CONVERGENCE RADII FOR EIGENVALUES OF TRI–DIAGONAL MATRICES

J. ADDUCI, P. DJAKOV, AND B. MITYAGIN

Abstract. Consider a family of infinite tri–diagonal matrices of the form $L + zB$, where the matrix L is diagonal with entries $L_{kk} = k^2$, and the matrix B is off–diagonal, with nonzero entries $B_{k,k+1} = B_{k+1,k} = k^\alpha$, $0 \le \alpha < 2$. The spectrum of $L + zB$ is discrete. For small $|z|$ the n-th eigenvalue $E_n(z)$, $E_n(0) = n^2$, is a well–defined analytic function. Let R_n be the convergence radius of its Taylor's series about $z = 0$. It is proved that

$$R_n \le C(\alpha) n^{2-\alpha} \quad \text{if } 0 \le \alpha < 11/6.$$

1. Introduction

Since the famous 1969 paper of C. Bender and T. Wu [2], branching points and the crossings of energy levels have been studied intensively in the mathematical and physical literature (e.g., [8, 1, 4, 3] and the bibliography there). In this paper our goal is to analyze – mostly along the lines of J. Meixner and F. Schäfke approach [10] – a toy model of tri–diagonal matrices.

We consider the operator family $L + zB$, where L and B are infinite matrices of the form

$$L = \begin{bmatrix} q_1 & 0 & 0 & 0 & \cdot \\ 0 & q_2 & 0 & 0 & \cdot \\ 0 & 0 & q_3 & 0 & \cdot \\ 0 & 0 & 0 & q_4 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & b_1 & 0 & 0 & \cdot \\ c_1 & 0 & b_2 & 0 & \cdot \\ 0 & c_2 & 0 & b_3 & \cdot \\ 0 & 0 & c_3 & 0 & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix} \tag{1.1}$$

with

$$q_k = k^2, \tag{1.2}$$

$$|b_k|, |c_k| \le M k^\alpha, \tag{1.3}$$

$$\alpha < 2. \tag{1.4}$$

Sometimes we impose a symmetry condition:

$$b_k = \bar{c_k}. \tag{1.5}$$

2000 *Mathematics Subject Classification.* 47B36 (primary), 47A10 (secondary).

Key words and phrases. tri–diagonal matrix, operator family, eigenvalues.

B. Mityagin acknowledges the support of the Scientific and Technological Research Council of Turkey and the hospitality of Sabanci University, April–June, 2008.

Under the conditions (1.2)–(1.4) the spectrum of $L+zB$ is discrete. If $\alpha<1$ then a standard use of perturbation theory shows that there is $r>0$ such that for $|z|<r$

$$Sp(L+zB)=\{E_n(z)\}_{n=1}^{\infty},\quad E_n(0)=n^2, \tag{1.6}$$

where each $E_n(z)$ is well–defined analytic function in the disc $\{z:|z|<r\}$.

If $\alpha\in[1,2)$, then in general there is no such $r>0$. But the fact that n^2 is a simple eigenvalue of L guarantees (see [9], Chapter 7, Sections 1-3) that for each n there exists $r_n>0$ such that, on the disc $\{z:|z|<r_n\}$, there are an analytic function $E_n(z)$ and an analytic eigenvector function $\varphi_n(z)$ with

$$(L+zB)\varphi_n(z)=E_n(z)\varphi_n(z),\ \ |z|<r_n, \tag{1.7}$$

$$\varphi_n(0)=e_n,\ \ E_n(0)=n^2. \tag{1.8}$$

Let

$$E_n(z)=\sum_{k=0}^{\infty}a_k(n)z^k \tag{1.9}$$

be the Taylor series of $E_n(z)$ about 0, and let R_n, $0<R_n\leq\infty$, be its radius of convergence. The asymptotic behavior of the sequence (R_n) is one of the main topics of the present paper.

It may happen that $R_n>r_n$. Then, by (1.9), $E_n(z)$ is defined in the disc $\{z:|z|<R_n\}$ as an extension of the analytic function (1.7) in $\{z:|z|<r_n\}$. But are its values $E_n(z)$ eigenvalues of $L+zB$ if z is in the annulus $r_n\leq|z|<R_n$? The answer is positive as one can see from the next considerations.

In a more general context let us define *Spectral Riemann Surface*

$$G=\{(z,E):\ \exists g\in Dom(L),\ g\neq 0\ |\quad (L+zB)g=Eg\}. \tag{1.10}$$

This notion is justified by the following statement (coming from K. Weierstrass, H. Poincare, T. Carlemann – see discussions on the related history in [6, 11, 7]).

Proposition 1. *If (1.1)–(1.4) hold, then there exists a nonzero entire function $\Phi(z,w)$ such that*

$$G=\{(z,w)\in\mathbb{C}^2:\Phi(z,w)=0\}. \tag{1.11}$$

Proof. The identity

$$(L+zB)g=wg,\quad g\neq 0,\quad g\in Dom(L) \tag{1.12}$$

is equivalent to

$$(1-A(z,w))h=0\quad\text{with}\quad h=L^{1/2}g\in Dom(L^{1/2}),\ h\neq 0, \tag{1.13}$$

where

$$A(z,w)=-zL^{-1/2}BL^{-1/2}+wL^{-1}. \tag{1.14}$$

Therefore, w is an eigenvalue of the operator $L+zB$ if and only if 1 is an eigenvalue of the operator $A(z,w)$.

On the space S_1 of trace class operators T the determinant

$$d(T) = \det(1 - T) \tag{1.15}$$

is well defined (see [6], Chapter 4, Section 1 or [12], Chapter 3, Theorem 3.4), and $1 \in Sp(T)$ if and only if $d(T) = 0$ (see [12], Theorem 3.5 (b)).

Of course, the second term L^{-1} in (1.14) is an operator of trace class (even in $S_p, p > 1/2$) by (1.2). But (1.3)–(1.4) imply that $L^{-1/2}BL^{-1/2}$ is in the Schatten class $S_p, p > 1/(2-\alpha)$; only $\alpha < 1$ would guarantee that it is of trace class.

However, (1.15) could be adjusted (see [6] Chapter 4, Section 2 or [12], Chapter 9, Lemma 9.1 and Theorem 9.2). Namely, for any positive integer $p \geq 2$ we set

$$d_p(T) = \det(1 - Q_p(T)) \tag{1.16}$$

where

$$Q_p(T) = 1 - (1-T)\exp\left(T + \frac{T^2}{2} + \cdots + \frac{T^{p-1}}{p-1}\right).$$

Then $Q_p(T) \in S_1$ if $T \in S_p$, so d_p is a well-defined function of $T \in S_p$ and $1 \in Sp(T)$ if and only if $d_p(T) = 0$.

In our context we define, with $A(z,w) \in$ (1.14) and $p > 1/(2-\alpha)$,

$$\Phi(z,w) = \det\left[(1 - Q_p(A(z,w))\right]. \tag{1.17}$$

Now, from Claim 8, Section 1.3, Chapter 4 in [6] it follows that $\Phi(z,w)$ is an entire function on $\mathbb{C}^2$.

The function Φ vanishes at (z,w) if and only if 1 is an eigenvalue of the operator $A(z,w)$, i.e., if and only if $(z,w) \in G$. This completes the proof. $\square$

In particular, the above Proposition implies that $\Phi(z, E_n(z)) = 0$ if $|z| < r_n$, so by analyticity and uniqueness $\Phi(z, E_n(z)) = 0$ if $r_n \leq |z| < R_n$. Equivalence of the two definitions (1.10) and (1.11) for the Spectral Riemann Surface G explains now that $E_n(z)$ is an eigenvalue function in the disc $\{z : |z| < R_n\}$.

Our main focus in the search for an understanding of the behavior of R_n will be on the special case where

$$0 \leq \alpha < 2, \tag{1.18}$$

$$b_k = \bar{c_k} = k^\alpha. \tag{1.19}$$

If $\alpha = 0$ in (1.19), we have the Mathieu matrices. They arise if Fourier's method is used to analyze the Hill–Mathieu operator on $I = [0, \pi]$

$$Ly = -y'' + 2a(\cos 2x)y, \quad y(\pi) = y(0), \quad y'(\pi) = y'(0).$$

In this case J. Meixner and F. W. Schäfke proved ([10], Thm 8, Section 1.5; [11], p. 87) the inequality $R_n \leq Cn^2$ and conjectured that the asymptotic $R_n \asymp n^2$ holds. This has been proved 40 years later by H. Volkmer [13].

But what can be said if $0 < \alpha < 2$? Proposition 4 in [5] shows that if (1.1)–(1.3) and (1.18) hold, then

$$R_n \geq cn^{1-\alpha}. \tag{1.20}$$

This estimate from below cannot be improved in the class (1.1)–(1.3), (1.18) as examples in Section 4 show. But in the special case (1.18)–(1.19) one could expect the asymptotic

$$R_n \asymp n^{2-\alpha}. \tag{1.21}$$

We show that

$$R_n \leq Cn^{2-\alpha},$$

at least for $0 < \alpha < 11/6$.

Notice that in the Hill–Mathieu case we have $\alpha = 0$, $b_k = 1$ $\forall k$, so the operator B is bounded, while it could be unbounded in the case $\alpha > 0$. We use the approach of Meixner and Schäfke [10], but complement it with an additional argument to help us deal with the cases where the operator B is unbounded (but relatively compact with respect to L). The main result is the following.

Theorem 2. *If the conditions (1.2) and (1.19) hold, then for each* $\alpha \in [0, \frac{11}{6})$ *there exist constants* $C_\alpha > 0$ *and* $N_\alpha \in \mathbb{N}$ *such that*

$$R_n \leq C_\alpha n^{2-\alpha}, \quad n \geq N_\alpha. \tag{1.22}$$

Proof is given in Section 3. It has two parts. In Section 2, we prove an upper bound for Taylor coefficients $|a_k(n)|$ in terms of k, n, R_n and α (see Theorem 3). In Section 3 we show how a certain lower bound on $|a_k(n)|$, in terms of k, n, and α, can be used to prove the desired inequality on particular subsets of $[0, 2)$. In the same section we provide such lower bounds for $|a_2(n)|, |a_4(n)|, \ldots, |a_{12}(n)|$. This general scheme could be used in an attempt to prove (1.22) for larger subsets of $[0, 2)$. One would then need to compute (and manipulate) $a_k(n)$ for values of $k > 12$. See Section 3 for details.

2. An upper bound for $|a_k(n)|$

In what follows in this section, suppose that n is a *fixed* positive integer.

Theorem 3. *In the above notations, and under the conditions (1.2) and (1.3), if*

(a) $\alpha \in [0, 2)$ *and (1.5) holds, or* *(b)* $\alpha \in [0, 1)$,

then

$$|a_k(n)| \leq C\rho^{-(k-1)} \left(n^\alpha + \rho^{\frac{\alpha}{2-\alpha}}\right), \quad 0 < \rho < R_n, \tag{2.1}$$

where $C = C(\alpha, M)$.

Proof. For $r > 0$, let

$$\Delta_r = \{z \in \mathbb{C} : |z| < r\}, \quad C_r = \{z \in \mathbb{C} : |z| = r\}.$$

Let us choose, for every $z \in \Delta_{R_n}$, an eigenvector $g(z) = (g_n(z))_{n=1}^{\infty}$ such that $\|g(z)\|_{\ell^2} = 1$ (this is possible by Proposition 1). Then

$$(L + zB)g(z) = E_n(z)g(z), \quad \|g(z)\|_{\ell^2} = 1, \tag{2.2}$$

which implies (after multiplication from the right by $g(z)$)

$$\ell(z) + zb(z) = E_n(z), \quad z \in \Delta_{R_n}, \tag{2.3}$$

where

$$\ell(z) := \langle Lg(z), g(z)\rangle = \sum_{k=1}^{\infty} k^2 |g_k(z)|^2, \tag{2.4}$$

and

$$b(z) := \langle Bg(z), g(z)\rangle = \sum_{k=1}^{\infty} \left(c_k g_k(z)\overline{g_{k+1}(z)} + b_k g_{k+1}(z)\overline{g_k(z)} \right). \tag{2.5}$$

The functions $\ell(z)$ and $b(z)$ are bounded if $|z| \leq \rho < R_n$. Indeed, by (2.4) we have $\ell(z) > 0$. By (2.5) and (1.3)

$$|b(z)| \leq \sum_{k=1}^{\infty} Mk^{\alpha} \left(|g_k(z)|^2 + |g_{k+1}|^2\right) \leq 2M \sum_{k=1}^{\infty} k^{\alpha} |g_k(z)|^2, \tag{2.6}$$

so, estimating the latter sum by Hölder's inequality, we get

$$|b(z)| \leq 2M(\ell(z))^{\alpha/2}. \tag{2.7}$$

Therefore, in view of (2.3).

$$\ell(z) \leq |E_n(z)| + |zb(z)| \leq |E_n(z)| + 2M\rho(\ell(z))^{\alpha/2}, \quad |z| \leq \rho.$$

Now, Young's inequality implies

$$\ell(z) \leq |E_n(z)| + (1 - \alpha/2)2^{\frac{\alpha}{2-\alpha}}(2M\rho)^{\frac{2}{2-\alpha}} + (\alpha/4) \cdot \ell(z),$$

so, in view of (1.18), $\ell(z)$ is bounded by

$$\ell(z) \leq 2|E_n(z)| + 2(1 - \alpha/2)2^{\frac{\alpha}{2-\alpha}}(2M\rho)^{\frac{2}{2-\alpha}}, \quad |z| \leq \rho.$$

By (2.7), the function $b(z)$ is also bounded if $|z| \leq \rho$.

Since in (2.2) the vectors $g(z)$, $z \in \Delta_{R_n}$, are chosen in an arbitrary way, we cannot expect the function $z \to g(z)$ to be continuous, or even measurable. But the functions $\ell(z)$ and $b(z)$ are measurable. The explanation of this fact is the only difference in the proof of (2.1) in the cases (a) and (b).

(a) The functions $\ell(z)$ and $b(z)$ are continuous on $\Delta_{R_n} \setminus (-R_n, R_n)$.

Indeed, in view of (2.5) the symmetry assumption (1.5) implies that the function $b(z)$ is real–valued. Therefore, from (2.3) it follows $yb(z) =$

$Im\, E_n(z)$ with $z = x+iy$, so $\ell(z)$ and $b(z)$ are continuous on $\Delta_{R_n} \setminus (-R_n, R_n)$ because

$$(2.8) \quad b(z) = \frac{1}{y}\, Im(E_n(z)), \quad \ell(z) = Re(E_n(z)) - \frac{x}{y}\, Im(E_n(z)), \quad y \neq 0.$$

(b) For every z such that $E_n(z)$ is a simple eigenvalue of $L + wB$ the values $\ell(z)$ and $b(z)$ are uniquely determined by (2.4) and (2.5) and do not depend on the choice of the vector $g(z)$ in (2.2). Therefore, the functions $\ell(z)$ and $b(z)$ are uniquely determined on the set

$$U = \{z \in \Delta_{R_n} : \ \ E_n(z) \ \text{ is a simple eigenvalue of } \ L + zB\}.$$

On the other hand, the set $\Delta_{R_n} \setminus U$ is at most countable and has no finite accumulation points (see Section 5.1 in [5]).

If $w \in U$, then it is known ([9], Ch.VII, Sect. 1-3, in particular, Theorem 1.7) that there is a disc $D(w,\tau)$ with center w and radius τ such that $E_n(z)$ is a simple eigenvalue of the operator $L + zB$ for $z \in D(w,\tau)$ and there exists an analytic eigenvector function $\psi(z)$ defined in $D(w,\tau)$, i.e.,

$$(L + zB)\psi(z) = E_n(z)\psi(z), \quad \psi(z) \neq 0, \quad z \in D(w,\tau).$$

Let $g(z) = \psi(z)/\|\psi(z)\|_{\ell^2}$ for $z \in D(w,\tau)$. Then the coordinate functions $g_k(z)$ are continuous, and by (2.4) the function $\ell(z)$, $z \in D(w,\tau)$, is a sum of a series of positive continuous terms. Therefore, the function $\ell(z)$ is lower semi–continuous in $D(w,\tau)$, so it is lower semi–continuous in U. Thus, $\ell(z)$ is measurable on Δ_{R_n}. By (2.3) we have $b(z) = (E_n(z) - \ell(z))/z$ for $z \neq 0$. Thus, $b(z)$ is measurable in Δ_{R_n} as well.

For each $\rho \in (0, R_n)$, consider the space $L^2(C_\rho)$ with the norm $\|\cdot\|_\rho$ defined by $\|f\|_\rho^2 = \frac{1}{2\pi}\int_0^{2\pi} |f(\rho e^{i\theta})|^2 d\theta$. The functions $\ell(z)$ and $b(z)$ are integrable on each circle C_ρ, $\rho < R_n$ because they are bounded and measurable on C_ρ.

From (2.7) and Hölder's inequality it follows that

$$(2.9) \qquad \|b(z)\|_\rho \leq 2M \|\ell(z)\|_\rho^{\alpha/2}.$$

Since $\ell(z) > 0$, by (2.3) and (2.7) we have

$$|Im\,(E_n(z) - n^2)| = |Im\,(zb(z))| \leq \rho |b(z)|.$$

Therefore,

$$(2.10) \qquad \|Im\,(E_n(z) - n^2)\|_\rho \leq \rho \cdot \|b(z)\|_\rho.$$

If f is an analytic function defined on Δ_{R_n} with $f(0) = 0$, then $\|Re(f)\|_\rho = \|Im(f)\|_\rho$. In particular, we have

$$\|Re\,(E_n(z) - n^2)\|_\rho = \|Im\,(E_n(z) - n^2)\|_\rho,$$

which implies, by (2.10),

$$(2.11) \qquad \|E_n(z) - n^2\|_\rho \leq \sqrt{2}\rho \cdot \|b(z)\|_\rho.$$

In view of (2.3) and (2.11), the triangle inequality implies

$$\|\ell\|_\rho \leq n^2 + \|E_n(z) - n^2\|_\rho + \|b(z)\|_\rho \leq n^2 + (1 + \sqrt{2})\rho \cdot \|b(z)\|_\rho.$$

Therefore, from (2.9) it follows that

$$\|\ell\|_\rho \le n^2 + 5M\rho\|\ell\|_\rho^{\alpha/2}. \tag{2.12}$$

Now, Young's inequality yields

$$5M\rho\|\ell\|_\rho^{\alpha/2} \le \Big(1-\alpha/2)(5M2^{\alpha/2}\rho\Big)^{\frac{2}{2-\alpha}} + \frac{\alpha}{4}\|\ell\|_\rho \le C_1\rho^{\frac{2}{2-\alpha}} + \frac{1}{2}\|\ell\|_\rho,$$

with $C_1 = \big(1-\alpha/2)(5M2^{\alpha/2}\big)^{\frac{2}{2-\alpha}}$. Thus, by (2.12), we have

$$\|\ell\| \le 2n^2 + 2C_1\rho^{\frac{2}{2-\alpha}}.$$

In view of (2.11) and (2.9), this implies

$$\|E_n(z) - n^2\|_\rho \le 3M\rho\left(2^{\alpha/2}n^\alpha + (2C_1)^{\alpha/2}\rho^{\frac{\alpha}{2-\alpha}}\right). \tag{2.13}$$

By Cauchy's formula, we have

$$a_k(n) = \frac{1}{2\pi i}\int_{\partial\Delta_\rho}\frac{E_n(\zeta)-n^2}{\zeta^{k+1}}d\zeta.$$

From (2.13) it follows that

$$|a_k(n)| \le \rho^{-k}\|E_n(z)-n^2\|_\rho \le 3M\rho^{-k+1}\left(2^{\alpha/2}n^\alpha + (2C_1)^{\alpha/2}\rho^{\frac{\alpha}{2-\alpha}}\right),$$

which implies (2.1) with $C = 3M(2+2C_1)^{\alpha/2}$. This completes the proof of Theorem 3. □

Remark. In fact, to carry out the proof of Theorem 3 we need only to know that there exists a pair of functions $\ell(z)$ and $b(z)$ which satisfy (2.3) and (2.7), and are integrable on each circle $C_\rho,\ \rho < R_n$. We explained that the pair defined by (2.2), (2.4) and (2.5) has these properties. In the case (a) of Theorem 3 the same argument could be used to define a pair of real analytic functions functions $\ell(z)$ and $b(z)$ which satisfy (2.3) and (2.7).

Indeed, by (1.5) the operator B is a self–adjoint, so $L+xB,\ x\in\mathbb{R}$, is self–adjoint as well. Thus, the function $E_n(z)$ takes real values on the real line and its Taylor's coefficients are real. Since the quotients $\frac{1}{y}Im(x+iy)^k,\ k\in\mathbb{N}$, are polynomials of y, it is easy to see by the Taylor series of $E_n(z)$ that $\frac{1}{y}Im(E_n(z))$ (defined properly for $y=0$) is a real analytic function in Δ_{R_n}. Therefore, if one defines a pair of functions $\tilde{\ell}(z)$ and $\tilde{b}(z)$ by (2.8), then (2.3) holds immediately, and (2.7) follows because on $\Delta_{R_n}\setminus(-R_n,R_n)$ these functions coincide with $\ell(z)$ and $b(z)$.

3. An upper bound for R_n

In this section we use (2.1) in the case of (1.19) to prove Theorem 2. Roughly speaking, the bound (1.22) will be achieved for $\alpha\in[0,\frac{11}{6})$ by inserting the known (from [5]) formulas for $a_2(\alpha,n),\dots,a_{12}(\alpha,n)$ into inequality (2.1). With our approach, using only $a_{2k},\ k\le 6$, it is possible to get good lower bounds only if $0\le\alpha<11/6$.

We begin with the following observation.

Lemma 4. *Suppose the conditions (1.2),(1.3) and (1.18) hold.*
(a) If for some fixed $k, n \in \mathbb{N}$ *and* $\alpha \in [0, 2 - \frac{2}{k})$ *we have* $a_k(n) \neq 0$*, then* $R_n < \infty$.
(b) If $R_n = \infty$*, then* $E_n(z)$ *is a polynomial such that* $\deg E_n(z) \leq \frac{\alpha}{2-\alpha}$.

Proof. Let $a = |a_k(n)| > 0$. Then, by Theorem 3,

$$a\rho^{k-1} \leq C\left(n^\alpha + \rho^{\frac{\alpha}{2-\alpha}}\right), \quad \forall \rho < R_n. \tag{3.1}$$

The condition $\alpha \in [0, 2 - \frac{2}{k})$ implies $k - 1 > \frac{\alpha}{2-\alpha}$; therefore, (3.1) fails for sufficiently large ρ. Thus, $R_n \leq \sup\{\rho : \ \rho \in (3.1)\} < \infty$, which proves (a).

If $R_n = \infty$, then (a) shows that $a_k(n) = 0$ for all k such that $k > \frac{\alpha}{2-\alpha}$. This proves (b).

□

Lemma 5. *Suppose that conditions (1.2) and (1.3) hold. If for some fixed* $k, n \in \mathbb{N}$, $A > 0$ *and* $\alpha \in [0, 2 - \frac{2}{k})$ *we have*

$$An^{k\alpha - 2(k-1)} \leq |a_k(n)|, \tag{3.2}$$

then

$$R_n \leq \tilde{C} n^{2-\alpha}, \tag{3.3}$$

where $\tilde{C} = \tilde{C}(\alpha, M, A, k)$.

Proof. It is enough to prove that

$$\rho \leq \tilde{C} n^{2-\alpha}, \quad \forall \rho \in (0, R_n). \tag{3.4}$$

Then (3.3) follows if we let $\rho \to R_n$.

By (2.1) we have

$$An^{k\alpha - 2(k-1)} \leq |a_k(n)| \leq 2C(\alpha, M)\rho^{-(k-1)} \max(n^\alpha, \rho^{\frac{\alpha}{2-\alpha}}).$$

If $n^\alpha \geq \rho^{\frac{\alpha}{2-\alpha}}$, then we get (3.4) with $\tilde{C} = 1$.

Suppose that $n^\alpha < \rho^{\frac{\alpha}{2-\alpha}}$. Then $\max(n^\alpha, \rho^{\frac{\alpha}{2-\alpha}}) = \rho^{\frac{\alpha}{2-\alpha}}$, so

$$A\rho^{k - \frac{2}{2-\alpha}} \leq 2C(\alpha, M)(n^{2-\alpha})^{k - \frac{2}{2-\alpha}}.$$

Thus, whenever $\alpha < 2 - 2/k$, this inequality implies (3.3) with $\tilde{C} = (2C/A)^\gamma$, where $\gamma = (2 - \alpha)/(k(2 - \alpha) - 2)$. □

According to the preceding lemma, all one needs in order to get an upper bound on R_n of the form (3.3) (or even to explain that R_n is finite) is to find a lower bound on $|a_k(n)|$ of the form (3.2) (or at least to explain that $a_k(n) \neq 0$). We now describe a technique to provide such lower bounds. Theorem 2 will follow when we get such lower bounds for $|a_2(n)|, \ldots, |a_{12}(n)|$.

Lemma 6. *Under conditions (1.4) and (1.19), for each fixed* $\alpha < 2$*, the coefficient* $a_k(n,\alpha)$ *can be written in the form*

$$a_k(n,\alpha) = n^{k\alpha-(k-1)} f_\alpha(1/n) \tag{3.5}$$

where

$$f_\alpha(w) = \sum_{j=0}^{\infty} P_k(j,\alpha) w^j$$

is analytic on the disk $|w| < 1/k$*, and* $P_k(j,\alpha)$ *are polynomials of* α*.*

Proof. We begin this proof by stating the equation (3.7) from [5]

$$a_k(n) = \frac{1}{2\pi i}\int_{\partial\Pi}\left(\sum_{|j-n|\le k}(\lambda - n^2)\langle R^0_\lambda (BR^0_\lambda)^k e_j, e_j\rangle\right) d\lambda, \tag{3.6}$$

where $R^0_\lambda = (\lambda - L)^{-1}$, e_j is the j^{th} unit vector, and Π is the square centered at n^2 of width $2n$. This formula appears in [5] only in the case of $\alpha \in [0,1)$, but its proof therein holds for $\alpha < 2$ as well. It follows from (1.1) that for each $j \in N$,

$$BR^0_\lambda e_j = \begin{cases} \frac{(j-1)^\alpha}{\lambda - j^2} e_{j-1} + \frac{j^\alpha}{\lambda - j^2} e_{j+1} & \text{if j} > 1 \\ \\ \frac{1}{\lambda - 1} e_2 & \text{if j} = 1. \end{cases}$$

So, $(\lambda - n^2)\langle R^0_\lambda (BR^0_\lambda)^k e_j, e_j\rangle$ can be written as a finite sum each of whose terms is of the form

$$\frac{\lambda - n^2}{\lambda - (n - j_0')^2}\prod_{i=1}^{k}\frac{(n - d_i')^\alpha}{\lambda - (n - j_i')^2}$$

with j_i' and d_i' integers satisfying $|j_i'|, |d_i'| < k$ for each i. So, from a residue calculation on (3.6), $a_k(n)$ can be written as a linear combination of terms of the form

$$(n - d_k)^\alpha \prod_{i=1}^{k-1}\frac{(n - d_i)^\alpha}{n^2 - (n - j_i)^2} \tag{3.7}$$

$$= Cn^{k\alpha-(k-1)}\left(1 - \frac{d_k}{n}\right)^\alpha \prod_{i=1}^{k-1}\left[\left(1 - \frac{d_i}{n}\right)^\alpha \left(1 - \frac{j_i}{2n}\right)^{-1}\right]$$

with $C = \prod_{i=1}^{k-1}(2j_i)^{-1}$ and $|j_i|, |d_i| < k$ for each i.

For $n > k$, we have $|d_i/n| < 1$ and $|j_i/(2n)| < 1$. Thus,

$$\left(1 - \frac{d_i}{n}\right)^\alpha = 1 - \alpha\left(\frac{d_i}{n}\right) + \frac{\alpha(\alpha-1)}{2}\left(\frac{d_i}{n}\right)^2 + \dots \tag{3.8}$$

$$\left(1 - \frac{j_i}{2n}\right)^{-1} = 1 + \left(\frac{j_i}{2n}\right) + \left(\frac{j_i}{2n}\right)^2 + \dots \tag{3.9}$$

are analytic functions of $z = 1/n$ whenever $n > k$. Combining (3.7) with (3.8)–(3.9), we deduce that $a_k(n)$ can be written as in (3.5) with $f_\alpha(z)$ analytic for $|z| < 1/k$. □

The preceding lemma guarantees that whenever $\alpha < 2$,

$$a_k(n,\alpha) = P_k(0,\alpha) n^{k\alpha-(k-1)} + O(n^{k\alpha-k}) \quad \text{as } n \to \infty.$$

When $a_2(n), \ldots, a_{12}(n)$ were computed (following the approach of [5, p.305–306]), an interesting phenomenon was observed. If $2 \le k \le 12$, then

$$P_k(j,\alpha) = 0 \ \text{ for each } \ 0 \le j \le k-2. \tag{3.10}$$

In particular, if (1.18) and (1.19) hold, then

$$a_k(n) = P_k(k-1,\alpha) n^{k\alpha-2(k-1)} + O(n^{k\alpha-2k+1}), \quad n \to \infty; \tag{3.11}$$

the polynomials $P_k(k-1,\alpha)$, $k = 2, 4, \ldots, 12$, are given in the following table.

k	$P_k(k-1,\alpha)$
2	$-\alpha + \frac{1}{2}$
4	$-\alpha^3 + \frac{9}{4}\alpha^2 - \frac{11}{8}\alpha + \frac{5}{32}$
6	$-\frac{9}{4}\alpha^5 + \frac{73}{8}\alpha^4 - \frac{27}{2}\alpha^3 + \frac{281}{32}\alpha^2 - \frac{147}{64}\alpha + \frac{9}{64}$
8	$-\frac{61}{9}\alpha^7 + \frac{2881}{72}\alpha^6 - \frac{6875}{72}\alpha^5 + \frac{33937}{288}\alpha^4 - \frac{11437}{144}\alpha^3 + \frac{64649}{2304}\alpha^2 - \frac{4507}{1024}\alpha + \frac{1469}{8192}$
10	$-\frac{1525}{64}\alpha^9 + \frac{23705}{128}\alpha^8 - \frac{353023}{576}\alpha^7 + \frac{648539}{576}\alpha^6 - \frac{5774039}{4608}\alpha^5 + \frac{7955297}{9216}\alpha^4 - \frac{6626165}{18432}\alpha^3 + \frac{6173425}{73728}\alpha^2 - \frac{148881}{16384}\alpha + \frac{4471}{16384}$
12	$-\frac{221321}{2400}\alpha^{11} + \frac{8544347}{9600}\alpha^{10} - \frac{1207947}{320}\alpha^9 + \frac{71029219}{7680}\alpha^8 - \frac{92577243}{6400}\alpha^7 + \frac{385333821}{25600}\alpha^6 - \frac{16162765}{1536}\alpha^5 + \frac{9344339}{1920}\alpha^4 - \frac{583689039}{409600}\alpha^3 + \frac{296768801}{1228800}\alpha^2 - \frac{12877899}{655360}\alpha + \frac{121191}{262144}$

Numerical computations tell us that in the following table, each inequality in the second column holds on the union of intervals shown in the first column.

Set	Inequality
$\alpha \in S_2 = \left[0, \frac{1}{4}\right] \cup \left[\frac{3}{4}, 1\right)$	$\|P_2(1,\alpha)\| > \frac{1}{8}$
$\alpha \in S_4 = \left[\frac{1}{4}, \frac{3}{4}\right] \cup \left[1, \frac{9}{8}\right] \cup \left[\frac{11}{8}, \frac{3}{2}\right)$	$\|P_4(3,\alpha)\| > \frac{1}{32}$
$\alpha \in S_6 = \left[\frac{9}{8}, \frac{11}{8}\right] \cup \left[\frac{25}{16}, \frac{5}{3}\right)$	$\|P_6(5,\alpha)\| > \frac{1}{200}$
$\alpha \in S_8 = \left[\frac{3}{2}, \frac{25}{16}\right] \cup \left[\frac{5}{3}, \frac{7}{4}\right)$	$\|P_8(7,\alpha)\| > \frac{1}{10}$
$\alpha \in S_{10} = \left[\frac{7}{4}, \frac{9}{5}\right)$	$\|P_{10}(9,\alpha)\| > \frac{1}{2}$
$\alpha \in S_{12} = \left[\frac{9}{5}, \frac{11}{6}\right)$	$\|P_{12}(11,\alpha)\| > 1$

Proof of Theorem 2. In view of (3.11) and the above table, there is a constant $A > 0$ such that, for each $\alpha \in [0, 2 - \frac{1}{6})$, we have

$$|a_k(n,\alpha)| > An^{k\alpha - 2(k-1)}, \quad n \geq N_\alpha. \tag{3.12}$$

Therefore, Lemma 5 implies that there exists a constant C_α such that

$$R_n \leq C_\alpha n^{2-\alpha} \quad \text{for } n \geq N_\alpha.$$

Thus, (1.22) holds for $n \in \mathbb{N}$, which completes the proof of Theorem 2.

4. General discussion

In this section we give a few examples to show that the order $1-\alpha$ of lower bound (1.20) for R_n is sharp in the class of matrices B with (1.2)–(1.4).

1. *A case in which* $R_n \sim n^{1-\alpha}$. Let $\alpha \in [0,1)$. Suppose now that in (1.1) we set

$$b_k = c_k = (2+(-1)^k)k^\alpha \tag{4.1}$$

$$q_k = k^2 \tag{4.2}$$

Then by [5], Section 7.5, p.35,

$$|a_2(n)| = \left|\frac{b_{n-1}c_{n-1}}{2n-1} - \frac{b_n c_n}{2n+1}\right| = \begin{cases} \left|\frac{9(n-1)^{2\alpha}}{2n-1} - \frac{n^{2\alpha}}{2n+1}\right| & \text{if } n \text{ is odd,} \\ \left|\frac{(n-1)^{2\alpha}}{2n-1} - \frac{9n^{2\alpha}}{2n+1}\right| & \text{if } n \text{ is even} \end{cases}$$

so

$$|a_2(n)| \geq c\, n^{2\alpha-1}, \quad c > 0.$$

In view of Lemma 4, this implies that $R_n < \infty$ for $\alpha \in [0,1)$.

Therefore, by (2.1) in Theorem 3, for each $\alpha \in [0,1)$, we have

$$n^{2\alpha-1} \leq |a_2(n)| \leq 2C(\alpha) R_n^{-1} \max(n^\alpha, R_n^{\frac{\alpha}{2-\alpha}}), \quad n \geq n_0. \tag{4.3}$$

If $n^\alpha \leq R_n^{\frac{\alpha}{2-\alpha}}$, then $R_n \geq n^{2-\alpha}$ and (4.3) gives $n^{2\alpha-1} \leq 2C(\alpha) R_n^{\frac{2\alpha-2}{2-\alpha}}$, which implies

$$2C(\alpha) \geq n^{2\alpha-1} R_n^{\frac{2-2\alpha}{2-\alpha}} \geq n^{2\alpha-1} n^{2-2\alpha} = n.$$

Therefore, we have $\max(n^\alpha, R_n^{\frac{\alpha}{2-\alpha}}) = n^\alpha \quad \text{for} \quad n > 2C(\alpha)$. So, (4.3) implies

$$R_n \leq 2C(\alpha) n^{1-\alpha} \quad \text{for} \quad n > 2C(\alpha).$$

On the other hand, by Proposition 4 of [5, p.296], we have $R_n \geq \frac{1}{8} n^{1-\alpha}$ for large enough n. Hence, we have shown that in the special case of (4.1)–(4.2),

$$R_n \asymp n^{1-\alpha}. \tag{4.4}$$

2. Of course we can simplify the example (4.1) by choosing

$$b_k = c_k = \left[1 + (-1)^{k-1}\right] k^\alpha \tag{4.5}$$

This ensures that $L + zB - E(z)I$ has the structure of a tri–diagonal matrix with 2×2 blocks along the diagonal. The m^{th} block will have the form

$$\begin{bmatrix} T-E & zb \\ zb & V-E \end{bmatrix}, \tag{4.6}$$

where

$$T = (2m-1)^2, \quad V = (2m)^2, \quad b = (2m-1)^\alpha, \quad m = 1, 2, \ldots.$$

It follows that the two eigenvalues corresponding to this block are

$$E(z) = \frac{1}{2}\left(T + V \pm \sqrt{(T-V)^2 + 4z^2b^2}\right).$$

So, the branching points of these branches of $E(z)$ occur at

$$z_{1,2} = \pm i \left(\frac{V-T}{2b}\right). \tag{4.7}$$

Hence, we have

$$z_{1,2}^m = \pm \frac{i(4m-1)}{2(2m-1)^\alpha} = \pm i(2m)^{1-\alpha}\left(1 + \frac{2\alpha - 1}{4m} + O(m^{-2})\right) \tag{4.8}$$

Therefore,

$$R_{2m-1} = R_{2m} \sim (2m)^{1-\alpha},$$

i.e., we have the same sharp order $1-\alpha$ as in (4.4).

3. This simplified example (4.5) is extreme in the sense that the spectral Riemann surface (SRS)

$$G(B) = \{(z,E) \in \mathbb{C}^2 : \quad (L + zB)f = Ef, \quad f \in \ell^2,\ f \neq 0\}$$

splits: it is a union of Riemann surfaces defined by determinants of the blocks (4.6), i.e.,

$$E^2 - E[(2m-1)^2 + (2m)^2] + (2m-1)^2(2m)^2 - z^2(2m-1)^2 = 0, \quad m \in \mathbb{N}.$$

In the case (4.1) we have no elementary reason to say anything about (ir)reducibility of the spectral Riemann surface $G(B)$ (see more about irreducibility of SRS in [5, 14]).

Nevertheless, we would conjecture that this surface $G(B)$ is *irreducible* if $B \in (4.1)$, or more generally, if

$$b_k = c_k \left(1 + \gamma(-1)^{k-1}\right) k^\alpha, \quad 0 \leq \gamma < 1. \tag{4.9}$$

If $\gamma = 0$ we proved in [5], Theorem 3, such irreducibility for $\alpha = 1/2$ and many but not all $\alpha's$ in $[0; 1/2]$.

If $1 \leq \alpha < 2$ let us choose in (4.6)

$$b = b_m = \frac{1}{B_m}(2m-1)^\alpha, \quad |B_m| \geq 1. \tag{4.10}$$

Then (4.7) holds, so by (4.8)

$$z_{1,2} = \pm i B_m (2m)^{1-\alpha}\left(1 + O(1/m)\right).$$

The sequence $\{B_m\}$ could be chosen in such a way that the set A of accumulation points for $\{z_{1,2}^m\}$ is the entire complex plane $\mathbb{C}$, or for any closed $K \subset \mathbb{C}$ with $K = -K$ we can make $A = K$.

4. Our argument in Section 2, uses Young's and Hölder's inequalities, i.e., the concavity of the function $x^{\alpha/2}$, $1 \leq x < \infty$, $0 \leq \alpha < 2$. It cannot be applied if $\alpha < 0$ although in this case the operator $B \in (1.3)$ is even compact. *Yet, we conjecture that $R_n \leq K(\alpha) n^{2-\alpha}$ holds both for $\alpha \in [\frac{11}{6}, 2)$ and $\alpha < 0$. Moreover, we expect that our conjecture (1.21) holds for $\alpha < 0$ as well.*

References

[1] G. Alvarez, Bender-Wu branch points in the cubic oscillator. J. Phys. A 28 (1995), no. 16, 4589–4598.

[2] C. M. Bender, T. T. Wu, Anharmonic oscillator. Phys. Rev. (2) 184 (1969), 1231–1260.

[3] C. M. Bender, M. Berry, P. N. Meisinger, V. M. Savage, M. Simsek, Complex WKB analysis of energy-level degeneracies of non-Hermitian Hamiltonians. J. Phys. A 34 (2001), no. 6, L31–L36.

[4] E. Delabaere and F. Pham, Unfolding the quartic oscilator, Ann. Phys. 261 (1997), 180–218.

[5] P. Djakov and B. Mityagin, Trace formula and spectral Riemann surfaces for a class of tri–diagonal matrices, J. Approx. Theory **139** (2006), 293-326.

[6] I. C. Gohberg and M. G. Krein, Introduction to the theory of linear nonselfadjoint operators, Volume 18, Translations of mathematical monographs, 1969, AMS.

[7] I. Gohberg, S. Goldberg and N. Krupnik, Traces and Determinants of Linear Operators, 2000, Birkhaeuser Verlag, Basel–Boston–Berlin.

[8] E. Harrell and B. Simon, The mathematical theory of resonances whose widths are exponentially small, Duke Math. J. 47 (1980), 845–902.

[9] T. Kato, Perturbation theory for linear operators, Springer Verlag, Berlin, 1980.

[10] J. Meixner and F. W. Schäfke, Mathieusche Funktionen und Spharoidfunktionen, Springer Verlag, 1954.

[11] J. Meixner, F. W. Schäfke and G. Wolf, Mathieu Functions and Spheroidal Functions and their Mathematical Foundations, Lecture Notes in Math. 837, Springer Verlag, 1980.

[12] B. Simon, Trace Ideals and their applications, London Math. Soc Lecture Notes 35, Cambridge Univ. Press, 1979.

[13] H. Volkmer, Quadratic growth of convergence radii for eigenvalues of two-parameter Sturm-Liouville equations. J. Differential Equations 128 (1996), 327–345.

[14] H. Volkmer, On Riemann surfaces of analytic eigenvalue functions. Complex Var. Theory Appl. 49 (2004), 169–182.

Department of Mathematics, The Ohio State University, 231 West 18th Ave, Columbus, OH 43210, USA

E-mail address: adducij@math.ohio-state.edu

Sabanci University, Orhanli, 34956 Tuzla, Istanbul, Turkey

E-mail address: djakov@sabanciuniv.edu

Department of Mathematics, The Ohio State University, 231 West 18th Ave, Columbus, OH 43210, USA

E-mail address: mityagin.1@osu.edu