy. By switching from proportionality to equal losses, it is possible to increase total investment. However, this switch causes a welfare loss in the society, both according to the egalitarian and utilitarian social welfare functions. A switch from proportionality to the equal awards principle always decreases total investment. It is also interesting to note that it also lowers egalitarian social welfare.

Finally, comparing equilibria at different risk-aversion profiles shows us that the three principles are very different in terms of the incentives that they provide for big versus small investors. The equal losses principle offers relatively better protection to the bigger (i.e. less risk averse) investors. The equal awards principle does the opposite. The proportionality principle strikes a compromise by offering the same proportion of their investment to every agent.

Note that our social welfare measures only consider the investors in the game. They do not take into account the welfare implications of investment in the rest of the economy (such as welfare effects of investment on consumers or future generations). This is an interesting question which is, unfortunately, out of the scope of our current model.

For tractability of the model, we use CARA utility functions. While the CARA family is widely used in economic modeling as well as finance, it is an open question whether our findings are replicated with other families of utility functions.

In our model, the agents simultaneously move to choose their investment levels. This is to model interactions where there are no structural order differences between the investors. It might be more appropriate to model other types of real life interactions by using a sequential version of our model. One complication is that with n agents of possibly heterogenous risk aversion levels, there are too many possible orders of moves. Some, however, might be more natural than the others.

In our model, the rate of return r and the probability of success p are independent of the agents' investment levels. It might be interesting to look at extensions of the model where these parameters, in some way, depend on the investment levels.

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7 Appendix A: Proofs

Proof. (Proposition 1) PRO, as a function of the investment levels, is defined as $PRO_i(s) = \beta s_i$. Then, by Equation 1, agent i's utility under PRO can be written as

$$U_i^{PRO}(s) = -pe^{-a_i r s_i} - (1-p)e^{a_i s_i (1-\beta)}.$$

Note that the payoff function of agent i is independent of the other agents' investment levels. The unconstrained maximizer of this expression is

$$\sigma_i(s_{-i}) = \frac{1}{a_i(r+1-\beta)} \ln\left(\frac{pr}{(1-p)(1-\beta)}\right).$$

The best response function of agent i can then be written as $b_i(s_{-i}) = \max\{0, \sigma_i(s_{-i})\}$. Note that this expression is independent of s_{-i}. So it in fact defines a *strictly dominant strategy* for each agent i. ∎

Proof. (Proposition 2) By Equation 1, agent i's utility under $AP[\alpha]$ becomes

$$\begin{aligned} U_i^{AP[\alpha]}(s_i, s_j) &= -pe^{-a_i r s_i} - (1-p)e^{-a_i\left(\frac{1+(n-1)\alpha}{n}\beta s_i + (1-\alpha)\frac{\beta}{n}\sum\limits_{N\backslash i} s_j\right) + a_i s_i} \\ &= -pe^{-a_i r s_i} - (1-p)e^{a_i s_i\left(\frac{n-\beta-(n-1)\alpha\beta}{n}\right) - a_i(1-\alpha)\frac{\beta}{n}\sum\limits_{N\backslash i} s_j} \end{aligned}$$

The unconstrained maximizer of this expression is

$$\sigma_i(s_{-i}) = \frac{n \ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right)}{a_i(n(1+r)-\beta-(n-1)\alpha\beta)} + \frac{(1-\alpha)\beta}{(n(1+r)-\beta-(n-1)\alpha\beta)} \sum_{N\backslash i} s_j.$$

Thus, agent i's best response is $b_i(s_{-i}) = \max\{0, \sigma_i(s_{-i})\}$. Since $\frac{(1-\alpha)\beta}{(n(1+r)-\beta-(n-1)\alpha\beta)} < 1$, there is a unique solution to the system $\{\sigma_i(s_{-i}) = s_i \mid i \in N\}$. Solving it gives,

$$s_i^* = \frac{n(1+r-\beta) + \beta(1-\alpha) + \beta(1-\alpha)a_i \sum_{N\backslash i} \frac{1}{a_j}}{a_i n(1+r-\beta)(1+r-\alpha\beta)} \ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right).$$

Note that the sign of s_i^* is equal to the sign of $\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right)$. If this is negative, $s_i^* < 0$ is not feasible. In this case, all agents choose zero investment. Alternatively, if $\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right) > 0$, the equilibrium is s^*. Finally, the condition $s_i^* \geq AP[\alpha]_i(s^*)$ simplifies to $\frac{\frac{1}{a_i}}{\sum_N \frac{1}{a_j}} \geq \frac{r\beta(1-\alpha)}{n(1-\alpha\beta)(1+r-\beta)}$ for each $i \in N$. Since $a_1 \leq ... \leq a_n$, this is equivalent to $\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}} \geq \frac{r\beta(1-\alpha)}{n(1-\alpha\beta)(1+r-\beta)}$. ∎

Proof. (Proposition 3) By Equation 1, agent i's utility under $LP[\alpha]$ becomes

$$U_i^{LP[\alpha]}(s_i, s_j) = -pe^{-a_i r s_i} - (1-p)e^{\left(\frac{(1-\beta)(1+(n-1)\alpha)}{n}\right)a_i s_i + \frac{(1-\alpha)(1-\beta)}{n}a_i \sum_{N\setminus i} s_j}.$$

The unconstrained maximizer of this expression is

$$\sigma_i(s_{-i}) = \frac{n\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right)}{a_i(nr + (1-\beta)(1+(n-1)\alpha))} - \frac{(1-\alpha)(1-\beta)}{(nr+(1-\beta)(1+(n-1)\alpha))}\left(\sum_{N\setminus i} s_j\right).$$

Thus, agent i's best response is $b_i(s_{-i}) = \max\{0, \sigma_i(s_{-i})\}$. Now, note that $\frac{n}{a_i(nr+(1-\beta)(1+(n-1)\alpha))} > 0$ and $\frac{(1-\alpha)(1-\beta)}{(nr+(1-\beta)(1+(n-1)\alpha))} > 0$. Thus, if $\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right) \leq 0$, agent i's best response is $b_i(s_{-i}) = 0$. Since this is true for every agent, the unique Nash equilibrium is then $(0, ..., 0)$.

Alternatively assume $\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right) > 0$. To calculate the equilibrium for this case, first note that the agents' best responses are ordered as $b_1 \geq ... \geq b_n$.[12] As a result, the equilibrium is of the form $s_1^* \geq ... \geq s_n^*$ where agents up to some $k \in N$ choose positive investment and the rest chooses zero investment. That is for some $k \in N$, $s_1^* \geq ... \geq s_k^* > 0 = s_{k+1}^* ... = s_n^*$. (If $k = n$, all agents choose positive investment in equilibrium.) To find such an equilibrium, we solve the system $\{\sigma_i(s_1, ..., s_k, 0, ..., 0) = s_i \mid i = 1, ..., k\}$ and obtain

$$s_i^* = \left(\frac{1}{a_i} - \frac{(1-\alpha)(1-\beta)}{(1-\beta)(1-\alpha)k + n(\alpha(1-\beta)+r)}\sum_{j=1}^{k}\frac{1}{a_j}\right)\frac{\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right)}{r+\alpha(1-\beta)}$$

for each $i \in \{1, ..., k\}$. Solving this system guarantees that agents $1, ..., k$ are playing best responses. To guarantee that agents $k+1, ..., n$ are playing best responses by choosing zero investment, it is sufficient to make sure that an equilibrium where agent $k+1$ also chooses positive investment is not feasible, that is,

$$\left(\frac{1}{a_{k+1}} - \frac{(1-\alpha)(1-\beta)}{(1-\beta)(1-\alpha)(k+1) + n(\alpha(1-\beta)+r)}\sum_{j=1}^{k+1}\frac{1}{a_j}\right) < 0.$$

This condition implies that for each $i \in \{k+1, ..., n\}$, we have $\sigma_i\left(s_{-i}^*\right) < 0$.

For the if and only if statement, first note that Inequality 3 implies

$$\frac{1}{a_n} - \frac{(1-\alpha)(1-\beta)}{(1-\beta)(1-\alpha)n + n(\alpha(1-\beta)+r)}\sum_{N}\frac{1}{a_j} \geq 0.$$

[12] More specifically, the unconstrained best responses all have the same slope and are ordered with respect to their intercepts. Thus, they are ordered as $\sigma_1 \geq \sigma_2 ... \geq \sigma_n$.

Thus by the first part of the proof, at the unique Nash equilibrium, $k = n$. That is, $s^* = (s_1^*, ..., s_n^*) > 0$. Also note that if $LP[\alpha]_i(s^*) \geq 0$ for each $i \in N$, we again have $s^* = (s_1^*, ..., s_n^*) > 0$.

Now since $s^* = (s_1^*, ..., s_n^*) > 0$, the condition $LP[\alpha]_i(s^*) \geq 0$ can be rewritten as

$$\frac{\frac{1}{a_i}}{\sum_N \frac{1}{a_j}} \geq \frac{(r+1)(1-\alpha)(1-\beta)}{n(1-\beta+r)(1-\alpha+\alpha\beta)}$$

for each $i \in N$. Since $a_1 \leq ... \leq a_n$, this condition is equivalent to Inequality 3, that is, $\frac{\frac{1}{a_n}}{\sum_N \frac{1}{a_j}} \geq \frac{(r+1)(1-\alpha)(1-\beta)}{n(1-\beta+r)(1-\alpha+\alpha\beta)}$. ■

Proof. (Theorem 1) There are three possible cases. First assume that $\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right) > 0$ for all $\alpha \in [0,1]$. Then, under all rules we have interior equilibria where agents all choose positive investment levels. Then, total investment under these rules is

$$\begin{aligned}
\sum_N s_i^* &= \sum_N \frac{n(1+r-\beta)+\beta(1-\alpha)+\beta(1-\alpha)a_i\sum_{N\backslash i}\frac{1}{a_j}}{a_i n(1+r-\beta)(1+r-\alpha\beta)} \ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right) \\
&= \left(\sum_N \frac{1}{a_i}\right)\frac{1}{r-\beta+1}\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right).
\end{aligned}$$

Let us look at its derivative with respect to α :

$$\frac{\partial\left(\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right)\frac{\sum_N \frac{1}{a_j}}{r-\beta+1}\right)}{\partial\alpha} = \frac{(n-1)\beta x}{(r-\beta+1)(n-\beta+\alpha\beta-n\alpha\beta)} > 0.$$

So, as $AP[\alpha]$ gets closer to PRO, its equilibrium total investment level increases.

Now assume that $\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha^*\beta)}\right) = 0$ for some $\alpha^* \in [0,1]$. Since the ln term is increasing in α, we have $\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right) > 0$ for all $\alpha \in (\alpha^*, 1]$. For these rules, the previous case shows that total investment is increasing in α. On the other hand, $\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right) < 0$ for all $\alpha \in [0, \alpha^*)$ and all of these rules induce zero investment in equilibrium.

Finally, assume that $\ln\left(\frac{npr}{(1-p)(n-\beta-(n-1)\alpha\beta)}\right) < 0$ for all $\alpha \in [0,1]$. Then all $AP[\alpha]$ rules induce zero investment in equilibrium. ■

Proof. (Theorem 2) There are three possible cases. First assume that $\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right) > 0$ for all $\alpha \in [0,1]$. Then, under all rules we have interior equilibria where agents all choose positive investment levels. Then, total investment under these rules is

$$\sum_N s_i^* = \frac{n^2(r+\alpha-\alpha\beta)\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right)\sum_N \frac{1}{a_i}}{A}$$

where $A = \left(\begin{array}{c} (nr+1-\beta+(n-1)(1-\beta)\alpha)^2 \\ +(nr+1-\beta+(n-1)(1-\beta)\alpha)(n-2)(1-\alpha)(1-\beta) \\ -(n-1)(1-\alpha)^2(1-\beta)^2 \end{array}\right)$. Let us look at its derivative with respect to α :

$$\begin{aligned}
&\frac{\partial\left(\frac{n^2(r+\alpha-\alpha\beta)\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right)\sum_N \frac{1}{a_i}}{\left((nr+1-\beta+(n-1)(1-\beta)\alpha)^2+(nr+1-\beta+(n-1)(1-\beta)\alpha)(n-2)(1-\alpha)(1-\beta)-(n-1)(1-\alpha)^2(1-\beta)^2\right)}\right)}{\partial\alpha} \\
= \; & \frac{-(n-1)\left(\sum_N \frac{1}{a_i}\right)}{(n\alpha-\alpha+1)(r-\beta+1)} < 0.
\end{aligned}$$

So, as $LP[\alpha]$ gets closer to PRO, its equilibrium total investment level decreases.

Now assume that $\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha^*)}\right) = 0$ for some $\alpha^* \in [0,1]$. Since the ln term is decreasing in α, we have $\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right) > 0$ for all $\alpha \in [0,\alpha^*)$. For these rules, the previous case shows that total investment is decreasing in α. On the other hand, $\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right) < 0$ for all $\alpha \in (\alpha^*, 1]$ and all of these rules induce zero investment in equilibrium.

Finally, assume that $\ln\left(\frac{npr}{(1-\beta)(1-p)(1+(n-1)\alpha)}\right) < 0$ for all $\alpha \in [0,1]$. Then all $LP[\alpha]$ rules induce zero investment in equilibrium. ■

Proof. (Lemma 4) Inserting the equilibrium investment levels into the agents' payoff functions, we obtain the following "indirect utility functions".

In equilibrium, PRO induces the welfare level $U_i^{PRO}(\epsilon(G^{PRO})) =$

$$\begin{cases} -p\left(\frac{(1-p)(1-\beta)}{pr}\right)^{\frac{r}{(r+1-\beta)}} - (1-p)\left(\frac{pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(r+1-\beta)}} & if \quad \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0, \\ -1 & \text{otherwise.} \end{cases}$$

In equilibrium, EA induces the welfare level $U_i^{EA}\left(\epsilon(G^{EA})\right) =$

$$\begin{cases} -p\left(\frac{(1-p)(2-\beta)}{2pr}\right)^{\frac{r(2+2r-\beta)a_j+r\beta a_i}{2a_j(r+1)(r-\beta+1)}} - (1-p)\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{2(1+r-\beta)a_j-r\beta\left(a_i+a_j\right)}{2a_j(r+1)(r-\beta+1)}} & if \quad \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0, \\ -1 & \text{otherwise.} \end{cases}$$

In equilibrium, EL induces the welfare level $U_i^{EL}\left(\epsilon(G^{EL})\right) =$

$$\begin{cases} -p\left(\frac{(1-p)(1-\beta)}{2pr}\right)^{\frac{a_i(1+2r-\beta)}{2(1+r-\beta)}\left(\frac{1}{a_i}-\frac{(1-\beta)}{(1+2r-\beta)}\frac{1}{a_j}\right)} - (1-p)\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)\left(a_i+a_j\right)}{2a_j(1+r-\beta)}} & if \quad \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > 0, \\ -1 & \text{otherwise.} \end{cases}$$

Item (i) holds since $U_i^{PRO}(\epsilon(G^{PRO}))$ is independent of a_i. To see items (ii) and (iii), we simply compare the welfare expressions for the two agents. ■

By Lemma 4, PRO produces the following egalitarian social welfare function.

$EG^{PRO}(p,r,\beta,a_1,a_2) = U_1^{PRO}(\epsilon\left(G^{PRO}\right)) = U_2^{PRO}(\epsilon\left(G^{PRO}\right)) =$

$$\begin{cases} -p\left(\frac{(1-p)(1-\beta)}{pr}\right)^{\frac{r}{(r+1-\beta)}} - (1-p)\left(\frac{pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(r+1-\beta)}} & if \quad \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0, \\ -1 & \text{otherwise.} \end{cases}$$

EA produces the following egalitarian social welfare function. We again use $U_1^{EA}\left(\epsilon\left(G^{EA}\right)\right) \leq U_2^{EA}\left(\epsilon\left(G^{EA}\right)\right)$ by Lemma 4. Thus, $EG^{EA}(p,r,\beta,a_1,a_2) = U_1^{EA}(\epsilon\left(G^{EA}\right)) =$

$$\begin{cases} -p\left(\frac{(1-p)(2-\beta)}{2pr}\right)^{\frac{r(2+2r-\beta)a_2+r\beta a_1}{2a_2(r+1)(r-\beta+1)}} - (1-p)\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{2(1+r-\beta)a_2-r\beta(a_1+a_2)}{2a_2(r+1)(r-\beta+1)}} & if \quad \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0, \\ -1 & \text{otherwise.} \end{cases}$$

Since by Lemma 4, $U_1^{EL}(\epsilon\left(G^{EL}\right)) \geq U_2^{EL}(\epsilon\left(G^{EL}\right))$, EL produces the egalitarian social welfare function $EG^{EL}(p,r,\beta,a_1,a_2) = U_2^{EL}(\epsilon\left(G^{EL}\right)) =$

$$\begin{cases} -p\left(\frac{(1-p)(1-\beta)}{2pr}\right)^{\frac{a_2(1+2r-\beta)}{2(1+r-\beta)}\left(\frac{1}{a_2}-\frac{(1-\beta)}{(1+2r-\beta)}\frac{1}{a_1}\right)} - (1-p)\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)(a_1+a_2)}{2a_1(1+r-\beta)}} & if \quad \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > 0, \\ -1 & \text{otherwise.} \end{cases}$$

The following lemma shows that the egalitarian social welfare under EL is maximized when $a_1 = a_2$.

Lemma 6 *For parameter values where EL induces an interior equilibrium (where all agents choose a positive investment level), that is, when* $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > 0$, *the egalitarian social welfare level under EL is decreasing in* a_2. *Thus, it is maximized when* $a_1 = a_2$ *and, is given by the expression*

$$EG^{EL}(p,r,\beta,a,a) = -p\left(\frac{(1-p)(1-\beta)}{2pr}\right)^{\frac{r}{r-\beta+1}} - (1-p)\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(1+r-\beta)}}.$$

Proof. To prove this, we let $a_2 = a_1 + x$ and differentiate EG^{EL} with respect to x. Now $\frac{\partial\left(EG^{EL}(p,r,\beta,a_1,a_1+x)\right)}{\partial x} = \frac{(\beta-1)}{2a_1(r-\beta+1)}\left(\frac{(1-p)}{\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{\frac{(\beta-1)(x+2a_1)}{2a_1(r-\beta+1)}}} + \frac{p}{\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{\frac{(x\beta-x+2ra_1)}{2a_1+2ra_1-2\beta a_1}}}\right)\ln\left(\frac{2pr}{(\beta-1)(p-1)}\right)$
is negative since
$\ln\left(\frac{2pr}{(\beta-1)(p-1)}\right) > 0$. Thus, EG^{EL} is maximized at $a_2 = a_1 = a$. The expression then becomes

$$EG^{EL}(p,r,\beta,a,a) = -p\left(\frac{(1-p)(1-\beta)}{2pr}\right)^{\frac{r}{r-\beta+1}} - (1-p)\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(1+r-\beta)}}.$$

■

A similar statement is true for EA.

Lemma 7 *For parameter values where EA induces an interior equilibrium (where all agents choose a positive investment level), that is, when* $\ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0$*, the egalitarian social welfare level under EA is decreasing in a_2. Thus, it is maximized when $a_1 = a_2$ and, is given by the expression*

$$EG^{EA}(p,r,\beta,a,a) = -p\left(\frac{(1-p)(2-\beta)}{2pr}\right)^{\frac{r}{r-\beta+1}} - (1-p)\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{1-\beta}{r-\beta+1}}.$$

Proof. To prove this, we let $a_2 = a_1 + x$ and differentiate EG^{EA} with respect to x. Now $\frac{\partial\left(EG^{EA}(p,r,\beta,a_1,a_1+x)\right)}{\partial x} = \frac{-r\beta a_1}{2(x+a_1)^2(r+1)(r-\beta+1)}A\left(\ln\frac{2pr}{(\beta-2)(p-1)}\right)$, where

$$\begin{aligned} A &= (1-p)\,2^{\frac{\left(x+a_1-x\beta+ra_1-\beta a_1+rx-\frac{1}{2}rx\beta-r\beta a_1\right)}{(x+a_1)(r+1)(r-\beta+1)}}\left(\frac{pr}{(\beta-2)(p-1)}\right)^{\frac{(2x+2a_1-2x\beta+2ra_1-2\beta a_1+2rx-rx\beta-2r\beta a_1)}{2(x+a_1)(r+1)(r-\beta+1)}} \\ &\quad+\left(\frac{1}{2}\right)^{\frac{r(2x+2a_1-x\beta+2ra_1+2rx)}{2(x+a_1)(r+1)(r-\beta+1)}} p\left(\frac{(\beta-2)(p-1)}{pr}\right)^{\frac{r(2x+2a_1-x\beta+2ra_1+2rx)}{2(x+a_1)(r+1)(r-\beta+1)}} \end{aligned}$$

is negative since $\ln\left(\frac{2pr}{(\beta-2)(p-1)}\right) > 0$. Thus, EG^{EA} is maximized at $a_2 = a_1 = a$. The expression then becomes

$$EG^{EA}(p,r,\beta,a,a) = -p\left(\frac{(1-p)(2-\beta)}{2pr}\right)^{\frac{r}{r-\beta+1}} - (1-p)\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{1-\beta}{r-\beta+1}}.$$

■

We next use the above lemmas to show that PRO induces higher egalitarian social welfare than both EA and EL.

Proof. (Theorem 3) Note that $a_1 \le a_2$. We first show that $EG^{PRO}(p,r,\beta,a) > EG^{EL}(p,r,\beta,a)$ is always true. Due to the assumption $\ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0$, and due to Lemma 6, we have $EG^{EL}(p,r,\beta,a_1,a_2) \le EG^{EL}(p,r,\beta,a_1,a_1) = -p\left(\frac{(1-p)(1-\beta)}{2pr}\right)^{\frac{r}{r-\beta+1}} - (1-p)\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(1+r-\beta)}}$ and $EG^{PRO}(p,r,\beta,a_1,a_2) = -p\left(\frac{(1-p)(1-\beta)}{pr}\right)^{\frac{r}{(r+1-\beta)}} - (1-p)\left(\frac{pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(r+1-\beta)}}$. The inequality $EG^{PRO}(p,r,\beta,a_1,a_2) > EG^{EL}(p,r,\beta,a_1,a_1)$ simplifies to $2^{\frac{-r}{r+1-\beta}} + \frac{r}{(1-\beta)}2^{\frac{(1-\beta)}{(r+1-\beta)}} - \frac{r+1-\beta}{(1-\beta)} > 0$ which is true for all $\beta \in (0,1)$ and $r \in (0,1]$. This proves $EG^{PRO}(p,r,\beta,a_1,a_2) > EG^{EL}(p,r,\beta,a_1,a_2)$.

We next show that $EG^{PRO}(p,r,\beta,a_1,a_2) > EG^{EA}(p,r,\beta,a_1,a_2)$ is always true. Now by assumption $\ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0$ and Lemma 7, we have $EG^{EA}(p,r,\beta,a_1,a_2) \le EG^{EA}(p,r,\beta,a_1,a_1) = -p\left(\frac{(1-p)(2-\beta)}{2pr}\right)^{\frac{r}{r-\beta+1}} - (1-p)\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{1-\beta}{r-\beta+1}}$ and $EG^{PRO}(p,r,\beta,a_1,a_2) = -p\left(\frac{(1-p)(1-\beta)}{pr}\right)^{\frac{r}{(r+1-\beta)}} - (1-p)\left(\frac{pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(r+1-\beta)}}$. The inequality $EG^{PRO}(p,r,\beta,a_1,a_2) > EG^{EA}(p,r,\beta,a_1,a_1)$ simplifies to the inequality $\frac{1-\frac{\beta}{2}+r}{\left(1-\frac{\beta}{2}\right)^{\frac{1-\beta}{r-\beta+1}}} - \frac{1-\beta+r}{(1-\beta)^{\frac{(1-\beta)}{(r+1-\beta)}}} > 0$ which is true for all $\beta \in (0,1)$ and $r \in (0,1]$. ■

The utilitarian social welfare function induced by PRO is $UT^{PRO}(p,r,\beta,a_1,a_2) =$

$$\begin{cases} -2p\left(\frac{(1-p)(1-\beta)}{pr}\right)^{\frac{r}{(r+1-\beta)}} - 2(1-p)\left(\frac{pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(r+1-\beta)}} & if \quad \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0, \\ -2 & \text{otherwise.} \end{cases}$$

EA produces the utilitarian social welfare $UT^{EA}(p,r,\beta,a_1,a_2) =$

$$\begin{cases} -\frac{p}{\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{r(2+2r-\beta)a_2+r\beta a_1}{2a_2(r+1)(r-\beta+1)}}} - (1-p)\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{2(1+r-\beta)a_2-r\beta(a_1+a_2)}{2a_2(r+1)(r-\beta+1)}} & \\ -\frac{p}{\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{r(2+2r-\beta)a_1+r\beta a_2}{2a_1(r+1)(r-\beta+1)}}} - (1-p)\left(\frac{2pr}{(1-p)(2-\beta)}\right)^{\frac{2(1+r-\beta)a_1-r\beta(a_1+a_2)}{2a_1(r+1)(r-\beta+1)}} & if \quad \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0, \\ -2 & \text{otherwise.} \end{cases}$$

Finally, EL produces the utilitarian social welfare $UT^{EL}(p,r,\beta,a_1,a_2) =$

$$\begin{cases} -p\left(\frac{(1-p)(1-\beta)}{2pr}\right)^{\frac{a_1(1+2r-\beta)}{2(1+r-\beta)}\left(\frac{1}{a_1}-\frac{(1-\beta)}{(1+2r-\beta)}\frac{1}{a_2}\right)} - (1-p)\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)(a_1+a_2)}{2a_2(1+r-\beta)}} & \\ -p\left(\frac{(1-p)(1-\beta)}{2pr}\right)^{\frac{a_2(1+2r-\beta)}{2(1+r-\beta)}\left(\frac{1}{a_2}-\frac{(1-\beta)}{(1+2r-\beta)}\frac{1}{a_1}\right)} - (1-p)\left(\frac{2pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)(a_1+a_2)}{2a_1(1+r-\beta)}} & if \quad \ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) \geq 0, \\ -2 & \text{otherwise.} \end{cases}$$

The following lemma shows that the utilitarian social welfare under EL is maximized when $a_1 = a_2$.

Lemma 8 *For parameter values where EL induces an interior equilibrium (where all agents choose a positive investment level), that is, when* $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) \geq 0$, *the utilitarian social welfare level under EL is decreasing in* a_2. *Thus, it is maximized when* $a_2 = a_1$ *and, is given by the expression*

$$UT^{EL}(p,r,\beta,a,a) = -2p\left(\frac{(1-\beta)(1-p)}{2pr}\right)^{\frac{r}{r-\beta+1}} - 2(1-p)\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{\frac{1-\beta}{r-\beta+1}}.$$

Proof. To prove this, we let $a_2 = a_1 + x$ and differentiate UT^{EL} with respect to x. $\frac{\partial\left(UT^{EL}(p,r,\beta,a,a+x)\right)}{\partial x} =$

$\frac{-(1-\beta)}{2a_1(x+a_1)^2(r-\beta+1)}\left(\ln\frac{2pr}{(1-\beta)(1-p)}\right)A$ where

$$A=\left(\begin{array}{c}\left(x^2+a_1^2\right)(1-p)\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{\frac{(1-\beta)(x+2a_1)}{2a_1(r-\beta+1)}}\\ -a_1^2(1-p)\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{\frac{(1-\beta)(x+2a_1)}{2(x+a_1)(r-\beta+1)}}\\ +2^{\frac{\left(\frac{1}{2}x+2a_1-\frac{1}{2}x\beta+ra_1-2\beta a_1\right)}{a_1(r-\beta+1)}}xa_1(1-p)\left(\frac{pr}{(\beta-1)(p-1)}\right)^{\frac{(1-\beta)(x+2a_1)}{2a_1(r-\beta+1)}}\\ -pa_1^2\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{-\frac{x(1+2r-\beta)+2ra_1}{2(x+a_1)(r-\beta+1)}}\\ +p(x+a_1)^2\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{\frac{x(1-\beta)-2ra_1}{2a_1(r-\beta+1)}}\end{array}\right)$$

Note that this derivative is negative if $x\geq 0$ and $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)>0$. Thus, UT^{EL} is maximized when $a_2=a_1$ (or equivalently, when $x=0$). Then, UT^{EL} becomes $UT^{EL}(p,r,\beta,a,a)=-2p\left(\frac{(1-\beta)(1-p)}{2pr}\right)^{\frac{r}{r-\beta+1}}-2(1-p)\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{\frac{1-\beta}{r-\beta+1}}$. ■

We next use the above lemma to show that PRO induces higher utilitarian social welfare than EL.

Proof. (Theorem 4) Note that $a_1\leq a_2$. We first show that $UT^{PRO}(p,r,\beta,a)>UT^{EL}(p,r,\beta,a)$ is always true. Due to the assumption $\ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right)>0$, and due to Lemma 8, we have $UT^{EL}(p,r,\beta,a_1,a_2)\leq UT^{EL}(p,r,\beta,a_1,a_1)=-2p\left(\frac{(1-\beta)(1-p)}{2pr}\right)^{\frac{r}{r-\beta+1}}-2(1-p)\left(\frac{2pr}{(\beta-1)(p-1)}\right)^{\frac{1-\beta}{r-\beta+1}}$. We will compare this expression to the utilitarian social welfare under PRO: $UT^{PRO}(p,r,\beta,a_1,a_2)=-2p\left(\frac{(1-p)(1-\beta)}{pr}\right)^{\frac{r}{(r+1-\beta)}}-2(1-p)\left(\frac{pr}{(1-p)(1-\beta)}\right)^{\frac{(1-\beta)}{(r+1-\beta)}}$. The inequality $UT^{PRO}(p,r,\beta,a_1,a_2)>UT^{EL}(p,r,\beta,a_1,a_1)$ simplifies to $2^{\frac{1-\beta}{r-\beta+1}}+\frac{(1-\beta)}{r}2^{\frac{-r}{r-\beta+1}}-\frac{(1-\beta+r)}{r}>0$ which is true for all $\beta\in(0,1)$ and $r\in(0,1]$. Therefore $UT^{PRO}>UT^{EL}$. ■

8 Appendix B: Constrained Equal Awards and Constrained Equal Losses Rules

In this section, we discuss the investment games induced by the Constrained Equal Awards (CEA) and the Constrained Equal Losses (CEL) rules, particularly in relation to their unconstrained counterparts, the EA and EL rules.

8.0.1 Constrained Equal Awards

For $\alpha = 0$, $AP[\alpha]$ becomes the Equal Awards rule, EA. This is the unconstrained version of a well-known rule from the axiomatic literature, called the Constrained Equal Awards rule, CEA, defined in Section 2. The function CEA_i is written below more explicitly for the case of two agents: if agent i invests too little (the first line), he gets full refund and if he invests too much (the third line), the other agent gets full refund. For in-between investment levels (the second line), the liquidation value of the firm is equally allocated between the two agents, similar to EA.

$$CEA_i\left(s_i, s_j\right) = \begin{cases} s_i & \text{if } s_i \leq \frac{\beta}{2-\beta} s_j, \\ \frac{\beta}{2}(s_i + s_j) & \text{if } \frac{\beta}{2-\beta} s_j \leq s_i \leq \frac{2-\beta}{\beta} s_j, \\ \beta(s_i + s_j) - s_j & \text{if } s_i \geq \frac{2-\beta}{\beta} s_j. \end{cases}$$

By Equation 1, the utility function of agent i is

$$U_i^{CEA}\left(s_i, s_j\right) = \begin{cases} -pe^{-a_i r s_i} - (1-p) & \text{if } s_i \leq \frac{\beta}{2-\beta} s_j, \\ -pe^{-a_i r s_i} - (1-p) e^{-a_i \frac{\beta}{2}(s_i + s_j) + a_i s_i} & \text{if } \frac{\beta}{2-\beta} s_j \leq s_i \leq \frac{2-\beta}{\beta} s_j, \\ -pe^{-a_i r s_i} - (1-p) e^{a_i (1-\beta)(s_i + s_j)} & \text{if } s_i \geq \frac{2-\beta}{\beta} s_j. \end{cases}$$

Note that, agent i's payoff function is different in each one of the three intervals.[13] To determine his best response to s_{-i}, agent i compares his payoffs from each one of his optimal choices in these intervals and picks the one(s) that yield the highest payoff. This leads to discontinuities in the agents' best responses. Thus, a Nash equilibrium does not exist for every parameter combination.[14] However, if it exists, it is identical to the unique equilibrium under EA.

Proposition 9 *There is no Nash equilibrium s^* where for some $i \in N$, $CEA_i(s^*) = s_i^* < \frac{1}{n}\beta \sum_N s_j^*$. Therefore, there is a unique Nash equilibrium under CEA. Also, if s^* is a Nash equilibrium under CEA, then s^* is also a Nash equilibrium under EA.*

Proof. Let $s \in S$ be such that for some $i \in N$, $s_i < \frac{\beta}{n} \sum_N s_j$. Differentiating, agent i's payoff function for this case, we obtain

$$\frac{\partial U_i^{CEA}(s_i, s_{-i})}{\partial s_i} = \frac{\partial\left(-pe^{-a_i r s_i} - (1-p)\right)}{\partial s_i} = a_i pre^{-a_i r s_i} > 0.$$

[13] The number of cases in agent i's utility function increases with the number of agents. For n agents, the precise number of cases is $1 + \sum_{k=1}^{n-1} \binom{n}{k}$.

[14] For example, let $N = \{1, 2\}$, $p = 0.8$, $\beta = 0.6$, $r = 0.5$, $a_1 = 0.4$, and $a_2 = 4$.

Therefore, no such s_i is a best response for agent i. The higher is s_i, the higher is his payoff. Thus there is no Nash equilibrium s^* where for some $i \in N$, $s_i^* < \frac{\beta}{n} \sum_N s_j^*$.

Now let s^* be a Nash equilibrium under CEA. Then, s^* is such that for each $i \in N$, $CEA_i(s^*) = \frac{\beta}{n} \sum_N s_j^*$. Thus,

$$U_i^{CEA}(s_i^*, s_{-i}^*) = U_i^{EA}(s_i^*, s_{-i}^*) = -pe^{-a_i r s_i^*} - (1-p)e^{-a_i\left(\frac{\beta}{n}\left(\sum_{N\setminus i} s_j^*\right)-(1-\frac{\beta}{n})s_i^*\right)}.$$

Therefore, if $i \in N$ is playing a best response with respect to U_i^{CEA}, he is also playing a best response with respect to U_i^{EA}. Thus, the unique equilibrium under CEA is also the unique equilibrium under EA. ■

Proposition 9 implies that analyzing the Nash equilibrium under EA also tells us about CEA.

8.0.2 Constrained Equal Losses

For $\alpha = 0$, $LP[\alpha]$ becomes the Equal Losses rule, EL. This is the unconstrained version of a well-known rule from the axiomatic literature, called the Constrained Equal Losses rule, CEL. The function CEL_i is written below more explicitly for the case of two-agents: if agent i invests too little (first line), he gets zero refund and if he invests too much (third line), the other agent gets zero refund. For in-between investment levels (the second line), the liquidation value of the firm is allocated to equate the agents' losses, similar to EL.[15]

$$CEL_i(s_i, s_j) = \begin{cases} 0 & \text{if } s_i \leq \frac{1-\beta}{1+\beta} s_j, \\ s_i - \frac{1-\beta}{2}(s_i + s_j) & \text{if } \frac{1-\beta}{1+\beta} s_j \leq s_i \leq \frac{1+\beta}{1-\beta} s_j, \\ \beta(s_i + s_j) & \text{if } s_i \geq \frac{1+\beta}{1-\beta} s_j. \end{cases}$$

By Equation 1, the utility function of agent i becomes

$$U_i^{CEL}(s_i, s_j) = \begin{cases} -pe^{-a_i r s_i} - (1-p)e^{a_i s_i} & \text{if } s_i \leq \frac{1-\beta}{1+\beta} s_j \\ -pe^{-a_i r s_i} - (1-p)e^{a_i \frac{1-\beta}{2}(s_i + s_j)} & \text{if } \frac{1-\beta}{1+\beta} s_j \leq s_i \leq \frac{1+\beta}{1-\beta} s_j \\ -pe^{-a_i r s_i} - (1-p)e^{a_i(1-\beta)s_i - a_i \beta s_j} & \text{if } s_i \geq \frac{1+\beta}{1-\beta} s_j \end{cases}$$

Similar to CEA, the best response correspondences are discontinuous and a Nash equilibrium does not exist for all parameter combinations.[16]

If it exists, however, for two agents, it can be one of the four following types. Note that, Type 1 and Type 2 equilibria under CEL are equal to the two types of equilibria obtained under EL.

[15] The number of cases in agent i's utility function increases with the number of agents. As in CEA, for n agents, the precise number of cases is $1 + \sum_{k=1}^{n-1} \binom{n}{k}$.

[16] For example, let $N = \{1, 2\}$, $p = 0.8$, $\beta = 0.8$, $r = 0.2$, $a_1 = 1.2$, and $a_2 = 1.8$.

Proposition 10 *Let $n=2$. If s^* is a Nash equilibrium of the investment game under CEL, then it has either one of the following forms.*
Type 1. $s_i^* = \frac{(2r-\beta+1)}{2r(r-\beta+1)}\left(\frac{1}{a_i} - \frac{(1-\beta)}{(1-\beta+2r)}\frac{1}{a_j}\right)\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right)$ *for both* $i \in N$ *or*
Type 2. $s_i^* = 0$ *for both* $i \in N$ *or*
Type 3. $s_i^* = \frac{1}{a_i(1+r)}\ln\left(\frac{pr}{1-p}\right)$ *and* $s_j^* = \frac{1}{a_j(r-\beta+1)}\ln\left(\frac{pr}{(1-\beta)(1-p)}\right) + \frac{\beta}{a_i(r+1)(r-\beta+1)}\ln\left(\frac{pr}{1-p}\right)$ *or*
Type 4. $s_i^* = 0$ *and* $s_j^* = \frac{1}{a_j(1+r-\beta)}\ln\left(\frac{pr}{(1-p)(1-\beta)}\right)$.

Proof. Note that $U_i^{CEL}(s_i, s_j)$ has three parts. For the two agents, there are two possible cases.
Case 1. In equilibrium both agents fall into the second part of their payoff function, which is identical to the one under EL. Then the equilibrium characterization is also identical to EL. The equilibrium is either of Type 1 or Type 2.
Case 2. In equilibrium, agent i falls into the first part of his payoff function and agent j falls into the third part. If the local maximizer of the first part of agent i's payoff function is zero, then the equilibrium is of Type 4. Otherwise, it is of Type 3. ■

Proposition 10 implies that analyzing the Nash equilibrium under EL also tells us about the first two types of equilibria under CEL.

Remark 1 *In a Type 4 equilibrium under CEL, only one agent invests and his choice is identical to what he would choose under PRO. For the same parameters, both agents choose positive investment under PRO. Thus, a Type 4 equilibrium under CEL induces less total investment than PRO.*

9 Appendix C: Numerical Welfare Comparisons

When comparing the social welfare levels induced by two bankruptcy rules, we will sometimes observe that the welfare ordering between them depends on the parameter values under consideration. In such cases, to compare the rules we will run a numerical analysis. We will use the following parameter values: $p \in \{0.01, 0.06, 0.11, ..., 0.96\}$, $\beta \in \{0.01, 0.06, 0.11, ..., 0.96\}$, $r \in \{0.01, 0.06, 0.11, ..., 0.96\}$, $a_1 \in \{0.1, 1.1, 2.1, ..., 50.1\}$, and $a_2 \in \{a_1, a_1 + 1..., 50.1\}$. This corresponds to 10608000 parameter combinations. For each parameter combination, we will calculate the equilibrium welfare levels induced by each bankruptcy rule and then compare these levels.

9.1 Egalitarian Social Welfare

Note that $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right)$. If $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) \leq 0$, all three rules lead to a boundary equilibrium (with zero investment) and trivially, to the same welfare level

$$EG^{EL}(p,r,\beta,a_1,a_2) = EG^{EA}(p,r,\beta,a_1,a_2) = EG^{PRO}(p,r,\beta,a_1,a_2) = -1.$$

Otherwise, we have the following cases.

Case 1: All three rules induce positive investment in equilibrium.

This case happens when $\ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0$. For this case, Theorem 3 shows that PRO induces higher Egalitarian social welfare than both EA and EL. The ordering between EA and EL depends on the parameter values. We thus carry out the following numerical comparison.

Numerical comparison of EA with EL: This case corresponds to 1294986 parameter combinations. Among them, EA induce a higher social welfare 949377 times (73 percent) and EL induce a higher social welfare 345609 times (27 percent). The two rules never induce the same welfare level.

Case 2: EA leads to zero investment in equilibrium.

This case happens when $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0 \geq \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right)$. Then, both PRO and EL lead to interior equilibria (with positive investment) and EA leads to a boundary equilibrium (with zero investment). Thus, $EG^{EA}(p,r,\beta,a_1,a_2) = -1$. We already showed in *Theorem 3* that PRO induces higher social welfare than EL when both induce positive investment in equilibrium. To compare PRO and EA, note that under PRO, each agent chooses a positive investment level as a dominant strategy, even though choosing zero investment and receiving the payoff -1 is feasible. Thus $EG^{PRO}(p,r,\beta,a_1,a_2) > -1 = EG^{EA}(p,r,\beta,a_1,a_2)$.

To sum up, PRO induces a higher social welfare than both EL and EA in this case too.

Numerical comparison of EA with EL: This case corresponds to 829058 parameter combinations. Among them, EA induce a higher social welfare 9437 times (1 percent) and EL induce a higher social welfare 819621 times (99 percent). The two rules never induce the same welfare level.

Case 3: Both EA and PRO lead to zero investment in equilibrium.

This case happens when $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > 0 \geq \ln\left(\frac{pr}{(1-p)(1-\beta)}\right)$. Then EA and PRO lead to boundary equilibria (with zero investment) and $EG^{EA}(p,r,\beta,a_1,a_2) = EG^{PRO}(p,r,\beta,a_1,a_2) = -1$. We numerically compare this payoff to $EG^{EL}(p,r,\beta,a_1,a_2)$.

Numerical comparison of EL with PRO and EA: This case corresponds to 481118 parameter combinations. In all of them, EL induces a higher social welfare than EA and PRO.

9.2 Utilitarian Social Welfare

Note that $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > \ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right)$. If $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) \leq 0$, all three rules lead to a boundary equilibrium (with zero investment) and trivially, to the same welfare level

$$UT^{EL}(p, r, \beta, a_1, a_2) = UT^{EA}(p, r, \beta, a_1, a_2) = UT^{PRO}(p, r, \beta, a_1, a_2) = -2.$$

Otherwise, we have the following cases.

Case 1: All three rules induce positive investment in equilibrium.

This case happens when $\ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right) > 0$. We showed in Theorem 4 that, in this case, PRO always induces a higher utilitarian social welfare than EL. The relationship between PRO and EA and between EA and EL changes with the parameters.

Numerical comparison of EA with EL: This case corresponds to 1294986 parameter combinations. Among them, EA induce a higher social welfare 851736 times (66 percent) and EL induce a higher social welfare 443250 times (34 percent). The two rules never induce the same welfare level.

Numerical comparison of EA with PRO: This case corresponds to 2726424 parameter combinations. Among them, PRO induce a higher social welfare 1659219 times (61 percent) and EA induce a higher social welfare 1067205 times (39 percent). The two rules never induce the same welfare level.

Case 2: EA leads to zero investment in equilibrium.

This case happens when $\ln\left(\frac{pr}{(1-p)(1-\beta)}\right) > 0 \geq \ln\left(\frac{pr}{(1-p)(1-\frac{\beta}{2})}\right)$. Then, both PRO and EL lead to interior equilibria and EA leads to a boundary equilibrium. Thus, $EG^{EA}(p, r, \beta, a_1, a_2) = -2$. We already showed in Theorem 4 that PRO induces higher social welfare than EL in this case. To compare PRO and EA, note that under PRO, each agent chooses a positive investment level as a dominant strategy, even though choosing zero investment and receiving the payoff -1 is feasible. Thus $UT^{PRO}(p, r, \beta, a_1, a_2) > -2 = UT^{EA}(p, r, \beta, a_1, a_2)$.

To sum up, PRO induces a higher social welfare than both EL and EA in this case.

Numerical comparison of EA with EL: This case corresponds to 829058 parameter combinations. Among them, EA induce a higher social welfare 544892 times (66 percent) and EL induce a higher social welfare 284166 times (34 percent). The two rules never induce the same welfare level.

Case 3: Both EA and PRO lead to zero investment in equilibrium.

This case happens when $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > 0 \geq \ln\left(\frac{pr}{(1-p)(1-\beta)}\right)$. Then EA and PRO lead to boundary equilibria (where agents choose zero investment) and $UT^{EA}(p,r,\beta,a) = UT^{PRO}(p,r,\beta,a) = -2$.

In terms of the probability of success, this case corresponds to the interval $p \in (\frac{1-\beta}{2r+1-\beta}, \frac{1-\beta}{r+1-\beta}]$. If $p = \frac{1-\beta}{2r+1-\beta}$, $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) = 0$ and under all three rules, Nash equilibrium investment is zero. Therefore, $UT^{EL}\left(\frac{1-\beta}{2r+1-\beta}, r, \beta, a_1, a_1\right) = -2$. If $p > \frac{1-\beta}{r+1-\beta}$, PRO induces positive investment in equilibrium. For in between p values we have $\frac{\partial\left(UT^{EL}(p,r,\beta,a_1,a_2)\right)}{\partial p} < 0$ if and only if $p \in (\frac{1-\beta}{2r+1-\beta}, \frac{1-\beta}{r+1-\beta})$ and $\frac{\partial\left(UT^{EL}(p,r,\beta,a_1,a_2)\right)}{\partial p} = 0$ if and only if $p = \frac{1-\beta}{r+1-\beta}$. Therefore, for each $p \in (\frac{1-\beta}{2r+1-\beta}, \frac{1-\beta}{r+1-\beta}]$, $UT^{EL}(p,r,\beta,a_1,a_1) < UT^{EL}\left(\frac{1-\beta}{2r+1-\beta}, r, \beta, a_1, a_1\right) = -2$. Since by Lemma 8, $UT^{EL}(p,r,\beta,a_1,a_1) \geq UT^{EL}(p,r,\beta,a_1,a_2)$, we have $UT^{EL}(p,r,\beta,a_1,a_2) < -2$. This is summarized in the following proposition.

Proposition 11 *Assume* $\ln\left(\frac{2pr}{(1-p)(1-\beta)}\right) > 0 \geq \ln\left(\frac{pr}{(1-p)(1-\beta)}\right)$. *Then*

$$UT^{EL}(p,r,\beta,a_1,a_2) < -2 = UT^{EA}(p,r,\beta,a_1,a_2) = UT^{PRO}(p,r,\beta,a_1,a_2).$$