Cauchy problem for a higher-order Boussinesq equation

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In this study we establish global well-posedness of the Cauchy problem for a higher-order Boussinesq equation.

1 Introduction

In this study we consider global existence and well-posedness of the following Cauchy problem for a higher-order Boussinesq equation:

$$u_{tt} - u_{xx} - u_{xxtt} + \beta u_{xxxxtt} = (g(u))_{xx},\ x \in R,\ t > 0 \tag{1}$$
$$u(x,0) = \phi(x),\ u_t(x,0) = \psi(x) \tag{2}$$

where β is a positive constant, $u(x,t)$ is a real-valued function of two real variables and $g(u)$ is a given function u. At the microscopic level the higher order Boussinesq equation was derived in [1] for the longitudinal vibrations of a dense lattice, in which a unit length of the lattice contains a large number of lattice points. Thus, in the above higher order Boussinesq equation which has been written in terms of the scaled variables, $u = u(x,t)$ is the longitudinal strain, t is time and x is a spatial coordinate along the length of the lattice. The same equation may also be derived at the macroscopic level using the continuum mechanics approach, in particular the nonlocal elasticity theory that includes the effect of long range interatomic forces.

The global existence of Cauchy problem for the generalized double dispersion equation where $\beta = 0$ and a linear term u_{xxxx} is included has been proved in [2]. It is therefore natural to ask how the higher-order dispersive term affects the global existence. In fact, the method presented in [2] for the generalized double dispersion equation was extended to the Cauchy problem (1)-(2) for the higher-order Boussinesq equation in [3]. This paper summarizes the results in [3]. Similar results also have been derived independently in [4].

In what follows $H^s = H^s(R)$ will denote the L^2 Sobolev space on R. For the H^s norm we use the Fourier transform representation $\|u\|_s^2 = \int (1+\xi^2)^s |\hat{u}(\xi)|^2 d\xi$. We use $\|u\|_\infty$ to denote the L^∞ norm.

2 Cauchy Problem

We first establish local well-posedness of the Cauchy problem (1)-(2) in the Sobolev space H^s with any $s > 1/2$. As in [2], the contraction mapping principle is used to prove the following theorem:

Theorem 2.1 *Let $s > 1/2$ and $\varphi \in H^s,\ \psi \in H^s,\ g \in C^k(R)$ with $g(0) = 0$ and $k = \max\{[s-1], 1\}$. Then there is some $T > 0$ such that the nonlinear Cauchy problem is well-posed with solution $u \in C^2([0,T], H^s)$ satisfying*

$$\max_{t\in[0,T]} (\|u(t)\|_s + \|u_t(t)\|_s) \le 2m(\|\phi\|_s + \|\psi\|_s).$$

Moreover we have regularity in the space variable:

Theorem 2.2 *Let $\varphi \in H^s,\ \psi \in H^s,\ g \in C^k(R)$ with $g(0) = 0$ and $k = \max\{[s-1], 1\}$. Suppose further that for some $1/2 < r < s$ and $T > 0$ we have a solution $u \in C^2([0,T], H^r)$. Then $u \in C^2([0,T], H^s)$.*

As usual the solution can be extended to the maximal time T_{max} which is finite only if blow-up occurs. Blow-up is characterized by the equivalent conditions

$$\limsup_{t\to T_{max}^-} [\|u(t)\|_s + \|u_t(t)\|_s] = \infty, \tag{3}$$

$$\limsup_{t\to T_{max}^-} \|u(t)\|_\infty = \infty. \tag{4}$$

Solutions of the Cauchy problem obey the energy identity. Below, we use the operator $\Lambda^{-1} w = F^{-1}[|\xi|^{-1} Fw]$ defined via Fourier transform F.

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Lemma 2.3 *Suppose that $g \in C(R)$, $G(u) = \int_0^u g(p)dp$, $\varphi \in H^1$, $\psi \in H^1$, $\Lambda^{-1}\psi \in H^1$ and $G(\varphi) \in L^1$. Then for the solution $u(x,t)$ of problem (1)-(2), we have the energy identity*

$$E(t) = \left\|\Lambda^{-1}u_t\right\|^2 + \|u\|^2 + \|u_t\|^2 + \beta\|u_{xt}\|^2 + 2\int_{-\infty}^{\infty} G(u)dx = E(0) \tag{5}$$

for all $t > 0$ for which the solution exists.

Finally we prove the following theorem about the global existence:

Theorem 2.4 *Assume that $s \geq 1$, $g \in C^k(R)$ with $g(0) = 0$ and $k = \max\{[s-1], 1\}$, $\varphi \in H^s$, $\psi \in H^s$, $\Lambda^{-1}\psi \in H^s$, $G(\varphi) \in L^1$, and $G(u) \geq 0$ for all $u \in R$, then the problem (1)-(2) has a unique global solution $u \in C^2([0,\infty), H^s)$.*

References

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