Strictly singular operators and isomorphisms of Cartesian products of power series spaces

By

P. B. Djakov*), S. Önal, T. Terzioğlu and M. Yurdakul

Abstract. V. P. Zahariuta, in 1973, used the theory of Fredholm operators to develop a method to classify Cartesian products of locally convex spaces. In this work we modify his method to study the isomorphic classification of Cartesian products of the kind $E_0^p(a) \times E_\infty^q(b)$ where $1 \leqq p, q < \infty$, $p \neq q$, $a = (a_n)_{n=1}^\infty$ and $b = (b_n)_{n=1}^\infty$ are sequences of positive numbers and $E_0^p(a)$, $E_\infty^q(b)$ are respectively ℓ^p-finite and ℓ^q-infinite type power series spaces.

Introduction. Let $(a_{ik})_{i,k\in\mathbb{N}}$ be a matrix of real numbers, such that $0 \leqq a_{ik} \leqq a_{i,k+1}$ for all i, k and $p \geqq 1$. We denote by $K^p(a_{ik})$ the ℓ^p-Köthe space defined by the matrix (a_{ik}), i.e. the space of all sequences of scalars $x = (x_i)$ such that

$$|x|_k := \left(\sum_i (|x_i| a_{ik})^p \right)^{1/p} < \infty \quad \forall k \in \mathbb{N}.$$

With the topology generated by the system of seminorms $\{|.|_k, k \in \mathbb{N}\}$ it is a Fréchet space. If $a = (a_i)$ is a sequence of positive numbers the Köthe spaces

$$E_0^p(a) = K^p \left(\exp\left(-\frac{1}{k} a_i \right) \right), \quad E_\infty^p(a) = K^p(\exp(k a_i))$$

are called, respectively, ℓ^p-finite and ℓ^p-infinite type power series spaces. They are Schwartz spaces if and only if $a_i \to \infty$.

Power series spaces play an important role in Functional Analysis because they provide sequence space representations for large classes of spaces of (analytic or $\mathbb{C}^\infty$) functions (see for more details [8, 9, 14]). Their isomorphic classification and structure properties were studied by Kolmogorov, Pelczynski, Mityagin and many other mathematicians and the question of isomorphic classification was solved completely in the Schwartzian case (see [8] for details) by the help of classical linear topological invariants, namely approximative and diametral dimensions.

For arbitrary (non-Schwartzian) spaces, Mityagin [10] obtained a complete isomorphic classification of ℓ^2-power series spaces (see also [12]). Moreover, he initiated in this paper a

method to construct new (generalized) linear topological invariants that are more powerful than approximative and diametral dimensions. Zahariuta [18] developed this method for Köthe spaces and obtained new results about isomorphic classification for some classes of Köthe spaces that include Cartesian and tensor products of power series spaces (for further developments see the survey [22]).

Another approach to the isomorphic classification of Cartesian products was Zahariuta's use of the theory of Fredholm operators [19, 20]. We modify Zahariuta's method (following [13], see also [21]) in order to extend its area of applications, and use the modified version to study the isomorphic classification of Cartesian products of the kind $E_0^p(a) \times E_\infty^q(b)$, where a, b are sequences of positive numbers and $p, q \in [1, \infty)$. Let us note that in [19], [20] a complete isomorphic classification of these spaces is obtained in the case where at least one of the sequences a, b tends to ∞ (i.e. at least one of the Cartesian factors is a Schwartz space). On the other hand, in the non-Schwartzian case a complete isomorphic classification of the spaces $E_0^1(a) \times E_\infty^1(b)$ is obtained in [2], [3] by using the appropriate linear topological invariants. In the same way one can characterize the isomorphisms of the spaces $E_0^p(a) \times E_\infty^p(b)$, where p is fixed, $p \in [1, \infty)$. Here we complete these results by studying the non-Schwartzian case for $p \neq q$.

Some of our results are presented without proof in [4].

Preliminaries. Let X and Y be locally convex spaces and $T : X \to Y$ be a continuous linear operator. The operator T is *bounded* (respectively *precompact*) if there exists a neighborhood U of zero in X such that $T(U)$ is bounded (respectively precompact) in Y. The operator T is *strictly singular* if its restriction on any closed infinite-dimensional subspace of X is not an isomorphism.

We write $(X, Y) \in \mathcal{B}$, $(X, Y) \in \mathcal{K}$, $(X, Y) \in \mathcal{SS}$, $(X, Y) \in \mathcal{BSS}$ if every continuous linear operator from X into Y is bounded, precompact, strictly singular, bounded and strictly singular, respectively. Since every precompact operator is bounded and strictly singular the relation $(X, Y) \in \mathcal{K}$ implies $(X, Y) \in \mathcal{BSS}$. The converse is not true. For example, if $1 \leqq p < q < \infty$ then $(\ell^p, \ell^q) \in \mathcal{BSS}$, but $(\ell^p, \ell^q) \notin \mathcal{K}$ since the identity mapping from ℓ^p to ℓ^q is not compact (see [7], Vol. I, Ch. 2. Sect. C).

A Köthe matrix (a_{ik}) is of type (d_1) or (d_2), respectively, if the following condition holds:

(d$_1$) $\exists k_0 \, \forall k \, \exists m, C : \quad a_{ik}^2 \leqq C a_{ik_0} a_{im};$

(d$_2$) $\forall k \, \exists m \, \forall \ell \, \exists C : \quad C a_{im}^2 \geqq a_{ik} a_{i\ell}.$

The corresponding Köthe spaces are referred as (d_1) or (d_2) spaces. It is easy to see that finite (respectively infinite) power series spaces are (d_2) (respectively (d_1)) spaces. V. P. Zahariuta [20] showed that $(X, Y) \in \mathcal{B}$ if X and Y are locally convex spaces with absolute bases, satisfying the conditions (d_2) and (d_1) respectively. Of course then $(X, Y) \in \mathcal{K}$ if X is a Schwartz space or Y is a Montel space. D. Vogt [15] studied the relation $(X, Y) \in \mathcal{B}$ for Fréchet spaces. Using his results (Satz 6.2 and Prop. 5.3 in [15]), one obtains that $(X, Y) \in \mathcal{B}$

if X and Y are Fréchet spaces satisfying the conditions $(\bar{\Omega})$ and (DN) respectively. One can easily see that $(d_2) \Rightarrow (\bar{\Omega})$ and $(d_1) \Rightarrow (DN)$, so the following statement holds.

Proposition 1. *If X is (d_2)-Köthe space and Y is (d_1)-Köthe space then $(X, Y) \in \mathscr{B}$. In particular, for any $p, q \in [1, \infty)$ we have $(E_0^p(a), E_\infty^q(b)) \in \mathscr{B}$.*

A locally convex space X is called *Mackey-complete* if for every bounded, absolutely convex, closed subset $A \subset X$ the linear span of A is a Banach space with unit ball A. It is easy to see that a sequentially complete locally convex space is Mackey-complete. In particular every Fréchet space is Mackey-complete.

An operator acting between two linear topological spaces is *Fredholm* if it is an open mapping with finite dimensional kernel and finite codimensional closed range. An operator T acting in a linear topological space X will be called *Riesz type* operator if $1_X - T$ is a Fredholm operator. The following proposition is due to V. V. Wrobel (see [16], Th. 3 and [17], Satz 1):

Proposition 2. *Bounded strictly singular operators between Mackey-complete spaces form an ideal of Riesz type operators.*

If E and $\tilde{E}$ are linear topological spaces with bases (e_i) and $(\tilde{e}_j)$ respectively, then an operator $T : E \to \tilde{E}$ is *quasidiagonal* if there exist scalars r_i and a mapping $j(i)$ such that $Te_i = r_i \tilde{e}_{j(i)} \; \forall i$. The spaces E and $\tilde{E}$ are *quasidiagonally isomorphic* (we write $E \overset{qd}{\cong} \tilde{E}$) if there exists an isomorphism $T : E \to \tilde{E}$ that is quasidiagonal.

Let $K^p(a_{ik})$ be a ℓ^p-Köthe space, $1 \leqq p < \infty$. For any strictly increasing sequence of integers $(j(i))$ the Köthe space $K^p(a_{j(i)k})$ is called *a basic subspace* of $K^p(a_{ik})$. Obviously each basic subspace is complemented. C. Bessaga [1] made the following conjecture:

If X is a (nuclear) Köthe space then every complemented subspace of X with basis is quasi-diagonally isomorphic to a basic subspace of X.

Modifying some ideas of M. Dragilev [5], C. Bessaga [1] proved his conjecture for stable nuclear finite or infinite power series spaces. Later many authors worked on this subject and Bessaga's conjecture was proved for wide classes of spaces. In particular, for power series spaces the following proposition holds (see [10, 11] and [6]).

Proposition 3. *If X is ℓ^p-finite (respectively ℓ^p-infinite) power series space then every complemented subspace of X with ℓ^p-absolute basis is quasi-diagonally isomorphic to a basic subspace of X.*

Finally we consider the following

Proposition 4. *If $1 \leqq p < q < \infty$ and $K^p(a_{ik}) \cong K^q(b_{ik})$ then the space $K^p(a_{ik})$ is nuclear.*

Proof. We consider two cases: $p < 2$ and $p \geqq 2$.

Let $p < 2$. It is easy to see that the space $K^p(a_{ik})$ is nuclear if and only if

$$\exists r \, \forall k \, \exists m : \quad \sum_{i=1}^{\infty} \left(\frac{a_{ik}}{a_{im}} \right)^r < \infty.$$

Since $(\ell^q, \ell^p) \in \mathscr{K}$ the space $K^p(a_{ik})$ is Schwartzian. Let $T : K^p(a_{ik}) \to K^q(b_{ik})$ be an isomorphism. Then we have

$$\forall\, k\; \exists\, k_1,\, m,\, C_1,\, C_2 : |x|_k \leqq C_1|Tx|_{k_1} \leqq C_2|x|_m.$$

Choose m big enough so that $a_{ik}/a_{im} \to 0$ (it is possible since the space is Schwartzian). We can assume without loss of generality that the sequence (a_{ik}/a_{im}) is decreasing (if not, one can reorder it).

Using the fact that the space ℓ^q is of type $q_1 = \min(2, q)$ (see [7], Vol. 2, p.72) we obtain, for any n, a θ_i, $i = 1, \ldots, n$, $\theta_i = 1$ or $\theta_i = -1$ such that

$$\frac{a_{nk}}{a_{nm}} n^{1/p} \leqq \left(\sum_{i=1}^{n} \left(\frac{a_{ik}}{a_{im}} \right)^p \right)^{1/p} = \left| \sum_{i=1}^{n} \theta_i \frac{e_i}{a_{im}} \right|_k \leqq C_1 \left| \sum_{i=1}^{n} \theta_i \frac{Te_i}{a_{im}} \right|_{k_1}$$

$$\leqq MC_1 \left(\sum_{i=1}^{n} \left(\frac{|Te_i|_{k_1}}{a_{im}} \right)^{q_1} \right)^{1/q_1} \leqq MC_2 n^{1/q_1},$$

where M is a constant. From here it follows that

$$\frac{a_{nk}}{a_{nm}} \leqq MC_2 n^{\frac{1}{q_1} - \frac{1}{p}},$$

therefore the sequence (a_{ik}/a_{im}) belongs to ℓ^r for $r > pq_1/(q_1 - p)$.

In the case $p \geqq 2$ we show that the space $K^q(b_{ik})$ is nuclear using the fact that ℓ^p has cotype p. Since T^{-1} is an isomorphism we have

$$\forall\, k\; \exists\, k_1,\, m,\, C_1,\, C_2 : |x|_k \leqq C_1|T^{-1}x|_{k_1} \leqq C_2|x|_m.$$

As in the first case, we can assume that the sequence (b_{ik}/b_{im}) is decreasing, and we obtain

$$\frac{b_{nk}}{b_{nm}} n^{1/p} \leqq \left(\sum_{i=1}^{n} \left(\frac{b_{ik}}{b_{im}} \right)^p \right)^{1/p} = \left(\sum_{i=1}^{n} \left| \frac{e_i}{b_{im}} \right|_k^p \right)^{1/p} \leqq C_1 \left(\sum_{i=1}^{n} \left| \frac{T^{-1}e_i}{b_{im}} \right|_{k_1}^p \right)^{1/p}$$

$$\leqq MC_1 \left| \sum_{i=1}^{n} \theta_i \frac{T^{-1}e_i}{b_{im}} \right|_{k_1} \leqq MC_2 \left| \sum_{i=1}^{n} \theta_i \frac{e_i}{b_{im}} \right|_m = MC_2 n^{1/q}.$$

From here the nuclearity follows as in the first case.

Modification of Zahariuta's method. We present now the modification of the method for isomorphic classification developed by Zahariuta in [20]. As usual we identify an operator $T = (T_{ij}) : E_1 \times E_2 \to F_1 \times F_2$ with the corresponding 2×2-matrix, whose entries are operators acting between the factors of the Cartesian products.

Lemma 1. *If* $T = (T_{ij}) : E_1 \times E_2 \to F_1 \times F_2$ *is an isomorphism such that* $T_{11} : E_1 \to F_1$ *is also an isomorphism then* $E_2 \cong F_2$.

Proof. Let $T^{-1} = (S_{ij})$. Consider the operators

$$S_{22} : F_2 \to E_2, \quad H : E_2 \to F_2,$$

where $H = T_{22} - T_{21}T_{11}^{-1}T_{12}$. Taking into account the fact that $T_{11}S_{12} + T_{12}S_{22} = 0$ we obtain

$$HS_{22} = T_{22}S_{22} - T_{21}T_{11}^{-1}T_{12}S_{22} = T_{22}S_{22} + T_{21}S_{12} = 1_{F_2}.$$

In an analogous way from $S_{21}T_{11} + S_{22}T_{21} = 0$ it follows that

$$S_{22}T_{22} - S_{22}T_{21}T_{11}^{-1}T_{12} = S_{22}T_{22} + S_{21}T_{12} = 1_{E_2}.$$

Hence the spaces E_2 and F_2 are isomorphic.

Theorem 1. *If $X_1, X_2, \tilde{X}_1, \tilde{X}_2$ are linear topological spaces such that $X_1 \times X_2 \cong \tilde{X}_1 \times \tilde{X}_2$ and each operator acting in X_1 that factors over $\tilde{X}_2$ is Riesz type operator then there exist a finite dimensional subspace L_1 in X_1 and complemented subspaces $E_1 \subset X_1$ and $M_1 \subset \tilde{X}_1$ such that*

$$X_1 \cong E_1 \times L_1, \quad \tilde{X}_1 \cong E_1 \times M_1, \quad M_1 \times \tilde{X}_2 \cong L_1 \times X_2.$$

Proof. Let $T = (T_{ij}) : X_1 \times X_2 \to \tilde{X}_1 \times \tilde{X}_2$ be an isomorphism, and let $T^{-1} = (S_{ij})$. Then we have $S_{11}T_{11} + S_{12}T_{21} = 1_{X_1}$, and since $S_{12}T_{21}$ is Riesz type operator $S_{11}T_{11}$ is a Fredholm operator. Thus the subspace $L_1 := \ker S_{11}T_{11}$ has finite dimension, the subspace $G = S_{11}T_{11}(X_1)$ is closed and has finite codimension. Let E_1 be a complementary subspace of L_1 in X_1 and π_G be a projection on G. Obviously the operator $A = S_{11}T_{11} : E_1 \to G$ is an isomorphism. We set $F_1 = T_{11}(E_1)$; then T_{11} maps E_1 into F_1 isomorphically. Moreover F_1 is a complemented subspace of $\tilde{X}_1$. Indeed, it is easy to see that the operator

$$P = T_{11}A^{-1}\pi_G S_{11} : \tilde{X}_1 \to \tilde{X}_1$$

is a projection on F_1.

Let $M_1 = P^{-1}(0)$ be the corresponding complementary subspace. Then we have

$$X_1 \cong E_1 \times L_1, \quad \tilde{X}_1 = F_1 \oplus M_1 \cong E_1 \times M_1,$$

so applying the lemma to $E_1 \times (L_1 \times X_2)$ and $F_1 \times (M_1 \times \tilde{X}_2)$ we obtain $L_1 \times X_2 \cong M_1 \times \tilde{X}_2$.

For any locally convex space X and any integer s, the symbol $X^{(s)}$ denotes an s-codimensional subspace of X if $s \geqq 0$ and a product of the kind $X \times L$, where $\dim L = -s$, if $s < 0$.

Corollary 1 (see [20], §6). *Retaining the assumptions of the theorem, if each operator acting in $\tilde{X}_1$ that factors over X_2 is Riesz type operator, then the subspace M_1 has finite dimension, so $\tilde{X}_1 \cong X_1^{(s)}$, $\tilde{X}_2 \cong X_2^{(-s)}$ with $s = \dim L_1 - \dim M_1$.*

Proof. By the assumption it follows immediately that each operator acting in M_1 that factors over X_2 is Riesz type operator. Applying the theorem to the isomorphism $M_1 \times \tilde{X}_2 \cong L_1 \times X_2$ we see that there exist complementary subspaces M_2 and M_3 of M_1 such that M_3 is finite dimensional and M_2 is isomorphic to a complemented subspace of L_1, hence M_1 has finite dimension.

Isomorphisms of Cartesian products of power series spaces. We begin with the following lemma.

Lemma 2. *If $X = \mathrm{proj}_k X_k$ and $Y = \mathrm{proj}_m Y_m$ are projective limits of normed spaces such that $\forall k, m \ (X_k, Y_m) \in \mathcal{SS}$ then each bounded operator $T : X \to Y$ is strictly singular.*

Proof. Suppose, on the contrary, there is a bounded operator $T : X \to Y$ that is not strictly singular. Then there exists an infinite-dimensional subspace $M \subset X$ such that the restriction $T|_M$ is an isomorphism. This means in particular that

$$\forall k \ \exists m(k), \ C_k : |x|_k \leq C_k |Tx|_{m(k)} \quad \forall x \in M.$$

On the other hand, since T is bounded, we have

$$\exists k_0 \ \forall m \ \exists D_m : |Tx|_m \leq D_m |x|_{k_0} \quad \forall x \in X,$$

therefore

$$|x|_{k_0} \leq C_{k_0} |Tx|_{m(k_0)} \leq C_{k_0} D_{m(k_0)} |x|_{k_0} \quad \forall x \in M,$$

i.e., if we consider T as an operator from X_{k_0} to $Y_{m(k_0)}$ then its restriction to M is an isomorphism. This contradicts the assumption $(X_{k_0}, Y_{m(k_0)}) \in \mathscr{SS}$, so the lemma is proved.

Theorem 2. *Let* $p \neq \tilde{q}$, $q \neq \tilde{p}$, $1 \leq p, q, \tilde{p}, \tilde{q} < \infty$, *and* $a, b, \tilde{a}, \tilde{b}$ *be sequences of positive numbers. Then the following conditions are equivalent:*

(i) $E_0^p(a) \times E_\infty^q(b) \cong E_0^{\tilde{p}}(\tilde{a}) \times E_\infty^{\tilde{q}}(\tilde{b})$;
(ii) *there exists an integer* s *such that*

$$E_0^{\tilde{p}}(\tilde{a}) \cong (E_0^p(a))^{(s)} \quad and \quad E_\infty^{\tilde{q}}(\tilde{b}) \cong (E_\infty^q(b))^{(-s)}.$$

Proof. By Proposition 1

$$(E_0^p(a), E_\infty^{\tilde{q}}(\tilde{b})) \in \mathscr{B} \quad and \quad (E_0^{\tilde{p}}(\tilde{a}), E_\infty^q(b)) \in \mathscr{B}.$$

We apply Lemma 2 to the pairs of spaces

$$E_0^p(a) = \text{proj}_k \ell^p \left(\exp\left(-\frac{1}{k} a_n\right)\right), \quad E_\infty^{\tilde{q}}(\tilde{b}) = \text{proj}_k \ell^{\tilde{q}} (\exp k \tilde{b}_n))$$

and $E_0^{\tilde{p}}(\tilde{a}), E_\infty^q(b)$.

Since $(\ell^p, \ell^q) \in \mathscr{SS}$ for $p < q$ and $(\ell^p, \ell^q) \in \mathscr{K}$ for $p > q$ (see [7], Vol. I, Ch. 2, Sect. C) the assumptions of Lemma 2 are fulfilled, hence

$$(E_0^p(a), E_\infty^{\tilde{q}}(\tilde{b})) \in \mathscr{BSS} \quad and \quad (E_0^{\tilde{p}}(\tilde{a}), E_\infty^q(b)) \in \mathscr{BSS}.$$

Now Corollary 1 completes the proof.

Corollary 2. *Under the assumptions of the theorem it follows from Proposition 4 that:*

if $p \neq \tilde{p}$ *the spaces* $E_0^p(a), E_0^{\tilde{p}}(\tilde{a})$ *are nuclear;*
if $q \neq \tilde{q}$ *the spaces* $E_\infty^q(b), E_\infty^{\tilde{q}}(\tilde{b})$ *are nuclear.*

In the next theorem we consider the case when one of the conditions $p \neq \tilde{q}$, $q \neq \tilde{p}$ does not hold.

Theorem 3. *Let* $a, b, \tilde{a}, \tilde{b}$ *be sequences of positive numbers and* $p, q, \tilde{q} \in [1, \infty)$, $p \neq \tilde{q}$. *Then*

(i) $E_0^p(a) \times E_\infty^q(b) \cong E_0^q(\tilde{a}) \times E_\infty^{\tilde{q}}(\tilde{b})$;

if and only if

(A) *when $\tilde{a}$ or b tends to* ∞, *there exists an integer s such that*

$$E_0^q(\tilde{a}) \cong (E_0^p(a))^{(s)} \quad \text{and} \quad E_\infty^{\tilde{q}}(\tilde{b}) \cong (E_\infty^q(b))^{(-s)};$$

(B) *when non tends to* ∞, *there exist complementary subsequences $\tilde{a}'$, $\tilde{a}''$ of $\tilde{a}$ and b', b'' of b such that $\tilde{a}''$, b'' are bounded and, further,*

$$E_0^p(a) \cong E_0^q(\tilde{a}') \quad \text{and} \quad E_\infty^{\tilde{q}}(\tilde{b}) \cong E_\infty^q(b').$$

Proof. As in Theorem 2 it follows

$$(E_0^p(a), E_\infty^{\tilde{q}}(\tilde{b})) \in \mathscr{BSS} \quad \text{and} \quad (E_0^q(\tilde{a}), E_\infty^q(b)) \in \mathscr{B}.$$

In the case (A) at least one of the spaces $E_0^q(\tilde{a})$, $E_\infty^q(b)$ is Schwartzian, therefore

$$(E_0^q(\tilde{a}), E_\infty^q(b)) \in \mathscr{K} \subset \mathscr{BSS}.$$

Thus the result follows from Corollary 1.

In the case (B) we have, by Theorem 1, a finite codimensional subspace of $E_0^p(a)$ is isomorphic to a complemented subspace of $E_0^q(\tilde{a})$. Since $\tilde{a}_i \nrightarrow \infty$ the space $E_0^q(\tilde{a})$ contains a basic subspace isomorphic to ℓ^q. Since any finite codimensional subspace of ℓ^q is isomorphic to ℓ^q, then $E_0^p(a)$ is isomorphic to a complemented subspace of $E_0^q(\tilde{a})$. By Proposition 3 there exist complementary subsequences $\tilde{a}'$, $\tilde{a}''$ of the sequence $\tilde{a}$ such that

$$E_0^q(\tilde{a}) = E_0^q(\tilde{a}') \oplus E_0^q(\tilde{a}'') \quad \text{and} \quad E_0^p(a) \cong E_0^q(\tilde{a}').$$

From Theorem 1 it follows that $E_\infty^q(b) \cong E_0^q(\tilde{a}'') \times E_\infty^{\tilde{q}}(\tilde{b})$, so again by Proposition 3 there exist complementary subsequences b', b'' of the sequence b such that

$$E_\infty^{\tilde{q}}(\tilde{b}) \cong E_\infty^q(b') \quad \text{and} \quad E_0^q(\tilde{a}'') \cong E_\infty^q(b'').$$

It is easy to see that the subsequences $\tilde{a}''$, b'' are bounded. Indeed, if on the contrary $\tilde{a}''$, b'' are not bounded, then, passing to subsequences and using Proposition 3, one would obtain subsequences $\tilde{a}'''$, b''' such that

$$E_0^q(\tilde{a}''') \cong E_\infty^q(b''')$$

and either $\tilde{a}''' \to \infty$ or $b''' \to \infty$. This is impossible because then any operator from $E_0^q(\tilde{a}''')$ to $E_\infty^q(b''')$ would be compact. Thus the spaces $E_0^q(\tilde{a}'')$ and $E_\infty^q(b'')$ are either finite dimensional, or they are isomorphic to ℓ^q.

Corollary 3. *Under the assumptions of Theorem 3, it follows from Proposition 4:*

in the case (A) *if $p \neq q$ the spaces $E_0^p(a)$, $E_0^q(\tilde{a})$ are nuclear, respectively if $q \neq \tilde{q}$ the spaces $E_\infty^q(b)$, $E_\infty^{\tilde{q}}(\tilde{b})$ are nuclear;*

in the case (B) *if $p \neq q$ then the space $E_0^p(a)$ is nuclear, respectively if $q \neq \tilde{q}$ then the space $E_\infty^{\tilde{q}}(\tilde{b})$ is nuclear.*

Using Mityagin's criterion for isomorphisms of power series spaces [12] one can easily obtain from Theorem 2 and Theorem 3 a complete characterization of isomorphisms

$$E_0^p(a) \times E_\infty^q(b) \cong E_0^{\tilde{p}}(\tilde{a}) \times E_\infty^{\tilde{q}}(\tilde{b}), \quad p \neq \tilde{q} \text{ or } q \neq \tilde{p}$$

in terms of sequences $a, b, \tilde{a}, \tilde{b}$. In particular we have

Theorem 4. *If $p \neq \tilde{q}$ or $q \neq \tilde{p}$ then the following conditions are equivalent:*

(i) $E_0^p(a) \times E_\infty^q(b) \cong E_0^{\tilde{p}}(\tilde{a}) \times E_\infty^{\tilde{q}}(\tilde{b})$;

(ii) $E_0^p(a) \times E_\infty^q(b) \overset{qd}{\cong} E_0^{\tilde{p}}(\tilde{a}) \times E_\infty^{\tilde{q}}(\tilde{b})$.

If all the spaces $E_0^p(a), E_0^{\tilde{p}}(\tilde{a}), E_\infty^q(b), E_\infty^{\tilde{q}}(\tilde{b})$ are non-Schwartz we have:

Theorem 5. *If each of the sequences $a, b, \tilde{a}, \tilde{b}$ does not tend to ∞ and $p \neq \tilde{q}$ or $q \neq \tilde{p}$ then*
$E_0^p(a) \times E_\infty^q(b) \cong E_0^{\tilde{p}}(\tilde{a}) \times E_\infty^{\tilde{q}}(\tilde{b})$ *if and only if* $\tilde{p} = p, \tilde{q} = q$ *and* $E_0^p(a) \overset{qd}{\cong} E_0^p(\tilde{a})$, $E_\infty^q(b) \overset{qd}{\cong} E_\infty^q(\tilde{b})$.

Let us note that the method used here does not work if $p = \tilde{q}$ and $q = \tilde{p}$. Analogs of Theorem 3 and Theorem 4 in the case $p = q = \tilde{p} = \tilde{q} = 1$ were obtained in [2], [3] by using the method of generalized linear topological invariants.

Finally let us note that our approach can be used to obtain analogous results for isomorphic classification of spaces $K^p(a_{ik}) \times K^q(b_{ik})$, where (a_{ik}) is (d_2)-matrix and (b_{ik}) is (d_1)-matrix.

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Anschriften der Autoren:

Plamen Borissov Djakov	Süleyman Önal, Murat Yurdakul	Tosun Terzioğlu
Department of Mathematics	Department of Mathematics	Sabanci University
Sofia University	Middle East Technical University	Istanbul
1164 Sofia	06531 Ankara	Turkey
Bulgaria	Turkey	