On Lelong-Bremermann Lemma

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Abstract. The main theorem of this note is the following refinement of the well-known Lelong-Bremermann lemma:

Let u be a continuous plurisubharmonic function on a Stein manifold Ω of dimension n. Then there exists an integer $m \leq 2n+1$, natural numbers p_s, and analytic mappings $G_s = \left(g_j^{(s)}\right) : \Omega \to \mathbb{C}^m$, $s = 1, 2, ...$, such that the sequence of functions

$$u_s(z) = \frac{1}{p_s} \max \left(\ln \left| g_j^{(s)}(z) \right| : \ j = 1, \ldots, m \right)$$

converges to u uniformly on each compact subset of Ω.

In the case, when Ω is a domain in the complex plane, it is shown that one can take $m = 2$ in the theorem above (section 3); on the other hand, for n-circular plurisubharmonic functions in $\mathbb{C}^n$ the statement of this theorem is true with $m = n+1$ (section 4). The last section contains some remarks and open questions.

1. Introduction

An important consequence of Oka's Theorem about characterization of domains of holomorphy in terms of pseudoconvexity is the result on the coincidence of the class $Psh(D)$ of all plurisubharmonic functions in a *pseudoconvex* domain with the class of all Hartogs functions in D (Bremermann [**5**], see also [**7, 16**]; the one-dimensional case has been investigated considerably earlier by Lelong [**11**]). In equivalent form this result says that every plurisubharmonic function in a pseudoconvex domain D is the regularized upper limit of some sequence $\alpha_i \ln |f_i(z)|$ with f_i analytic in D and $\alpha_i > 0$.

An immediate corollary of the above result is the following statement known also as Lelong-Bremermann Lemma:

Proposition 1. *Let u be a continuous plurisubharmonic function on a pseudoconvex domain D. Then for each compact subset K of D and $\varepsilon > 0$ there exists a natural number N, an analytic mapping $F = (f_i) : D \to \mathbb{C}^N$, and numbers $\alpha_i > 0$ such that*

$$|u(z) - \max\{\alpha_i \ln |f_i(z)| : i = 1, ..., N\}| < \varepsilon, \qquad z \in K.$$

For more recent related results of this type in various modes of convergence we refer the reader to [**10**] and [**6**]. Proposition 1 does not say anything about the

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behavior of the numbers $N = N(K, \varepsilon)$. However information about the bound for N is welcomed in certain investigations like attempts to approximate simultaneously a pluriregular pair by a sequence of similar analytic polyhedral pairs (see, [**12, 13, 18**]).

The main result of this paper (Theorem 1) says that the number N in the above proposition can be taken $\leq 2n+1$, where n is the dimension of D. The proof is based on a generalization of the reduction argument given in [**18**], Lemma 2, combined with a perturbation argument for smooth mappings.

In Section 3 we examine the one-dimensional case in more detail and prove that for the set of all continuous subharmonic functions the least upper bound of the number of analytic functions involved is just two.

In Section 4 we consider some special classes of plurisubharmonic functions for which the number N can be better estimated. In the last section we give final remarks and discuss some unsolved questions.

2. Main theorem

THEOREM 1. *Let u be a continuous plurisubharmonic function on a Stein manifold Ω of dimension n. Then there is a sequence of analytic mappings $G_i = \left(g_j^{(i)}\right) : \Omega \to \mathbb{C}^{2n+1}$ and a sequence of natural numbers p_i such that the sequence*

$$\frac{1}{p_i} \max\left\{\ln\left|g_j^{(i)}(z)\right| : \ j = \overline{1, 2n+1}\right\}, \ i \in \mathbb{N} \tag{2.1}$$

converges to $u(z)$ uniformly on each compact subset of Ω.

PROOF. Fix a compact subset K of Ω and $\delta > 0$. In view of Lelong-Bremermann lemma we can find analytic on Ω functions f_j and $\alpha_j > 0$, $j = 1, \ldots N$ such that

$$u(z) - \frac{\delta}{4} \leq v(z) \leq u(z), \qquad z \in K, \tag{2.2}$$

where $v(z) := \max\left\{\alpha_j \ln|f_j(z)| : j = \overline{1, N}\right\}$. The natural number N, in general, depends upon K and δ. Since $v(z)$ is continuous on K, we can assume, without loss of generality, that $\alpha_j = \frac{1}{q}$, $1 \leq j \leq N$ with some natural number q.

Set $k = 2n+2$ and suppose that $N \geq k$. Consider the set $\mathcal{J}_k$ of all k-tuples $J = (j_1, \ldots j_k)$ such that $1 \leq j_1 < \ldots < j_k \leq N$ and introduce the set

$$\Delta^k := \{w = (w_\nu) \in \mathbb{C}^k : |w_1| = \ldots = |w_\nu| = \ldots = |w_k|\},$$

For each $J = (j_\nu) \in \mathcal{J}_k$ we define the mapping $\Phi_J : \Omega \times \Delta^k \to \mathbb{C}^k$ by the formula $\Phi_J(z, w) := (f_{j_\nu}(z) - w_\nu)$. Since the real dimension of the manifold $\Omega \times \Delta^k$ is $2n+k+1$, in view of Sard's Theorem, the closed set $\Phi_J\left(K \times \Delta^k\right)$ in $\mathbb{C}^k = \mathbb{R}^{2k}$ has Lebesgue measure zero and hence is nowhere dense in $\mathbb{C}^k$. Therefore all the sets

$$S_J := \left\{\zeta = \left(\zeta_j\right) \in \mathbb{C}^N : \left(\zeta_{j_\nu}\right) \in \Phi_J\left(K \times \Delta^k\right)\right\}, \ J = (j_\nu) \in \mathcal{J}_k,$$

are closed and nowhere dense in $\mathbb{C}^N$. So the set $S = \bigcup\limits_{J \in \mathcal{J}_k} S_J$ is also closed and nowhere dense in $\mathbb{C}^N$. Thus for each $\varepsilon > 0$ there is $\eta = \left(\eta_j\right) \in \mathbb{C}^N \setminus S$ with

$$\max\left\{\left|\eta_j\right| : j = \overline{1, N}\right\} < \varepsilon. \tag{2.3}$$

Then the mapping $h = (h_j) := \left(f_j + \eta_j\right)$ has the property:

$$\{z \in K : |h_{j_1}(z)| = \ldots = |h_{j_\nu}(z)| = \ldots = |h_{j_k}(z)|\} = \varnothing \tag{2.4}$$

for every $J = (j_\nu) \in \mathcal{J}_k$. Due to (2.3), (2.2), one can choose ε sufficiently small to provide the estimate

$$|u(z) - w(z)| < \frac{\delta}{2},\ z \in K, \tag{2.5}$$

where $w(z) := \frac{1}{q} \max \{\ln |h_j(z)| : 1 \le j \le N\}$.

Now set $m = k - 1 = 2n + 1$. The property (2.4) helps us to use an idea from [**18**] how to reduce the number of functions from N to m. Namely, we construct a sequence of mappings $h^{(s)} = \left(h_r^{(s)}\right)_{r=1}^{m} \in A(\Omega)^m$, $s \in \mathbb{N}$, by the formula

$$h_r^{(s)}(z) := \sum_{J=(j_\nu)\in\mathcal{J}_r} \left(h_{j_1}(z) \cdot \ldots \cdot h_{j_r}(z)\right)^{s\frac{m!}{r}},\ 1 \le r \le m \tag{2.6}$$

and consider the sequence of functions

$$w_s(z) := \frac{1}{qsm!} \max\left(\ln\left|h_r^{(s)}(z)\right| :\ r = 1, \ldots, m\right),\ s \in \mathbb{N}. \tag{2.7}$$

We shall show that there is $S_0 > 0$ such that

$$|u(z) - w_s(z)| < \delta,\ z \in K,\ s \ge S_0. \tag{2.8}$$

It is easily seen that $\left|h_r^{(s)}(z)\right| \le 2^N \max\left\{|h_j(z)|^{sm!} : 1 \le j \le N\right\}$. Hence, taking into account (2.5), we get the estimate from above

$$w_s(z) \le u(z) + \frac{\delta}{2} + \frac{N\ln 2}{qsm!} \le u(z) + \delta,\quad z \in K,\ s \ge S_1 \tag{2.9}$$

with some $S_1 > 0$. Now we will estimate the sequence (2.7) from below. Fix $z \in K$. Then there is $r = r(z) \le m$ and $J = (j_\nu) \in \mathcal{J}_r$ such that

$$|h_{j_1}(z)| = \ldots = |h_{j_r}(z)| > |h_i(z)|,\ i \notin J.$$

We choose an open neighborhood $U_z \Subset \Omega$ of z so that

$$d(z) := \max_I \sup_{\zeta\in U_z} \left\{\left|\frac{h_{i_1}(\zeta) \cdot \ldots \cdot h_{i_r}(\zeta)}{h_{j_1}(\zeta) \cdot \ldots \cdot h_{j_r}(\zeta)}\right|\right\} < 1,$$

where the outer maximum is taken over all r-tuples $I = (i_\nu) \in \mathcal{J}_r$, $I \ne J$. By continuity, we can suppose also that U_z is such that the conditions

$$|w(z) - w(\zeta)| < \sigma,\ (1-\sigma)|h_j(z)| < |h_j(\zeta)|,\ \zeta \in U_z,\quad j \in J$$

hold with $\sigma > 0$ (this number will be chosen later). Then the inequality

$$\begin{aligned} \left|h_r^{(s)}(\zeta)\right| &\ge |h_{j_1}(\zeta) \cdot \ldots \cdot h_{j_r}(\zeta)|^{\frac{sm!}{r}} \left(1 - \sum_{I\ne J}\left|\frac{h_{i_1}(\zeta) \cdot \ldots \cdot h_{i_r}(\zeta)}{h_{j_1}(\zeta) \cdot \ldots \cdot h_{j_r}(\zeta)}\right|^s\right)^{\frac{m!}{r}} \\ &\ge \left((1-\sigma)|h_{j_1}(z)|\right)^{sm!}\left(1 - 2^N\ d(z)^s\right)^{\frac{m!}{r}} \end{aligned}$$

holds for all $\zeta \in U_z$ with $r = r(z)$. Thus, for $\zeta \in U_z$, we have

$$\begin{aligned} w_s(\zeta) &= \frac{1}{qsm!}\max\left(\ln\left|h_j^{(s)}(\zeta)\right| :\ j = \overline{1,m}\right) \ge \frac{1}{qsm!}\ln\left|h_r^{(s)}(\zeta)\right| \\ &\ge \frac{1}{q}\left(\ln(1-\sigma) + \ln|h_{j_1}(z)|\right) + \frac{1}{qsr}\ln\left(1 - 2^N\ d(z)^s\right) \end{aligned}$$

for sufficiently large s. Since $w(z) = \frac{1}{q} \ln |h_{j_1}(z)| \geq w(\zeta) - \sigma$, we can choose $\sigma = \sigma(z)$ and $S = S(z)$ so that

$$w_s(\zeta) \geq w(\zeta) - \delta/2,\ \zeta \in U_z, \quad s \geq S.$$

A compactness argument together with (2.5) now gives S_2 such that

$$w_s(\zeta) \geq w(\zeta) - \delta/2 \ \geq u(\zeta) - \delta,\ \zeta \in K,\ s \geq S_2.$$

Taking into account (2.9), this yields (2.8) with $S_0 = \max\{S_1, S_2\}$. Hence, setting $g_j := h_j^{(s)}$, $j = 1, \ldots, m$, and $p = qsm!$ with some $s \geq S_0$, we obtain an analytic mapping $G = (g_j)_{j=1}^{2n+1} : \Omega \to \mathbb{C}^{2n+1}$ and a natural number p such that

$$\sup_{z\in K} \left| u(z) - \frac{1}{p} \max\left(\ln |g_j(\zeta)| :\ j = \overline{1, 2n+1}\right) \right| \leq \delta. \tag{2.10}$$

Now consider an exhaustion of the Stein manifold by compact sets $\{K_i\}_{i=1}^{\infty}$ and a sequence of positive numbers $\{\delta_i\}_{i=1}^{\infty}$ that converges to zero. Let $G_i = \left(g_j^{(i)}\right) : \Omega \to \mathbb{C}^{2n+1}$ and p_i be constructed as above for $K = K_i$ and $\delta = \delta_i$, $i \in \mathbb{N}$. Then, due to (2.10), the sequence (2.1) converges to u uniformly on each compact subset of the Stein manifold Ω. □

3. Approximation of subharmonic functions

Here we show that in the one-dimensional case Theorem 1 is true with $N = 2$. First we consider two lemmas. During the preparation of this paper for publication, we became aware about the result (see, preprint [**3**], Theorem 1.2), which is somehow stronger than Lemma 1 below. We decided to keep our proof of this lemma since it is more direct and does not use the Yulmukhamedov Lemma (see, e.g., [**3**], Lemma A).

Lemma 1. *Suppose that μ is a positive Borel measure with a compact support K and the potential*

$$v(z) := \int_K \ln |\zeta - z| \ d\mu(\zeta) \tag{3.1}$$

is continuous on $\mathbb{C}$. Then there exist sequences of polynomials P_s and Q_s of a common degree N_s such that the sequence

$$v_s(z) := \frac{M}{N_s} \max\{\ln |P_s(z)|, \ln |Q_s(z)|\}, \tag{3.2}$$

where $M = \mu(K)$, converges to $v(z)$ uniformly on $\mathbb{C}$.

Proof. Without loss of generality we suppose that $v \in C^2(D)$, hence $\mu = w\, d\lambda$ where w is a continuous function in $\mathbb{C}$, vanishing outside of K, and λ is the Lebesgue measure on $\mathbb{C} = \mathbb{R}^2$. Let $d_s := 2^{-s}$. Given $s \in \mathbb{N}$ and $\alpha = (\alpha_1, \alpha_2) \in \mathbb{Z}^2$ we denote by $\Delta_s(\alpha)$ the square

$$\{x + iy : \alpha_1 d_s < x \leq (\alpha_1 + 1) d_s; \alpha_2 d_s < y \leq (\alpha_2 + 1) d_s\} \tag{3.3}$$

and set $\Lambda_s(\alpha) = \frac{1+i}{2^{s+1}} + \Delta_s(\alpha)$. Let $a_s(\alpha)$ be a center of the square (3.3), $b_s(\alpha)$ its upper-right vertex (that is the center of the square $\Lambda_s(\alpha)$) and A_s be the set of all $\alpha \in \mathbb{Z}^2$ provided that the distance of the square $\Delta_s(\alpha)$ from K does not exceed $2d_s$. The last assumption implies the conditions:

$$K \cap \Delta_s(\alpha) = K \cap \Lambda_s(\alpha) = \varnothing,\ \text{if } \alpha \notin A_s. \tag{3.4}$$

For $\alpha \in A_s$ we choose non-negative integers $m_s(\alpha)$ and $n_s(\alpha)$ so that the inequalities

$$|Mm_s(\alpha) - 8^s \mu(\Delta_s(\alpha))| \le M; \; |Mn_s(\alpha) - 8^s \mu(\Lambda_s(\alpha))| \le M \tag{3.5}$$

hold with $M = \mu(K)$ and $\sum m_s(\alpha) = \sum n_s(\alpha) = 8^s$. We show that the sequences of polynomials

$$P_s(z) = \prod_{\alpha \in A_s} (z - a_s(\alpha))^{m_s(\alpha)}; \; Q_s(z) = \prod_{\alpha \in A_s} (z - b_s(\alpha))^{n_s(\alpha)}$$

are sought-for ones with $N_s = 8^s$.

We introduce E_s (respectively F_s) as a set of all points $z \in \mathbb{C}$ with the distance $\ge 2^{-(s+2)}$ from all zeros of the polynomial P_s (respectively, Q_s). By the construction we have $E_s \cup F_s = \mathbb{C}$.

Using the notation $p_s(z) := \frac{M}{N_s} \ln |P_s(z)|$, we want to show that

$$|v(z) - p_s(z)| \le \varepsilon(s), \; z \in E_s \tag{3.6}$$

with $\varepsilon(s) \to 0$ as $s \to \infty$.

First we prove this estimate with $\widetilde{p}_s(z) := \sum_{\alpha \in A_s} \mu(\Delta_s(\alpha)) \ln |a_s(\alpha) - z|$ instead of $p_s(z)$. Fix $z \in E_s$ and introduce the notation:

$$A'_s := \left\{ \alpha \in A_s : |a_s(\alpha) - z| \ge \sqrt{d_s} \right\}, \; A''_s := A_s \setminus A'_s, \; B_s := \cup_{\alpha \in A''_s} \Delta_s(\alpha).$$

Then

$$\begin{aligned} |v(z) - \widetilde{p}_s(z)| &\le \sum_{\alpha \in A_s} \int_{\Delta_s(\alpha)} |\ln |\zeta - z| - \ln |a_s(\alpha) - z|| \, d\mu(\zeta) \\ &\le I_1 + I_2 + I_3, \end{aligned}$$

where

$$I_1 := - \sum_{\alpha \in A'_s} \int_{\Delta_s(\alpha)} \ln \left(1 - \frac{|\zeta - a_s(\alpha)|}{|a_s(\alpha) - z|} \right) d\mu(\zeta),$$

$$I_2 := C \int_{B_s} |\ln |\zeta - z|| \, d\lambda(\zeta),$$

$$I_3 := C \, \lambda(B_s) \max \{ |\ln |a_s(\alpha) - z| \, | : z \in B_s \}$$

with $C := \max \{ w(z) : z \in K \}$, where w is defined in the very beginning of the proof. Since $|\zeta - a_s(\alpha)| \le d_s < 1/2$ if $\zeta \in \Delta_s(\alpha)$, we have $I_1 \le 2M\sqrt{d_s} =: \varepsilon_1(s)$. On the other hand, $I_2 \le 2\pi C \int_0^{2\sqrt{d_s}} \rho |\ln \rho| \, d\rho =: \varepsilon_2(s)$. Finally, due to the definition of E_s, we have $|a_s(\alpha) - z| \ \ge d_s/4$, therefore $I_3 \le 4\pi C d_s \ (s+2) \ln 2 =: \varepsilon_3(s)$.

Now we set $R_s := 4^s \max \{ |z| : z \in K \}$. Taking into account that $\# A_s \le C' 4^s$ with some constant C' and applying (3.5), we obtain the estimate:

$$\begin{aligned} |p_s(z) - \widetilde{p}_s(z)| &\le \ln R_s \sum_{\alpha \in A_s} \left| \mu(\Delta_s(\alpha)) - \frac{Mm_s(\alpha)}{8^s} \right| \\ &\le MCC' 4^{-s} \ln R_s =: \varepsilon_4(s) \end{aligned}$$

for all $z \in E_s$ such that $|z| \le R_s$. Combining the above estimates, we obtain that the estimate (3.6) holds for all $z \in E_s$ such that $|z| \le R_s$ with $\varepsilon(s) := \sum_{j=1}^4 \varepsilon_j(s)$, which tends to 0 as $s \to \infty$.

The function $\psi(z) := v(z) - p_s(z)$ is harmonic for $|z| > R_s$ and $\psi(z) = c \ln z + h(z)$, where $h(z)$ is harmonic at ∞. But

$$c = \mu(K)\left(1 - 8^{-s} \sum_{\alpha \in L_s} m_s(\alpha)\right) = 0.$$

Therefore, by the maximum principle, the estimate (3.6) is true also if $|z| > R_s$. In the same way one can prove the estimate

$$|v(z) - q_s(z)| \le \delta(s), \ z \in F_s \tag{3.7}$$

for $q_s(z) := \frac{M}{N_s} \ln |Q_s(z)|$ with $\delta(s) \to 0$ (we may assume that $\varepsilon(s) = \delta(s)$). Then, combining (3.6), (3.7), we obtain that

$$q_s(z) - 2\varepsilon(s) \le p_s(z) \le q_s(z) + 2\varepsilon(s), \ z \in E_s \cap F_s. \tag{3.8}$$

Moreover, due to the maximum principle, the right inequality in (3.8) is true also on every disc $\{|z - a_s(\alpha)| < d_s/4\}$, while the left one is so in every disc $\{|z - b_s(\alpha)| < d_s/4\}$. Since $v_s(z) = \max\{p_s(z), q_s(z)\}$, we obtain that the estimate $|v(z) - v_s(z)| \le 3\varepsilon(s)$ holds for all $z \in \mathbb{C}$, what completes the proof. □

Lemma 2. *Let Ω be a domain in $\widehat{\mathbb{C}}$ with the boundary $\partial\Omega$ formed by a finite number of disjoint smooth Jordan curves and $M = \{a_1, \dots, a_k, \dots, a_m\}$ be a finite subset of $\widehat{\mathbb{C}} \setminus \Omega$, containing just one point from each connected component . Suppose that φ is a continuous on $\overline{\Omega}$ and harmonic in Ω function. Then for each $\delta > 0$ there is a function $g \in A\left(\widehat{\mathbb{C}} \setminus M\right)$, such that $g(z) \neq 0$ in $\widehat{\mathbb{C}} \setminus \Omega$ and a natural number q such that*

$$\left|\varphi(z) - \frac{1}{q} \ln |g(z)|\right| < \delta, \ z \in \overline{\Omega}. \tag{3.9}$$

Proof. By Keldysh's approximation theorem (see, e.g., [**2**], 7.9), for each $\delta > 0$ there is a function $\psi(z)$ harmonic in $\widehat{\mathbb{C}} \setminus M$ and such that

$$|\varphi(z) - \psi(z)| < \delta/2, \ z \in \overline{\Omega}.$$

On the other hand, due to Logarithmic Conjugation Theorem (see, e.g., [**1**], 9.15), there exist $s \in A\left(\widehat{\mathbb{C}} \setminus M\right)$ and real numbers $b_1, \dots, b_k, \dots, b_m$ such that

$$\psi(z) = \operatorname{Re} s(z) + b_1 \ln |z - a_1| + \ldots + b_m \ln |z - a_m|$$

for all $z \in \widehat{\mathbb{C}} \setminus M$. Now, choosing integers p_k and $q \in \mathbb{N}$ so that

$$\max\left\{\sum_{k=1}^{m} \left|b_k - \frac{p_k}{q}\right| |\ln |z - a_m|| : z \in \overline{\Omega}\right\} < \frac{\delta}{2},$$

we obtain (3.9) with $g(z) = (z - a_1)^{p_1} \cdot \ldots \cdot (z - a_1)^{p_m} \exp ps(z)$. □

Theorem 2. *Let u be a continuous subharmonic function in a domain $D \subset \widehat{\mathbb{C}}$, $D \neq \widehat{\mathbb{C}}$. Then for any compact subset $K \subset D$ and each $\varepsilon > 0$ there exist functions f_1, f_2, analytic in D, and $\alpha > 0$ such that*

$$|u(z) - \alpha \max\{\ln |f_1(z)|, \ln |f_2(z)|\}| < \varepsilon, \ z \in K. \tag{3.10}$$

Proof. We may assume that $\infty \notin D$, otherwise we apply a change of variables $w = \frac{1}{z-a}$ with $a \notin D$. On the other hand, we may suppose, without loss of generality, that $K = \overline{\Omega}$, where Ω is a domain with the boundary $\partial\Omega$ formed by a finite number of disjoint smooth Jordan curves. By Riesz Representation Theorem (see, e.g., [**14**]) there exist a unique Borel measure μ in D and a function $\varphi \in h(\Omega)$ such that

$$u(z) = \int_K \ln|\zeta - z| \, d\mu(\zeta) + \varphi(z), \ z \in \Omega.$$

It is easy to see that the function $v(z) := \int_K \ln|\zeta - z| \, d\mu(\zeta)$ is continuous on K, hence (by [**9**], Theorem 5.1) in C. So, the function φ is continuously extendible onto K. Therefore, applying Lemmas 1, 2 with $\delta = \varepsilon/2$, we choose $\alpha > 0$, a couple of polynomials (P, Q) and a function $g \in A(D)$, so that the relation (3.10) will be fulfilled if we set

$$f_1(z) := P(z) \, e^{\alpha g(z)}; \quad f_2(z) := Q(z) \, e^{\alpha g(z)},$$

what completes the proof. □

4. Approximation of n-circular plurisubharmonic functions

Let D be a pseudoconvex n-circular (Reinhardt) domain in $\mathbb{C}^n$. Any plurisubharmonic function $u : D \to [-\infty, \infty)$ depending only on moduli of coordinates (we will call such functions *n-circular*) is convex with respect to the variables $t_\nu = \ln|z_\nu|$. It is easily seen that this function can be represented in the form

$$u(z) = \sup\left\{\sum_{\nu=1}^{n} \alpha_\nu \ln|z_\nu| - \beta : (\alpha_1, \ldots, \alpha_\nu, \ldots, \alpha_n; \beta) \in M\right\} \tag{4.1}$$

with some set $M = M(u) \subset \mathbb{R}^{n+1}$. Notice that if D is complete then α_ν are always non-negative in (4.1).

Theorem 3. *Let u be a continuous n-circular plurisubharmonic function in a pseudoconvex n-circular domain D. Then there exists a sequence of rational mappings $Q^{(r)} = \left(Q_j^{(r)}\right) : \mathbb{C}^n \to \mathbb{C}^{n+1}$ such that the sequence*

$$u_r(z) = \max\left\{\ln\left|Q_j^{(r)}(z)\right| : j = \overline{1, n+1}\right\}, \quad r \in \mathbb{N} \tag{4.2}$$

converges to $u(z)$ uniformly on each compact subset of D. If the domain D is complete, then all $Q_j^{(r)}$ are polynomial.

Proof. Fix any compact set $K \subset D$. Since u is continuous, we conclude from the representation (4.1), following [**18**], that for each $\delta > 0$ there is a function

$$v(z) = \frac{1}{q} \sup\left\{\sum_{\nu=1}^{n} m_{\nu,j} \ln|z_\nu| + \beta_j : j = \overline{1, N}\right\}, \tag{4.3}$$

where $m_{\nu,j}$ are integers and q is a natural number, such that

$$|u(z) - v(z)| < \delta, \ z \in K.$$

We will use the notation

$$l_j(t) := \sum_{\nu=1}^{n} m_{\nu,j} \ln t_\nu + \beta_j, \quad \mathbb{R}_+^n \tag{4.4}$$

where $t = (t_\nu) \in \mathbb{R}^n_+ = [0, +\infty)^n$.

Now we want to choose numbers ξ_j so that $|\xi_j| < \delta$, $j = 1, N$, and for each $z \in D$ the supremum in the expression

$$w(z) := \frac{1}{q} \sup \left\{ l_j \left((|z_\nu|) \right) - \xi_j : j = \overline{1, N} \right\} \tag{4.5}$$

is attained for no more than $n+1$ indices j. Set $k = n+2$ and denote by $\mathcal{J}_k$ the set of all k-tuples $J = \{j_1, \ldots, j_k\}$ such that $1 \le j_1 < \ldots < j_k \le N$. Given $J \in \mathcal{J}_k$, we set $\Omega_J := \left\{ t \in \mathbb{R}^n_+ : l_j(t) > -\infty,\ j \in J \right\}$ and introduce the real analytic mapping $\Phi_J : \Omega_J \times \mathbb{R} \to \mathbb{R}^k$ so that $\Phi_J(t, \tau) := \left(l_{j_s}(t) - \tau \right)_{s=1}^k$. Then, applying the considerations similar to the used in the proof of Theorem 1, we conclude that there is a nowhere dense set $S \subset \mathbb{R}^N$ such that for each $J \in \mathcal{J}_k$ and every $t \in \Omega_J$ there are at least two different among the numbers $l_j(t) - \xi_j$, $j \in J$ for any choice of $(\xi_j) \in \mathbb{R}^N \setminus S$, in particular, we can assume that $|\xi_j| < \delta$ for all j. Taking into account that the supremum in (4.3) can be attained only if $l_j((|z_\nu|)) > -\infty$, we conclude now that for any $z \in D$ the supremum in (4.5) is attained for no more than $n+1$ indices j.

The function (4.5) can be written in the form

$$w(z) = \frac{1}{q} \sup \left\{ \ln |h_j(z)| : j = \overline{1, N} \right\},$$

where $h_j(z) := e^{\beta_j - \xi_j} \sqcap_{j=1}^n z_\nu^{m_{\nu,j}}$, $j = \overline{1, N}$, are analytic in D. Now, applying the construction (2.6) with $m = n+1$ to this functions we obtain the sequence (2.7) converging to the function $w(z)$ uniformly on the compact K. Finally, taking a sequence of compact sets K_r exhausting D, $\delta_r \to 0$ and choosing properly $s = s(r)$, we obtain the desired sequence $u_r(z)$ with rational functions $Q_j^{(r)}(z)$, by the construction. If the domain D is complete, then in the above considerations all integers $m_{\nu,j}$ must be non-negative, therefore the functions $h_j(z)$ and hence the functions $Q_j^{(r)}(z)$ will be polynomials. □

Using the representation of Green pluripotential as a plurisubharmonic n-circular function ([**17**], Proposition 1.4.3) yields

Corollary 1. *Suppose that K is a pluriregular polynomially convex n-circular compact subset of a logarithmically convex bounded n-circular domain D. Then the Green pluripotential*

$$\omega(z) := \limsup_{\zeta \to z} \left\{ \sup \left\{ u(\zeta) : u \in Psh(D);\ u|_K \le 1;\ u < 1 \text{ in } D \right\} \right\} \tag{4.6}$$

can be approximated by some sequence (4.2), with polynomial mappings $Q^{(r)}$, uniformly on any compact subset of D.

5. Final remarks

5.1 The following problem arises in connection with Theorems 1,2.

Problem 1. *What is the smallest upper bound for N that works for the class of all continuous plurisubharmonic functions on a given manifold?*

Naturally, such a bound is expected to be $\le 2n$, but in order to prove that some more delicate methods are needed. For certain specific classes of plurisubharmonic functions the technique used here might yield more refined bounds as it was shown in the previous section.

5.2 Another important example has been considered in [**18**]: using the construction similar to (2.6) in a combination with some algebraic considerations (see Lemma 6 there), it is shown there that, in the conditions of Corollary 1, the function (4.6) *can be approximated uniformly on any compact subset of* $D \setminus K$ *by a sequence*

$$u_r(z) = \frac{1}{q_r} \ln \max \left\{ \left| Q_j^{(r)}(z) \right| : j = \overline{1,n} \right\}, \quad r \in \mathbb{N},$$

with polynomial mappings $\left(Q_j^{(r)}\right) : \mathbb{C}^n \to \mathbb{C}^n$ and natural numbers q_r. Further reducing of the number of polynomials $Q_j^{(r)}$ to $n = \dim D$ is due to the fact that the pluripotential (4.6) is a *maximal* plurisubharmonic function ([**15**]) in the annulus $D \setminus K$.

5.3 Suppose that a sequence

$$u_r(z) = \frac{1}{q_r} \ln \max \left\{ \left| f_j^{(r)}(z) \right| : j = \overline{1,n} \right\}, \quad r \in \mathbb{N}, \tag{5.1}$$

with $f_j^{(r)}$ analytic on a given Stein manifold D and $q_r \in \mathbb{N}$, converges uniformly on each compact subset of a subdomain $G \subset D$ to a continuous function $u(z)$. Then it is easily seen that u must be a *maximal* plurisubharmonic function in D, that is $(dd^c u(z))^n \equiv 0$ in D ([**4, 15**]). In connection with **5.2** the following general question arises

Problem 2. *Does the above approximation property characterize the maximality of a continuous plurisubharmonic function? In other words, given Stein manifold* D *and a continuous function* $u \in Psh(D)$ *such that* $(dd^c u(z))^n \equiv 0$ *in a domain* $G \subset D$*, does there exist a sequence (5.1) with* $f_j^{(r)} \in A(D)$ *, converging uniformly on each compact subset of* G *to the function* $u(z)$*?*

As follows from Lemma 2, the question is answered positively for any plane domain. For several variables the positive answer to this question is known only in some particular cases, like that considered in **5.2.**

5.4 Let u be a plurisubharmonic function on a Stein manifold Ω, $\dim \Omega = n$, which can be represented in the form $u(z) = \limsup_{\xi \to z} \limsup_{s \to \infty} \frac{\ln |h_s(\xi)|}{q_s}$, $z \in \Omega$ with $q_s > 0$ and h_s from a given subalgebra L of $A(\Omega)$, that contains the constants. Then Proposition 1 is true with all the functions f_i belonging to the algebra L. Examining the proof of Theorem 1, one can see that all the functions $g_j^{(s)}$ in (2.1) will belong to the algebra L as well. Therefore Theorem 1.2 of Gamelin-Sibony [**8**], combined with such a version of Theorem 1, yields

Theorem 4. *Let* D *be a bounded domain in* $\mathbb{C}^n$ *with smooth boundary, such that* $\overline{D}$ *has a Stein neighborhood basis. Let* u *be a real-valued continuous function on* $\overline{D}$ *that is plurisubharmonic on* D*. Then* u *can be approximated uniformly on* $\overline{D}$ *by functions of the form* $\max \left\{ \alpha_\nu \ln |g_\nu| : \nu = \overline{1, 2n+1} \right\}$*, where* $\alpha_\nu > 0$ *and* g_ν *are analytic in some neighborhood of* $\overline{D}$*,* $\nu = \overline{1, 2n+1}$*.*

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