

# Stability of Flip and Exchange Symmetric Entangled State Classes under Invertible Local Operations

Z. Gedik\*

*Faculty of Engineering and Natural Sciences, Sabanci University, Tuzla, Istanbul 34956, Turkey*

(Dated: October 14, 2009)

Flip and exchange symmetric (FES) many-qubit states, which can be obtained from a state with the same symmetries by means of invertible local operations (ILO), constitute a one-parameter family of curves in the Hilbert space. Eigenstates of FES ILOs correspond to vectors that cannot be transformed to other FES states. This means equivalence classes of states under ILO can be determined in a systematic way for an arbitrary number of qubits. Furthermore, for entangled states, at the boundaries of neighboring equivalence classes, one can show that when the fidelity between the final state after an ILO and a state of the neighboring class approaches unity, probability of success decreases to zero. Therefore, the classes are stable under ILOs.

PACS numbers: 03.67.-a, 03.65.Ud, 03.67.Bg, 03.67.Mn

Keywords: qubits, quantum entanglement, ILO, LOCC, SLOCC

For composite systems, quantum states which cannot be written as a product of the states of individual subsystems have been recognized since the early days of quantum mechanics [1–3]. This phenomenon, called entanglement, is at the center of quantum information theory since it is a key resource in quantum processes [4]. A natural class of operations suitable for manipulating entanglement is that of local operations and classical communication (LOCC) [5, 6]. For probabilistic transformations, the condition of certainty can be removed to allow conversion of the states through stochastic local operations and classical communication (SLOCC) [7]. This coarse graining simplifies the equivalence classes labeled by continuous parameters in case of the local unitary operations. Two states are equivalent under SLOCC if an invertible local operation (ILO) relating them exists [8]. For example, in case of three qubits,  $|\psi\rangle$  and  $|\phi\rangle$  are in the same equivalence class if  $|\phi\rangle = A \otimes B \otimes C |\psi\rangle$  where  $A$ ,  $B$  and  $C$  are invertible operators corresponding to each party.

All entangled pure states of two qubits can be converted to the Einstein-Podolsky-Rosen (EPR)  $(1/\sqrt{2})(|00\rangle + |11\rangle)$  state under SLOCC [8, 9]. In other words, there is a single equivalence class. For three qubits, by calculating transformed states explicitly, it has been shown that there are two inequivalent states ( $|GHZ\rangle$  and  $|W\rangle$ ) [8]. Starting from four qubits, equivalence classes of multipartite systems are labeled by at least one continuous parameter [8]. Using the isomorphism  $SU(2) \otimes SU(2) \simeq SO(4)$  from Lie group theory, nine different ways of entanglement have been found for four qubits [10]. There are no complete classifications for five or more qubits. However, for exchange symmetric  $n$ -qubit states, entanglement classification under SLOCC has been achieved by introducing two parameters called diversity degree and degeneracy configuration.

Also, the number of families has been shown to grow as the partition function of the number of qubits [11].

The main purpose of the present work is to study stability of SLOCC equivalence classes under ILO in the FES subspace whose dimension is approximately  $n/2$ . Such a restriction may seem to be an oversimplification when compared to the  $2^n$  dimensional Hilbert space problem. However, FES states are important if one considers for example bosonic qubits where exchange symmetry is essential. The classification method to be presented is distinguished in two ways: It is systematic and it gives information about neighboring equivalence classes and their relative sizes. More important, it justifies the SLOCC classification in the sense that at the boundaries of different classes, probabilities to end up with states arbitrarily close to the states of neighboring classes are shown to be vanishing.

A FES  $n$ -qubit state  $|\psi\rangle$  satisfies  $X^{\otimes n}|\psi\rangle = |\psi\rangle$  where

$$X = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (1)$$

is the qubit flip operator and  $P_{ij}|\psi\rangle = |\psi\rangle$ . Here  $P_{ij}$  is the exchange operator for the  $i^{\text{th}}$  and the  $j^{\text{th}}$  qubits. The three qubit state ( $|GHZ\rangle = (1/\sqrt{2})(|000\rangle + |111\rangle)$ ) is FES, and ( $|W\rangle = (1/\sqrt{3})(|100\rangle + |010\rangle + |001\rangle)$ ) can be made FES with a simple ILO. How near  $|GHZ\rangle$ -type and  $|W\rangle$ -type states can be is one of the questions addressed in this study.

Imposing exchange symmetry greatly simplifies the classification problem. All local operators become identical and thus will be denoted by  $M$ . Qubit flip symmetry reduces the number of parameters, i.e., entries of the  $2 \times 2$  matrix  $M$ , from four to two. Hence, FES ILOs can be written as

$$M = \begin{pmatrix} a & b \\ b & a \end{pmatrix}, \quad (2)$$

where  $a^2 \neq b^2$ . Since  $a$  and  $b$  cannot simultaneously vanish,  $M$  can be simplified further through division by

---

\*Electronic address: [gedik@sabanciuniv.edu](mailto:gedik@sabanciuniv.edu)

$a$  or  $b$ . For  $a \neq 0$

$$M(t) = f(t) \begin{pmatrix} 1 & t \\ t & 1 \end{pmatrix}, \quad (3)$$

where  $t^2 \neq 1$ , and the function  $f(t) \neq 0$  will be shown to be bounded. All of the results of the current work remain the same for the  $b \neq 0$  case where the diagonal and the anti-diagonal of  $M(t)$  are simply interchanged.

Let  $|\psi(0)\rangle$  be a normalized arbitrary  $n$  qubit FES state. All equivalent normalized states can then be written as

$$|\psi(t)\rangle = \frac{M^{\otimes n}(t)|\psi(0)\rangle}{\langle\psi(0)|(M^\dagger M)^{\otimes n}|\psi(0)\rangle}, \quad (4)$$

which are FES as well. They lie on a curve parametrized by  $t$  provided that  $t$  is real. As  $t$  changes from  $-\infty$  to  $\infty$ , excluding  $\pm 1$ ,  $|\psi(t)\rangle$  traces the curve. However, if  $|\psi(0)\rangle$  turns out to be an eigenstate of  $M^{\otimes n}(t)$ , no ILO will alter it or by definition  $|\psi(0)\rangle$  will form an equivalence class by itself. Eigenstates of  $M^{\otimes n}$  are of the form  $\bigotimes_{k=1}^n |\pm\rangle_k$  where  $|\pm\rangle = (1/\sqrt{2})(|0\rangle \pm |1\rangle)$ , and number of  $|+\rangle$  and  $|-\rangle$  states in the Kronecker product are  $p$  and  $q = n - p$ , respectively. Flip symmetric ones are those with even  $q$ . Eigenvalues are given by

$$\lambda_{pq} = f^n(t)(1+t)^p(1-t)^q, \quad (5)$$

and they are  $n!/p!q!$  fold degenerate. The eigenstate  $|\psi_{pq}\rangle$  denotes the FES state obtained by evaluating the symmetric linear combination of degenerate eigenstates. In practice, the only constraint on  $M$  is  $(M^\dagger M)^{\otimes n} / \langle\psi(0)|(M^\dagger M)^{\otimes n}|\psi(0)\rangle \leq I$  if  $M$  is to come from a positive operator-valued measure. Here the denominator gives the probability of obtaining the final state  $|\psi(t)\rangle$  from  $|\psi(0)\rangle$ . For a given set of transformations  $\{M_i\}$  leading to different final states, the normalization condition is  $\sum_i M_i^\dagger M_i = I$  with the immediate consequence  $|f(t)| \leq 1$ .

For two qubits, possible even  $q$  values are 0 and 2. Both are nondegenerate and hence the corresponding eigenstates are separable. Therefore, there are no entangled FES states unreachable from the EPR state by means of ILO, which is a very well known result [9].

In case of three qubits, allowed  $q$  values are the same as above, but this time while  $\lambda_{30}$  corresponds to a separable state  $|S\rangle = |+++ \rangle$ ,  $\lambda_{12}$  is threefold degenerate and it is easy to see that the eigenvector  $|\psi_{12}\rangle = (1/\sqrt{3})(|+--\rangle + |+-+\rangle + |-+-\rangle)$  is equivalent to the  $|W\rangle$  state. Hence,  $|W\rangle$  is distinguished from other three-qubit entangled states, for example from  $|GHZ\rangle$ , in that it is unreachable via ILO which was again noticed earlier [8]. Having real expansion coefficients in the computational basis,  $|GHZ\rangle$  can be written as  $|GHZ\rangle = \cos\theta|\psi_{12}\rangle + \sin\theta|\psi_{30}\rangle$  with  $\theta = \pi/6$ . In other words,  $|GHZ\rangle$  lies on the geodesic connecting the separable  $|S\rangle$  state and the  $|W\rangle$  state transformed into FES form by means of ILOs. For  $|\psi(0)\rangle = |GHZ\rangle$ ,  $|\psi(t)\rangle$  is again on the same geodesic and approaches the

FES  $|W\rangle$  state when  $t \rightarrow -1$ , where the probability  $\langle GHZ|(M^\dagger M)^{\otimes 3}|GHZ\rangle = |f(t)|^6[(1+t^2)^3 + 8t^3]$  goes to zero. Such a tradeoff between fidelity and conversion probability was pointed out and experimentally implemented by Walther *et al.* [12]. On the other hand,  $|\psi(t)\rangle$  tends to  $|S\rangle$  as  $t \rightarrow 1$ . Thus, almost all FES three qubit states are equivalent to  $|GHZ\rangle$  under ILO while  $|W\rangle$  and  $|S\rangle$  are two neighbors of this equivalence class. GHZ-equivalent states are stable under ILO in the sense that the nearer the final state  $|\psi(t)\rangle$  to  $|W\rangle$ , the less is the probability of success.

Allowed  $q$  values for four qubits are 0, 2 and 4. The first and the third are separable  $|\psi_{40}\rangle$  and  $|\psi_{04}\rangle$  states, respectively. The only interesting one is  $|\psi_{22}\rangle$  which is nothing but  $G_{0,-1,0,1}$  in the notation of Ref. [10]. Since there are three different eigenstates, the FES subspace is a sphere. It is possible to show that all one-parameter family of curves start and end on  $|\psi_{40}\rangle$  and  $|\psi_{04}\rangle$ , and they either do not pass through  $|\psi_{22}\rangle$  or the probability decays to zero as  $|\psi(t)\rangle$  approaches  $|\psi_{22}\rangle$ . The four qubit problem is the simplest non-trivial case in the sense that there is more than one curve; there are in fact infinitely many curves. Among the nine classes of four qubit states, the only FES one is  $G_{abcd}$  with  $b = a - d$  and  $c = 0$ , and it corresponds to a great circle on the sphere passing through  $|\psi_{22}\rangle$  and making equal angles with  $|\psi_{40}\rangle$  and  $|\psi_{04}\rangle$  [10]. Thus, all FES four qubit states are equivalent to a  $G_{a,a-d,0,d}$  state under ILO and  $G_{0,-1,0,1}$  is a distinguished entangled state in that it is unreachable starting from the ones with  $a \neq 0$ .  $G_{a,a-d,0,d}$  corresponds to the canonical form  $(|GHZ_4\rangle + \mu|D_4^{(2)}\rangle)/\sqrt{1+|\mu|^2}$  of Ref. [11] where  $|D_4^{(2)}\rangle$  is the Dicke state with two  $|0\rangle$  components and  $\mu = \sqrt{3}(a-d)/(a+d)$ .

In the case of five qubits, there are two entangled FES eigenstates:  $|\psi_{32}\rangle$  and  $|\psi_{14}\rangle$ . On the other hand,  $|\psi_{42}\rangle$  and  $|\psi_{24}\rangle$  are two SLOCC inequivalent six qubit entangled states. The five qubit case is the same as the four qubit problem in the sense that both are three dimensional spaces. In general an odd number qubit problem is equivalent to one less even case. For example, it is possible to see without any calculation that  $|\psi_{32}\rangle$  is a FES entangled state which cannot be reached by ILOs starting from the others. Furthermore, one-parameter family of curves join  $|\psi_{50}\rangle$  to  $|\psi_{14}\rangle$ , but conversion probability decreases to zero as  $|\psi(t)\rangle$  tends to  $|\psi_{14}\rangle$ . In case of six qubits, curves extend from  $|\psi_{60}\rangle$  to  $|\psi_{06}\rangle$ . They either do not pass through  $|\psi_{42}\rangle$  and  $|\psi_{24}\rangle$  or if they reach these two states, probability for such processes vanish.

For  $n$  qubits, the most general FES state can be written as

$$|\psi(0)\rangle = \sum_{q=0}^{2\lfloor n/2 \rfloor} c_{pq} |\psi_{pq}\rangle \quad (6)$$

where the sum runs over even  $q$  so that there will be  $\lfloor n/2 \rfloor + 1$  terms and  $\sum |c_{pq}|^2 = 1$ . Here,  $\lfloor x \rfloor$  stands for the largest integer less than or equal to  $x$ . One or two

of  $|\psi_{pq}\rangle$  states will be separable, for odd and even  $n$ , respectively. Application of a FES ILO gives

$$|\psi(t)\rangle \propto \sum c_{pq}(1+t)^p(1-t)^q|\psi_{pq}\rangle \quad (7)$$

which lies on the sphere  $S^{[n/2]}$ . On the surface of the sphere, a subset described by  $[n/2] - 1$  free parameters will be enough to obtain all FES states by means of ILOs. Clearly, provided that  $c_{0n}c_{n0} \neq 0$ ,  $\lim_{t \rightarrow 1} |\psi(t)\rangle = |\psi_{n0}\rangle$  and  $\lim_{t \rightarrow -1} |\psi(t)\rangle = |\psi_{n-2[n/2], 2[n/2]}\rangle$ , the latter being an entangled state for odd  $n$ . A representative subset can be obtained by taking one point from each such curve connecting the two states. For example, in case of four qubits this is achieved by  $G_{a,a-d,0,d}$  which cuts all the curves from  $|\psi_{40}\rangle$  to  $|\psi_{04}\rangle$ .

Let  $n$  be even so that both  $|\psi_{n0}\rangle$  and  $|\psi_{0n}\rangle$  are separable. Entangled states  $|\psi_{pq}\rangle$  ( $pq \neq 0$ ) will be stable in the sense that either no curve will pass through them or even if there is a curve containing  $|\psi_{pq}\rangle$  (for  $t = 1$  or  $t = -1$ ), probability of success will be decreasing to zero as  $|\psi(t)\rangle$  tends to  $|\psi_{pq}\rangle$ . An easy way to see this behavior is as follows. Let  $c_{pq}$  and  $c_{p'q'}$  ( $n > p > p' > 0$ ) be the only non-vanishing coefficients so that the curve becomes the geodesic connecting  $|\psi_{pq}\rangle$  and  $|\psi_{p'q'}\rangle$ . Since  $|\psi(t)\rangle \propto (1+t)^{p'}(1-t)^q[c_{pq}(1+t)^{p-p'}|\psi_{pq}\rangle + c_{p'q'}(1-t)^{q'-q}|\psi_{p'q'}\rangle]$ ,  $t \rightarrow -1$  and  $t \rightarrow 1$  limits correspond to  $|\psi_{p'q'}\rangle$  and  $|\psi_{pq}\rangle$  states, respectively. The vectors vanish in these two limiting cases and therefore probabilities for these events to occur go to zero. If more than two expansion coefficients are non-vanishing, it is not possible to reach all eigenstates  $|\psi_{pq}\rangle$  in the sense that  $|\psi_{pq}\rangle$  has a small enough neighborhood which does not contain any points of the curve. Only those curves where  $p$  is the largest or smallest of the subscripts of non-vanishing  $c_{pq}$  coefficients pass through  $|\psi_{pq}\rangle$ .

The odd qubit  $n = 2m + 1$  problem is equivalent to the even  $n = 2m$  case in the sense that both are  $d = m + 1$  dimensional problems and there is a one-to-one correspondence between the components of the vectors in two spaces given by

$$c_{2(m-k), 2k} \longleftrightarrow c_{2(m-k)+1, 2k}(1+t), \quad (8)$$

where  $k = 0, 1, 2, \dots, d$ . The factor  $1+t$  does not change the vector since they are all to be normalized but probabilities do change. While  $|\psi_{0,2m}\rangle$  is a reachable and separable state, the corresponding odd space partner  $|\psi_{1,2m}\rangle$  is entangled and cannot be approached arbitrarily closely since the probability decays to zero for such processes.

Stability or unreachability properties are specific to entangled eigenstates of  $M^{\otimes n}$ . For separable states probabilities of being approached are in general nonzero. This is understandable since local measurements are enough to collapse the whole wave function into a separable one.

In conclusion, a systematic method to classify FES  $n$ -qubit entangled states has been presented. It has been shown that ILOs result in a one-parameter family of curves in the Hilbert space. Some entangled states, namely eigenstates of FES ILOs, embedded in other entangled states, have been found to be either totally unreachability, i.e., no curves pass through them, or even if they are on a curve, the probability decays to zero as they are approached. This observation is important because it justifies SLOCC classification. As one approaches to a boundary between two different classes, probabilities get smaller and smaller. Since probability is a continuous function, the same results must hold for states not necessarily FES but in the vicinity of FES ones. Finally, FES entangled  $n$ -qubit states given above are also good tests for algebraic invariants proposed to distinguish SLOCC equivalence classes [13, 14].

## Acknowledgments

This work has been partially supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) under Grant 107T530. The author would like to thank G. Karpat and C. Saçlıoğlu for helpful discussions and the Institute of Theoretical and Applied Physics at Turunç where part of this research has been done.

- 
- [1] E. Schrödinger, *Naturwiss.* **23**, 807 (1935); **23**, 823 (1935); **23**, 844 (1935) Translation: *Proc. APS*, **124**, 323 (1980).
  - [2] A. Einstein, B. Podolsky, and N. Rosen, *Phys. Rev.* **47**, 777 (1935).
  - [3] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, *Rev. Mod. Phys.* **81**, 865 (2009).
  - [4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (Cambridge University Press, Cambridge, England, 2000).
  - [5] C.H. Bennett, G. Brassard, S. Popescu, B. Schumacher, J. A. Smolin, and W. K. Wootters, *Phys. Rev. Lett.* **76**, 722 (1996).
  - [6] C.H. Bennett, D. P. DiVincenzo, J. A. Smolin, and W. K. Wootters, *Phys. Rev. A* **54**, 3824 (1996).
  - [7] C. H. Bennett, S. Popescu, D. Rohrlich, J. A. Smolin, and A. V. Thapliyal, *Phys. Rev. A* **63**, 012307 (2001).
  - [8] W. Dür, G. Vidal, and J. I. Cirac, *Phys. Rev. A* **62**, 062314 (2000).
  - [9] H. K. Lo and S. Popescu, *Phys. Rev. A* **63**, 022301 (2001).
  - [10] F. Verstraete, J. Dehaene, B. De Moor, and H. Verschelde, *Phys. Rev. A* **65**, 052112 (2002).
  - [11] T. Bastin, S. Krins, P. Mathonet, M. Godefroid, L. Lamata, and E. Solano, *Phys. Rev. Lett.* **103**, 070503 (2009).

- [12] P. Walther, K.J. Resch, and A. Zeilinger, Phys. Rev. Lett. **94**, 240501 (2005).      [14] J.G. Luque and J.Y. Thibon, Phys. Rev. A **67**, 042303 (2003).
- [13] A.A. Klyachko, arXiv:quant-ph/0206012v1 (2002).